1. INTRODUCTION

In many incoherent-imaging scenarios, the resolution achieved is severely degraded by atmospheric turbulence. A variety of clever predetection and postdetection correction approaches have been developed to reduce or remove the effects of turbulence-induced blur. Predetection correction, afforded by adaptive optics (AO), is clearly preferable to postdetection correction with regard to SNR. The recent proliferation of AO systems attests to the enormous success of this technology. However, AO systems remain complex and expensive and there will always be residual aberrations due to imperfect wavefront sensing, deformable-mirror fitting errors, and servo-lag errors. When there is a need to remove residual aberration effects after AO correction or when AO is not available, postdetection correction methods have an important role. These methods include shift-and-add [1], speckle imaging [2], multiframe blind deconvolution [3], deconvolution from wavefront sensing [4], and phase-diverse speckle (PDS) [5,6].

Predetection and postdetection correction methods have been developed and exercised largely under a space-invariant imaging model wherein the form of the point-spread function (PSF) is viewed as unchanged over the field of view (FOV), allowing the image to be expressed as a convolution of the object distribution with the PSF. Although nearly all imaging systems exhibit space-variant blur, the space-invariant model is well suited to imaging scenarios for which the isoplanatic angle is sufficiently large. For example, the isoplanatic angle in astronomical imaging at visible wavelengths is commonly ≈5 arc sec, typically yielding many diffraction-limited resolution elements across an isoplanatic patch. In such cases, both predetection and postdetection methods can be exercised under a space-invariant model to give images having reasonable space-bandwidth products. If larger image formats are required, mosaicking methods can be invoked. Some promising mosaicking techniques based on a bispectrum approach have been successfully applied to space-variant imaging scenarios with mild space variance [7–10].

The problem of correcting for turbulence-induced blur is significantly more complicated when the space variance becomes more pronounced, as manifested by a smaller isoplanatic angle and the attendant reduction in the number of resolution elements across the isoplanatic angle. The origin of the space-variant effects associated with imaging through volume turbulence is easily understood and is depicted in Fig. 1. Light emanating from spatially separated object points will encounter different volumes of turbulence that induce different aberrations, thereby yielding PSFs with differing shapes. The degree of space variance is largely dictated by the geometry of the imaging scenario. Space-variant effects are particularly severe in horizontal-path imaging, slant-path (air-to-ground or ground-to-air) geometries, and ground-based imaging of low-elevation satellites or astronomical objects. In these geometries, the isoplanatic angle can be comparable to or even smaller than the diffraction-limited resolution angle. Clearly, space-invariant methods used in conjunction with mosaicking will fail in this regime. We are driven to consider methods that fundamentally model space variance.

Multiconjugate AO systems have been proposed to ac-
complish predetection correction of pronounced space-variant blur. These systems accommodate space-variant effects by placing multiple deformable mirrors at optical conjugates to differing range locations. The deformable mirrors essentially model the volume turbulence with the equivalent of a finite number of phase screens distributed along the optical path. Proposed multiconjugate systems escalate the complexity of conventional AO systems, requiring multiple beacons, multiple wavefront sensors, sophisticated reconstructors, multiple deformable mirrors, and advanced closed-loop control [11–13].

Similarly, researchers have recently considered adapting postdetection methods to accommodate pronounced space-variant blur. The space-variant analog of shift-and-add, which can appropriately be called dewarp-and-add, has been articulated and demonstrated [14]. Both of these methods use a sequence of short-exposure images. Whereas shift-and-add seeks to correct the global tilt in each short-exposure image (appropriate for space-invariant blur), dewarp-and-add corrects the space-variant tilt, which manifests itself as a warping distortion in each frame. Correction of space-variant tilt significantly improves image quality, although higher-order aberrations persist and diffraction-limited performance may not be achieved. Solar astronomers have developed a related procedure, referred to as destretching [15]. Another interesting postdetection method using multiframe image data is based on the concept of “local lucky shots” [16,17]. The basic idea is to utilize an image-quality metric for deciding when and how much to fuse the local image content into an accumulated synthetic image. Results to date for this technology are promising, but proposed methodologies have yet to utilize dewarping or image deconvolution as part of the method.

It is also interesting to consider generalizing the method of phase diversity [18,19] to accomplish postdetection correction of pronounced space-variant blur. Proof-of-concept simulations using space-variant blur have been reported for a one-dimensional blur [20] and for a more physically based (turbulence-induced) two-dimensional blur [21]. However, in many realistic scenarios, a phase-diversity data set may not provide sufficient signal to successfully estimate both a diffraction-limited image and a complicated space-variant blur function. This suggests investigating the use of a richer data set provided by PDS and generalizing the method to accommodate pronounced space-variant blur. Note that PDS can be viewed as a generalization of another popular technique known as multi-frame blind deconvolution (MFBD) – we prefer PDS, which has been shown in various theoretical studies to provide improved performance over MFBD, in particular with regard to aberration estimation [22–26].

