Lives of the Stars Lecture 6: Stellar evolution

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In tonight’s lecture

• Changes on the main sequence
  – gradual changes during the star’s life

• The beginning of the end
  – what happens when the star runs out of hydrogen

• Different fates
  – how low-mass and high-mass stars end differently

• The evidence for stellar evolution
  – star clusters
A lingering death

You recall from our discussion of stellar structure that main sequence stars are in a *dynamic equilibrium*: they are winning the battle against gravity, but only at the cost of burning fuel: hydrogen.

What happens when the star runs out of fuel? That leads to massive changes in the star’s structure and appearance: *stellar evolution*. Tonight we will look at the profound changes a star goes through during the last stages of its life.
The fact that gravity determines the structure of the star explains why it is the star’s mass which is by far the most important variable in what it looks like.

This is exemplified in the Herzsprung-Russell diagram, the most obvious feature of which — the main sequence — is a mass sequence, not a time sequence. An F-type star formed as an F-type star, and will remain an F-type star during the entire main sequence stage.
Stars do change somewhat while they are on the main sequence. While a star is on the main sequence, it is burning hydrogen in its core. Helium is formed, and gradually builds up as a sort of ash: *helium poisoning*.

Helium can’t burn, because it needs a temperature of about 100 million degrees before it can ignite. So it just builds up in the core.
As each fusion converts four H atoms to one He atom, the core of the star has fewer atoms in it, so the pressure goes down. Gravity squeezes the core more tightly, which increase the temperature, which increase the rate of fusion. This produces more energy, which makes the outside layers of the star expand a bit, which makes the star a bit brighter and a bit cooler.
These changes in the Sun will have a profound effect on the Earth. When the Sun began its main sequence life about 5 billion years ago, it was only 70% as bright as it is now. In another 5 billion years, it will be roughly twice as bright, which will raise the average temperature of the Earth at least 19° C.

In fact, we have a hard time reconciling these predicted temperatures with the geological record (the “faint young sun paradox”).
How long a star can burn hydrogen on the main sequence depends on two things:

• how much hydrogen it has; and
• how fast it burns it

The first is just the star’s mass, and the second the star’s luminosity.

But we know how those two are related: we derived the mass-luminosity relation when we were looking at stellar models in Lecture 3.
Here it is again: as we increase the mass of the star, the luminosity increases \textbf{enormously}: a factor of 10 increase in mass corresponds to a factor of 3000 increase in luminosity.
We can write this dependence of the star’s luminosity on its mass as

\[ L \propto M^{3.5} \]

so the lifetime of a star goes like

\[ t \propto \frac{M}{L} \propto \frac{M}{M^{3.5}} = M^{-2.5} \]

In other words, a factor of 10 increase in mass corresponds to a **decrease** in the lifetime of the star by a factor of 300.
So we can make a table showing the lifetime for stars of different masses:

<table>
<thead>
<tr>
<th>Spectral type</th>
<th>Mass (Sun=1)</th>
<th>Luminosity</th>
<th>Years on main sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td>O5</td>
<td>40</td>
<td>405,000</td>
<td>1x10^6</td>
</tr>
<tr>
<td>B0</td>
<td>15</td>
<td>13,000</td>
<td>11x10^6</td>
</tr>
<tr>
<td>A0</td>
<td>3.5</td>
<td>80</td>
<td>440x10^6</td>
</tr>
<tr>
<td>F0</td>
<td>1.7</td>
<td>5.4</td>
<td>3x10^9</td>
</tr>
<tr>
<td>G0</td>
<td>1.1</td>
<td>1.4</td>
<td>8x10^9</td>
</tr>
<tr>
<td>K0</td>
<td>0.8</td>
<td>0.46</td>
<td>17x10^9</td>
</tr>
<tr>
<td>M0</td>
<td>0.5</td>
<td>0.08</td>
<td>56x10^9</td>
</tr>
</tbody>
</table>
Interestingly, the amount of energy released per kilogram is almost identical for all types of stars.

\[
\text{energy per kilogram} = \frac{L_t}{M} \\
\propto M^{3.5} \times M^{-2.5}/M \\
= \text{constant}
\]

Massive stars are much more luminous, but live for much less time, so the amount of energy extracted per kilogram is identical to the faintest \( M \) stars.