In this article, we report on a postdetection correction method that seeks to correct tilt and higher-order space-variant aberrations, with the goal of reconstructing near-diffraction-limited imagery in the presence of pronounced space-variant blur. Our approach has been to generalize the PDS technique by using a physically motivated distributed-phase-screen model. In addition, we present evidence that phase diversity could be used as a beaconless wavefront sensor in a multiconjugate AO system when imaging extended scenes.

In the following section, we briefly review the conventional (space-invariant) PDS algorithm, discuss its generalization to accommodate space-variant scenarios, and formally state the problem of imaging through volume turbulence with PDS. In Section 3, we describe an algorithm that seeks a regularized maximum-likelihood estimate to solve this problem. Simulation results that demonstrate the reconstruction of near-diffraction-limited imagery under both matched and mismatched model assumptions are presented in Section 4. In Section 5 we present evidence that phase diversity could be used as a beaconless wavefront sensor in a multiconjugate AO system. Conclusions are drawn in Section 6.

2. PHASE-DIVERSE SPECKLE

PDS [5,27,28] is an image-acquisition and processing technique that blends the strengths of phase-diversity imaging and speckle (short-exposure) data. In phase-diversity imaging [18,19], two (or more) images are simultaneously collected: a conventional image that is degraded by unknown aberrations and a second (diversity) image that is intentionally defocused by a known amount. PDS data are recorded when a pair of short-exposure images (conventional and diversity) is collected for each of many realizations of aberration, as depicted in Fig. 2. Maximum-likelihood estimation is used to jointly estimate a fine-resolution representation of the object and an aberration function for each realization. PDS has been successfully used in a variety of up-looking scenarios for which the blur is essentially space-invariant over reasonable fields of view. These include the imaging of binary stars [29], artificial satellites [30], and

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Fig. 1. (Color online) Volume turbulence induces space-variant blur. Light emanating from spatially separated object points will encounter different volumes of turbulence, thereby yielding PSFs with differing shapes.

Fig. 2. (Color online) Data-collection scheme for phase-diverse speckle imaging. Short-exposure in- and out-of-focus image pairs are collected for each of multiple turbulence realizations.
solar features [5,6]. Of particular interest is the repeated application of PDS processing to neighboring patches of solar-astronomy data, where the patch size is selected to insure space-invariant processing. These individual reconstructions can be subsequently mosaicked into a near-diffraction-limited reconstruction over a FOV much larger than an isoplanatic patch [31,32]. The successful application of PDS in these space-invariant or mildly space-variant applications has motivated us to generalize PDS to accommodate cases exhibiting pronounced space variance.

In many scenarios, the space variance is so severe that mosaicking methods will fail. Indeed, many realistic scenarios yield isoplanatic patches comparable in size to a single resolution element. In these regimes we utilize a distributed-phase-screen model to approximate the effects of volume turbulence. We also appeal to geometric-optics and weak-scattering approximations for the propagation of the radiation through the volume turbulence. Although these approximations are valid in many applications, diffraction and scintillation effects can also be included with a wave-propagation model when required. The distributed-phase-screen construct is often used to simulate the propagation of light through volume turbulence [33].

Figure 3 illustrates the distributed-phase-screen model for the case of three phase screens. Note that the phase-aberration function that accumulates in the pupil will depend on the field angle and will be the sum of shifted and magnified versions of the three phase screens, where the shifts are proportional to the field position, and the magnification depends on the cone angle and the phase-screen location. For the general case of $L$ phase screens distributed along the propagation path, the $j$th realization of this accumulated phase function is a projection of the in-

$$s_{jk}(x,x') = \frac{|h_{jk}(x-x',x')|^2}{\sum_{x''}|h_{jk}(x'',x')|^2},$$

with the coherent impulse response $h_{jk}$ defined as

$$h_{jk}(x,x') = F[H_{jk}(u,x')],$$

and where the coherent transfer function $H_{jk}$ is given by

$$H_{jk}(u,x') = |H(u)| \exp(i[\theta_{jk}(u,x') + \theta_{jk}(u)]).$$

In these equations, $|H(u)|$ is a binary clear-aperture function, $\theta_{jk}(u)$ is the known diversity function for the $k$th phase diversity (PD) channel, and $F[\cdot]$ represents the discrete Fourier-transform operation given by

$$F(c(u)) = \sum_{u} c(u) \exp \left( \frac{i 2 \pi (x,u)}{N} \right),$$

where $x$ and $u$ are two-dimensional coordinates taking values in the $N \times N$ integer grid $[0, \ldots, N-1] \times [0, \ldots, N-1]$. Note that the space-variant PSF $g_{jk}$ depends on the aberration parameter vector $a_{j}$, which in turn specifies $\phi_{j}(u,x')$. The PDS image-formation process is well approximated by the discrete sum equation

$$g_{jk}(x) = \sum_{x'} f(x') s_{jk}(x,x'),$$

where $x'$ denotes image-plane field position, $x$ denotes collection array position, $f$ is the object properly magnified to

$$x' = -x' f_{o} / (R - f_{o}),$$

where $f_{o}$ is the focal length of the optical system. Invoking this transformation, we find that the $j$th realization of the accumulated phase function in Eq. (1) corresponding to pupil-position vector $u$ and image-plane field-position $x'$ vector is given by

$$\phi_{j}(u,x') = \sum_{l=1}^{L} \phi_{jl} \left[ 1 - \frac{z_{l}}{R} \right] u - \left( \frac{R - f_{o}}{R f_{o}} \right) z x' \right].$$

Within this projection framework, a general prescription is given in [35] for optimally simulating the effects of volume turbulence with a fixed number of phase screens. This is important in that it provides an established framework and a criterion for optimizing a phase-screen configuration (placement and strength) for specific imaging geometries.

Each of the $L$ phase screens is parameterized with coefficients via the expansion

$$\phi_{jl}(u) = \sum_{m} \alpha_{jlm} \psi_{ml}(u),$$

where $\{\psi_{ml}; m\}$ is a convenient set of basis functions for the $l$th phase screen. Notationally, for each realization $j$, $\alpha_{j}$ will denote the whole set of coefficients $\{\alpha_{jlm}\}$. The resultant phase-aberration function, given in Eq. (3), then generates a space-variant PSF. A discrete approximation to the space-variant PSF is mathematically related to the resultant phase-aberration function by

$$\phi_{jl}(u,x') = \sum_{l=1}^{L} \phi_{jl} \left[ 1 - \frac{z_{l}}{R} \right] u - \left( \frac{R - f_{o}}{R f_{o}} \right) z x' \right].$$

This is important in that it provides an established framework and a criterion for optimizing a phase-screen configuration (placement and strength) for specific imaging geometries.

![Figure 3](https://example.com/figure3.png)

Fig. 3. (Color online) Distributed-phase-screen model used to approximate the effects of volume turbulence.
the image plane, the index $k$ denotes the diversity channel, and $g_{jk}$ denotes the “noiseless image.” Though not explicitly incorporated here, important real-system effects, such as the camera modulation transfer function (MTF), optical bandwidth, and differences in channel transmission and magnification, are easily included in the model.

The actual PDS detected-image data are denoted by $d_{jk}(x)$, which are noisy versions of the “noiseless images” $\tilde{g}_{jk}(x)$. A full PDS data set is composed of

$$\{d_{jk}(x), \begin{cases} j = 1, 2, \ldots, J \\ k = 1, 2, \ldots, K \end{cases} \}, \quad (10)$$

and it is these data that are processed in the PDS algorithm. Our problem can now be stated. Given the sequence of $JK$ detected short-exposure images $\{d_{jk}\}$, the diversity phase functions $\{\theta_k\}$, and the clear-aperture function $[H]$, we want to estimate a single undegraded object and the parameters for all realizations of each of the $L$ phase screens.

3. MAXIMUM-LIKELIHOOD IMAGE RECONSTRUCTION

Our PDS processing algorithm is based on the sound theoretical framework of maximum-likelihood estimation (MLE) [36]. To accommodate both photon and read noise in the data, we employ a Poisson/Gaussian mixture model for $d_{jk}$, as given by

$$d_{jk}(x) = \mathcal{P}[g_{jk}(x)] + \epsilon_{jk}(x), \quad (11)$$

where $g_{jk}(x)$ is the noiseless image as given in Eq. (9), $\mathcal{P}[\mu]$ denotes a Poisson random variable with mean of $\mu$, and $\epsilon_{jk}(x)$ is a stationary white Gaussian noise process with mean zero and variance $\sigma^2_{\epsilon}$. In addition, we make the standard assumption of statistical independence between all Poisson and Gaussian noise contributions. Let $\tilde{d}_{jk}(x)$ and $\tilde{g}_{jk}(x)$ denote the original quantities with the constant $\sigma^2_{\epsilon}$ added, that is,

$$\tilde{d}_{jk}(x) = d_{jk}(x) + \sigma^2_{\epsilon}, \quad (12)$$

$$\tilde{g}_{jk}(x) = g_{jk}(x) + \sigma^2_{\epsilon}; \quad (13)$$

then

$$\tilde{d}_{jk}(x) = \mathcal{P}[\tilde{g}_{jk}(x)]. \quad (14)$$

Relation (14) states that the data, corrupted by both photon and read noise and with an appropriate bias added, can be approximated by a Poisson statistical model. Moderate SNR is a sufficient condition for invoking the assumption of Poisson distribution in relation (14) (see [37]). Based on this statistical approximation, the log-likelihood function is given by (ignoring constant terms)

$$\mathcal{L}(f, \{\alpha_j\}) = \sum_{j=1}^{J} \sum_{k=1}^{K} \sum_{x} \{d_{jk}(x)\ln[\tilde{g}_{jk}(x)] - \tilde{g}_{jk}(x)\} \quad (15)$$

where in the last equality, we have invoked that PSF $s_{jk}$, as given in Eq. (5), is normalized to sum to 1 (in the $x$ argument). The MLE are the values of the object $f$ and aberration parameters $\{\alpha_j\}$ that maximize the log-likelihood function. The actual estimation algorithms are based on maximizing the above objective function using an appropriate nonlinear optimization algorithm. A natural choice is a gradient-based optimization scheme because of its search efficiency.

We have derived closed-form expressions for the gradient of the log-likelihood function. The partial derivative with respect to the object parameter $f(x')$ is given by

$$\frac{\partial \mathcal{L}(f, \{\alpha_j\})}{\partial f(x')} = \sum_{j=1}^{J} \sum_{k=1}^{K} \frac{\tilde{d}_{jk}(x)}{\tilde{g}_{jk}(x)} \frac{\partial \tilde{g}_{jk}(x, x')}{\partial f(x')} - JK. \quad (17)$$

The partial derivative with respect to the aberration parameter $\alpha_{jlm}$ is given by

$$\frac{\partial \mathcal{L}(f, \{\alpha_j\})}{\partial \alpha_{jlm}} = \sum_{k=1}^{K} \sum_{x} \frac{\partial \tilde{d}_{jk}(x)}{\partial \alpha_{jlm}} \left( \sum_{x'} f(x') \frac{\partial \tilde{g}_{jk}(x, x')}{\partial \alpha_{jlm}} \right), \quad (18)$$

with

$$\frac{\partial \tilde{g}_{jk}(x, x')}{\partial \alpha_{jlm}} = -2 S_o \left( h_{jk}^*(x - x', x') \frac{\phi_{jlm}}{F_{o}} \left( 1 - \frac{z^2}{R} \right) u - \left( \frac{R - f_{o}}{F_{o}} \right) z_0 x' \right) H_{jk}(u, x') \exp \left( -\frac{(i2\pi(x', u))}{N} \right), \quad (19)$$

where $*$ denotes complex conjugation, $J$ denotes the imaginary part, $S_o$ denotes the denominator in Eq. (5), and $\phi_{jlm}, h_{jk}, H_{jk}$ are defined as in Eqs. (4), (6), and (7), respectively. See [19] for the analogous gradient expressions in the space-invariant case.

For the space-variant PDS algorithm, we utilize these gradient expressions within a limited-memory quasi-Newton (LMQN) optimization scheme [38]. Under this scheme, we optimize the log-likelihood function jointly over both aberration and object-pixel parameters.

Because the sensitivity of the objective function may be wildly different for aberration parameters relative to that of object-pixel parameters, we used a single scale factor to normalize the relative effects of these two varieties of parameters. This scaling may be viewed as a first-order search preconditioning strategy. The scale factor was found empirically by considering a range of values in a Monte Carlo experiment and selecting a value based on convergence success and rate of convergence. Although nonnegativity could readily be invoked to constrain the search, we did not exercise this capability in our simula-
tions because all of our simulated objects were extended scenes on a pedestal so no object pixel approached a zero value. A nonnegativity constraint has significant value for use with objects that include near-zero values, such as a satellite on a black background.

An additional feature of our algorithm is that we estimate an object field that is larger than the FOV in order to accurately model objects that extend beyond the FOV and account for the extended nature of the PSFs. This “guardband” technique has been used with great success in space-invariant scenarios [5].

Finally, we employ a sieve regularization strategy in which the object is represented as a convolution of a rectangular array of delta functions with a Gaussian kernel function having a variable width parameter $\tau$ [5,39]. The parameter $\tau$ specifies the half-maximum width of the Gaussian kernel via the relationship of half-maximum width = $0.833 \tau$. In our simulation results, $\tau$ was selected via experimentation with several different values. The sieve regularization obviates any need to regularize via an iterative-algorithm stopping criterion.

Though not presented here, the MLE approach can readily be adapted to include $a$ priori statistical models on the turbulence-induced aberrations. The details of such an extension leading to Bayesian maximum $a$ posteriori estimation for the space-invariant scenario are given in [40].

4. SIMULATION RESULTS: PDS IMAGE RECONSTRUCTIONS

We have performed a variety of simulation experiments designed to approximate a realistic horizontal-path scenario. In this paper, we will present three sets of simulation results for PDS image reconstruction. The prominent distinctions between the three sets are outlined in Table 1. Essentially, these simulation results represent three different levels of mismatch between the model employed in generating the simulated data and the model assumed within the PDS processing algorithm. In the first experiment, the simulated data were generated with a geometric optics (GeOp) propagation model applied to three phase screens distributed between the object and the telescope. In this experiment, the model for data generation is perfectly matched to the model used in the PDS estimation.

For the second experiment, the simulated data were generated with GeOp propagation using 20 phase screens distributed along the line of sight; these data were again processed by a PDS estimation algorithm which employed only three phase screens under a GeOp propagation model. Thus, there was a model mismatch between the number of phase screens used to create the data and the number used to process the data.

For the first two experiments described, the data were generated using a GeOp propagation model. Whereas this kind of model can be quite good in cases of light-to-moderate turbulence, a wave-propagation model is required to accommodate diffraction and scintillation effects observed with increased turbulence strength. Accordingly, in our third simulation experiment, we employed Fresnel propagation from screen to screen (over 20 phase screens distributed along the line of sight) to create the PDS data. As before, the reconstruction was performed under a GeOp propagation model using only three phase screens. Thus, in this third experiment there was a model mismatch (between data generation and PDS reconstruction) for both the number of phase screens and the propagation model.

All of the simulation experiments performed were designed to approximate a common realistic horizontal-path imaging scenario. Parameters characterizing this scenario are summarized in Table 2. The range was $R = 5.0 \text{ km}$, the wavelength was chosen as $\lambda = 0.5 \text{ m}$, the aperture diameter was $D = 12.5 \text{ cm}$, and we assumed a narrow band incoherent illumination. In this scenario, the diffraction-limited resolution ($R_A/D$) on the object is $2.0 \text{ cm}$. We limited our object to $40 \times 40$ Nyquist-spaced pixels because of the stressing computations of the space-variant imaging equations. The actual FOV was $30 \times 30$ pixels, thus providing a $5$-pixel guardband. The object was a familiar human face with a scale selected to approximately match the $2.0 \text{ cm}$ resolution. The number of phase screens utilized in all of the PDS processing algorithms was three, and they were optimally placed for accuracy in approximating the volume turbulence [35]. More specifically, they were placed so as to minimize the field- and pupil-averaged mean-squared-error between the accumulated phase from a “true” volume turbulence phase and the accumulated phase from the three-phase screen model. The diversity channel was set at $0.5$ waves of defocus (peak-to-valley), which was found via simulation to be approximately optimal. The mean signal levels and read-noise levels for both channels were $\sim 11 \times 10^5$ photoelectrons (per pixel) and three electrons, respectively, suggesting that the primary noise source was

<table>
<thead>
<tr>
<th>Simulation Experiment</th>
<th>Data Generation Model</th>
<th>Number of Phase Screens</th>
<th>PDS Reconstruction Model</th>
<th>Number of Phase Screens</th>
</tr>
</thead>
<tbody>
<tr>
<td>GeOp Matched</td>
<td>Geometric Optics</td>
<td>3</td>
<td>Geometric Optics</td>
<td>3</td>
</tr>
<tr>
<td>GeOp Mismatched</td>
<td>Geometric Optics</td>
<td>20</td>
<td>Geometric Optics</td>
<td>3</td>
</tr>
<tr>
<td>Fresnel Mismatched</td>
<td>Fresnel</td>
<td>19</td>
<td>Geometric Optics</td>
<td>3</td>
</tr>
</tbody>
</table>
signal-dependent photon noise. The sieve parameter was set at \( \tau = 1.0 \) for the first simulation experiment and at \( \tau = 1.33 \) for both of the mismatched simulation experiments.

A. GeOp Matched Simulation

In this subsection, we present the results of the simulation experiment where the model used in the data generation matches the model used in the PDS algorithm. Recall that the object is located 5 km from the telescope. We assume that the intervening turbulence is homogeneous, with a strength specified by the structure parameter \( C_n^2 = 6.6 \times 10^{-15} \text{ m}^{-1/3} \). The Fried seeing parameter for this scenario is found to be \( r_o = 1.79 \text{ cm} \), and the isoplanatic angle, as defined by Fried, is \( \theta_o = 1.7 \text{ \mu rad} \), which translates into an isoplanatic patch of only 6.5 mm on the target. Note that the reported \( r_o \) parameter is for spherical waves. Care should be used in interpreting the isoplanatic angle since this is an asymptotic quantity [41] that may be overly pessimistic in this geometry. Nevertheless, the isoplanatic-angle calculation suggests that the simulations correspond to a highly space-variant regime, since there are approximately three isoplanatic patches per resolution element.

We modeled the volume turbulence with three phase screens optimally distributed along the 5 km optical path. The exact locations of the phase screens relative to the pupil as prescribed by the optimality theory in [35] are shown in Table 3 along with the incremental \( r_o \) values as determined via projection-based calculations. These incremental \( r_o \) values are derived by geometry of the object and the plane under consideration. It is interesting that in this 5 km horizontal-path geometry with uniform turbulence, the optimal phase screen locations are significantly nonuniform. Based on the incremental \( r_o \) values, Kolmogorov realizations were generated for each of the three phase screens via standard methods.

Each phase screen was then approximated with a sinusoidal basis set. Part of the motivation for using a sinusoidal basis set, as opposed to the more traditional Zernike-polynomial basis set, is that they offer significant computational advantages owing to their separability. In particular, the separability allows much faster computation of the two-dimensional phase screen values than would be possible for the more complicated nonseparable Zernike basis functions.

Computation of the accumulated pupil phase function in Eq. (1) requires sampling of each phase-screen basis function after applying an appropriate magnification and shift. Each field position requires a different shift, which is generally not an integer multiple of the pupil sampling. For this reason, it is not possible to store the basis functions as a single two-dimensional array of sampled values, as is common in space-invariant imaging applications. A straightforward approach is to simply evaluate the basis functions on the fly for each object-pupil pixel pair. This can be computationally expensive, especially when complicated functions such as Zernike polynomials are used. Alternatively, the basis-function samples can be precomputed and stored for each object-pupil pixel pair, but the storage required makes this impractical for many applications. Our approach uses separable basis functions that permit the manageable storage of precomputed vectors that can be combined efficiently via an outer product to yield the contribution from each basis function to the accumulated phase. Note that this is an implementation issue only. In principle, any set of basis functions can be used.

As a compromise between fidelity and keeping the computations manageable, we approximated each of the three phase screens with a linear combination of 36 sinusoidal basis functions, so there is a total of 108 phase-screen parameters characterizing each volume-turbulence realization. For each point in the object array we obtained an accumulated pupil phase function by projecting the three phase screens with a cone-beam projection; the corresponding PSF \( s_{jk}(x, x') \) is then derived via Eqs. (3)–(7). Note that in the diversity channel \( (k = 2) \) an additional 0.5

<table>
<thead>
<tr>
<th>Screen Number</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Location (km)</td>
<td>0.644</td>
<td>2.043</td>
<td>3.984</td>
</tr>
<tr>
<td>Incremental ( r_o ) (cm)</td>
<td>2.7</td>
<td>3.1</td>
<td>7.5</td>
</tr>
</tbody>
</table>
wave of quadratic defocus is added to the net pupil phase before computing the PSF.

Noiseless images were formed via Eq. (9). We simulated a full $40 \times 40$ array in the image plane, and we kept only the central $30 \times 30$ subarray to represent the image data, thus accommodating objects that extend beyond the detector FOV. As previously described, the algorithm was run on a total of 10 image pairs (corresponding to 10 different atmospheric realizations), five of which are shown in Fig. 4. As can be seen from these images, the space-variant blur is so severe that there are virtually no recognizable features in the data.

To better illustrate the space-variant nature of the blur, we show in Fig. 5 the conventional images of a five-point object as seen through the same distributed phase screens used to generate the data in Fig. 4. The space variance is clearly manifest in both distortion (PSF position) and the variations in PSF shape. This type of degradation is particularly challenging for computer estimation and correction.

The ten image pairs were processed through the PDS algorithm to generate phase-screen estimates for each of the ten realizations in addition to a single object estimate. In this processing, we actually estimate the object over the full $40 \times 40$ pixel region, which extends beyond the FOV, though we only have confidence in the center $30 \times 30$ pixels shown in Fig. 6. For comparison, we also show the true object and a diffraction-limited image representing the image that would have been collected in the absence of any aberrations. In Fig. 6, we also show upsampled versions of all the images in order to allow for a better interpretation of the results. It is astonishing that the restored object estimate has so much fine detail given the nonliteral nature of the raw data. Quantitatively, the normalized mean-squared error between the restoration and the original (true), after doing a subpixel registration, is 0.054.

The results in Fig. 6 show that the PDS algorithm successfully restores the object. We are also interested in determining whether the PDS algorithm is estimating the distributed-phase-screen aberration coefficients accurately. To answer this question, we show in Fig. 7 scatter plots of the estimated aberration coefficients versus true aberration coefficients for each of the three phase screens and for five realizations. As can be seen in this figure, the scatter plots cluster fairly tightly along the 45° diagonal, indicating that the PDS algorithm also provides reasonably accurate aberration estimates. It is curious that the aberration estimates tend to be most accurate for the phase screen closest to the pupil and be progressively less accurate for screens closer to the object. We have observed this trend in other simulation experiments involving different numbers of phase screens and different aberration levels.

The object, diffraction-limited image, and PDS object estimate for GeOp matched case. Both Nyquist-sampled (upper row) and up-sampled (lower row) versions of these images are given. PDS achieves near-diffraction-limited resolution.
B. GeOp Mismatched Simulation

The results of the GeOp matched experiments are very encouraging. However, real volume turbulence is a continuum that we are approximating with a discrete set of phase screens, and it is important to assess the accuracy of this approximation as it relates to PDS image reconstruction. To this end, we conducted a simulation experiment in which the simulated image data were generated according to a GeOp propagation model using 20 phase screens (thereby closely approximating volume turbulence) and processed with a PDS algorithm assuming only a three-phase-screen model. The exact placement and strength (incremental $r_o$ value) of the 20 phase screens was again done according to theory for optimally approximating volume turbulence as developed in [35].

As in the GeOp matched simulation experiment, we simulated phase screens based on the Kolmogorov theory and approximated each phase screen by projecting onto a set of sinusoidal basis functions. For the GeOp mismatched experiment, the number of basis functions per phase screen was increased to 100 so as to better approximate the volume turbulence. For each of the 20 phase screens, there were 100 phase aberration parameters corresponding to the 100 coefficients for the basis functions, thus bringing the total number of aberration coefficients to 2000 for each volume-turbulence realization. We generated a total of ten PD image pairs, including noise, each corresponding to a volume-turbulence realization. We generated data with comparable levels of space-variant degradation beyond even the GeOp matched case.

![Fig. 7. (Color online) Scatter plots—from the matched case—of PDS-estimated aberration coefficients versus true aberration coefficients for phase screens 1, 2, 3, and five realizations. Here the numbering is such that phase screen 1 is closest to the pupil and phase screen 3 is closest to the object. The index $j$ on the plots refers to aberration realization. These results show that in the matched case, the PDS algorithm generates relatively accurate aberration coefficients.](image-url)
variant degradation in this case, the structure parameter for the volume turbulence was reduced to $C_n^2 = 2.1 \times 10^{-15} \text{ m}^{-2/3}$ resulting in an $r_o = 3.45 \text{ cm}$. Simulated data for five realizations are shown in Fig. 8. Again the data appear to be very degraded. We ran the three-phase-screen PDS algorithm on the ten PD pairs of images. With this algorithm we are estimating 300 aberration parameters for each turbulence realization when in fact 2000 parameters were used to generate the data. The resultant PDS object estimate is shown in Fig. 9. The PDS image reconstruction is again very good with the normalized mean-squared error between the restoration and the original (true) being 0.134. This result represents an important milestone since it shows that even in the presence of what might appear to be a significant mismatch, the three-phase-screen PDS algorithm is sufficiently robust to operate on extremely degraded image data corresponding to realistic volume-turbulence scenarios and reconstruct fine-resolution images. Apparently the three-phase-screen model is sufficiently rich to capture the space-variant blur produced by continuous volume turbulence.

C. Fresnel Mismatched Simulation

The results reported thus far have utilized a GeOp propagation model for both image-data generation and PDS processing. A more realistic approach for generating the image data would be to use a Fresnel-propagation model, thus capturing diffraction and scintillation effects. We conducted a simulation experiment in which image data were generated by modeling the volume turbulence with 19 phase screens and mathematically applying the more realistic model of Fresnel propagation from screen to screen. The PDS algorithm, based on a three-phase screen GeOp model, was then applied to these data to produce image reconstructions. Thus in this third simulation experiment, there is a model mismatch for both the number of phase screens and the propagation model. The assumptions for this simulation experiment were identical to those of the mismatched GeOp experiment, with the following exceptions: (1) In going from mismatched GeOp to Fresnel, the aberration strength was reduced slightly ($C_n^2 = 1.35 \times 10^{-15} \text{ m}^{-2/3}$) in order to achieve image data with comparable space-variant degradations, and (2) we operated on 20 realizations instead of ten. A subset of the Fresnel-propagated image data is provided in the first and third rows of Fig. 11 below, where the conventional images are seen to be similar to the GeOp-propagated data. The resultant image reconstructions in the mismatched Fresnel case are shown in the fourth column of Fig. 9. In this case, the normalized mean-squared error...
between the restoration and the original (true) is 0.147. There is clearly a perceptible loss in quality relative to the image reconstructions from GeOp data. Nevertheless, the facial features, though not as clear, are certainly recognizable and constitute a dramatic improvement over the raw data. This result represents another important milestone in that it shows the PDS algorithm can generate near-diffraction-limited image reconstructions even in certain scenarios where the volume turbulence causes scintillation.

### 5. MULTICONJUGATE AO WITH PDS

The simulation experiments from Section 4 demonstrate the utility of using a space-variant PDS algorithm for postdetection image reconstruction. Since the space-variant PDS algorithm also estimates the aberrations, it can be used for doing predetection correction as well. Specifically, it can be used as a (generalized) wavefront sensor within a multiconjugate AO architecture as depicted in Fig. 10. To implement this idea, we adopt a sliding-window scheme. Short-exposure PD image pairs continually accumulate to form a time series of data. At time \( t_0 \), a window of the \( J \) most-recently accumulated image pairs is processed using the PDS algorithm. The window size \( J \) is selected to provide sufficient data to achieve accurate estimates while insuring that the object is virtually unchanged over these realizations. Distributed-phase-screen estimates for the most recent realization, corresponding to time \( t_0 \), are then applied as phase corrections in the appropriate conjugate planes via deformable mirrors or spatial phase modulators. Given sufficient computational resources, the sliding-window construct affords near-real-time correction.

To demonstrate this predetection correction concept, we conducted a simulation experiment that reused the Fresnel-propagation mismatched simulation data in a sliding-window context. Recall that the salient points for the mismatched Fresnel experiment are that (a) PD data were generated using Fresnel propagation and 19 phase screens, each generated with a projected Kolmogorov turbulence model and parameterized by 100 coefficients; (b) the volume-turbulence realizations are statistically independent, and (c) for each realization, the PDS algorithm generates phase-screen estimates for each of three phase screens with 100 parameters for each phase screen.

For this demonstration we utilized ten (out of 20) volume-turbulence realizations and for this application demonstration we envision a sliding window that consists of ten realizations. The distributed-phase-screen estimate for the most recent realization can then be used to correct the phase at the three corresponding conjugate planes. In practice this would be accomplished with deformable mirrors. Here we do it through simulation. The simulated multiconjugate-AO-compensated image is then formed using Fresnel propagation over the 19 phase screens in addition to the three compensating phase screens. A single compensated image is then derived from the ten-realization sliding window.

Since all ten realizations are statistically independent, we can rearrange the order of which realization is considered to be the “most recent” within the sliding window. There are ten such cases and accordingly, there are ten compensated images that can be derived from this data set. These ten multiconjugate-AO-compensated images are displayed in Fig. 11.

It is clear that in all cases significant improvement is achieved by the multiconjugate compensation. We view these results as representative of how the proposed multiconjugate AO system would operate in “capture mode.” We conclude from these results that the three distributed-phase-screen estimates are physically meaningful in the sense that they successfully model the 19-phase-screen volume turbulence in the mismatched Fresnel experiments. While the scatter plots in Fig. 7 indicate that the estimated phase screens in the matched GeOp experiments are physically meaningful, it was not clear at the outset that this would also be the case in the mismatched experiments.

![Fig. 10. (Color online) Schematic for proposed PDS multiconjugate adaptive-optics system. PD image pairs continually accumulate to form a time series of data. At time \( t_0 \), a window of the \( J \) most-recently accumulated image pairs is processed using the PDS algorithm. Distributed-phase-screen estimates for the most recent realization, corresponding to time \( t_0 \), are then applied as phase corrections in the appropriate conjugate planes. The sliding-window construct affords near-real-time correction.](image-url)
Fresnel experiments. The three phase screens could conceivably have provided a nonphysical model to accommodate the space-variant PSFs needed for image reconstruction. The clear improvement in the compensated images in Fig. 11 demonstrates the potential utility of PDS for predetection correction in addition to postdetection correction.

The results in Fig. 11 represent the performance of PDS multiconjugate AO in open-loop (or capture) mode. This is because none of the image data had any compensation applied to it. In closed-loop operation (maintenance mode), the collected PDS data (current and past PD image pairs) used in the phase-screen estimation would already be compensated and hence would be of a higher quality than is shown from this simulation experiment. Clearly, the performance in closed-loop operation will only improve.

Although the multiconjugate-AO-compensated image data are a significant improvement over the conventional data, they fall short of achieving diffraction-limited quality. This can be attributed in part to the use of three phase screens to approximate what is essentially a continuous volume turbulence (19 phase screens). We can anticipate further loss in performance in real systems due to servo-lag and deformable mirror fitting errors. A natural way to mitigate these shortfalls is to combine predetection and postdetection correction. This hybrid strategy can be accomplished by applying PDS object estimation to the multiconjugate-compensated images, thus producing final image products which are of the highest quality and resolution. It is well known that postdetection correction performs better if predetection correction can be invoked first [30,42].

6. CONCLUSION

We find the results for predetection and postdetection correction of space-variant blur using PDS to be very promi-
ising. We have reviewed simulation results which show that the PDS algorithm can be effective in correcting severe space-variant blur. In the case of postdetection correction, we showed via realistic simulation experiments that the three-phase-screen PDS image-reconstruction algorithm effectively recovers near-diffraction-limited images, starting with imagery in which there is virtually no recognizable detail. We then presented a novel PDS multiconjugate-AO-compensation concept and demonstrated in simulation the potential for this concept in a realistic horizontal-path imaging scenario. Accordingly, PDS is a candidate beaconless multiconjugate wavefront sensor. Ultimately, the most effective PDS imaging system for scenarios with space-variant blur will almost certainly be some combination of predetection and postdetection processing.

Closed-loop wavefront correction has already been demonstrated with phase-diverse wavefront sensing [43–45]. However, the computations required for the space-variant PDS algorithm are considerably greater than are required for space-invariant PD or PDS, as can be ascertained from the imaging equations presented in Section 2. The increase in computational load over space-invariant PDS is roughly proportional to the number of pixels in the object. In the case of a 40 × 40 object this factor for increased computational load is ∼1600. This is because the convolution theorem cannot be used with space-variant blur. Consequently, for each iteration of the MLE algorithm, the PSF must be calculated for each point in the object. To date, we have been able to obtain significant improvements in computational efficiency by exploiting various symmetries and redundancies, but we believe that there is much more that can be gained through algorithm advances. In addition, the PDS algorithm computations parallelize naturally. With these considerations the use of PDS for predetection correction of space-variant blur is challenging but plausible. Reducing and managing computations remains an area of active research for our group. Other areas of ongoing research include accommodating broadband and scintillated data in the algorithm as we apply space-variant PDS algorithms to real data.

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REFERENCES