This is, of course, because all stars are getting their energy from the same source: nuclear reactions.
The beginning of the end

When the core runs out of hydrogen, fusion stops so the core contracts and heats up. This heats the unprocessed hydrogen just outside the core, which then ignites in a spherical shell, burning outwards. This sudden increase of energy forces the outer layers of the star to expand dramatically, so the star becomes enormously large and red: a red giant. Stars like the Sun can increase in radius by a factor of 10–100, while more massive stars can increase by a factor of 1000.
On the HR diagram, the star moves dramatically upwards (getting much brighter) and somewhat to the right (getting cooler). This is called the giant branch.

For the first time, the star changes its spectral type. A G-type star may end up as a high-K or low-M giant.
Although the H-fusion shell can force the envelope to expand, it cannot stop the collapse of the core. This continues to collapse, until the inner 12% of the mass is compressed to very high density. At such extreme densities, the gas stops obeying the ideal gas laws.
The collapse of the core is stopped by electron degeneracy, which might better be called quantum pressure. Electrons in the core can only have certain energies, and quantum mechanics states that two electrons cannot occupy the same energy level. As the gas becomes very dense, all the available energy levels are occupied. Since the electrons cannot change their energy, the gas cannot be compressed, so it begins to behave like hardened steel.

Low-density gas (non-degenerate)  High-density gas (degenerate)
This enables the core to resist the inward pressure of the outer layers of the star.

However, now that the gas can no longer be squeezed, the star has lost its thermostat. Recall that a star on the main sequence regulates its energy production by way of the ideal gas laws:

core loses energy $\rightarrow$ pressure drops

$\rightarrow$ star contracts

$\rightarrow$ temperature increases

$\rightarrow$ energy production increases

$\rightarrow$ equilibrium is restored
In a degenerate gas, however, pressure does not depend on temperature. If a star supported by degeneracy pressure loses energy, the pressure remains unchanged. The star does not contract, and there is no increase in temperature to counteract the loss of energy.

Similarly, if the star heats up, the pressure does not increase so as to expand the core and lower the temperature, so the temperature can keep growing.
The helium flash

The core continues to contract and the temperature keeps rising. Once it reaches 100 million K, helium can ignite to form carbon. Two nuclei temporarily form an unstable atom of beryllium; if a third helium nucleus reacts with beryllium, it forms a carbon atom.

In low mass stars (up to 3 solar masses), the turn-on of helium fusion is very sudden: the helium flash.
Because the core is degenerate, when helium ignites and produces more energy, the temperature of the core rises but the core does not expand. This means that the rate of energy production keeps going up, which keeps increasing the temperature, in a runaway process. In a few hours the temperature jumps to a billion degrees, producing more energy than the entire galaxy.

But this energy doesn’t escape the star: it all goes into removing the electron degeneracy. Now the core can behave like a perfect gas again, so the star’s thermostat has been restored: the core can expand and cool.
Quickly the P-T thermostat brings the reaction under control, and the star begins fusing helium steadily in the core.

More massive stars don’t have a helium flash, because their cores are not degenerate when helium fusion begins, so the runaway process never takes place: helium fusion begins gradually.
Aside: Why does He fusion need high temperatures?

Helium nuclei contain four positive charges, so in order to fuse together they have to overcome a much higher electromagnetic (Coulomb) barrier than two hydrogen nuclei (one positive charge each). Fusing heavier elements takes even higher temperatures.

\[
\begin{align*}
4H & \rightarrow \text{He} & 5 \text{ million K} \\
3\text{He} & \rightarrow \text{C} & 100 \text{ million K} \\
\text{C} + \text{He} & \rightarrow \text{O} & 600 \text{ million K} \\
2\text{C} & \rightarrow \text{Mg} & 1,000 \text{ million K}
\end{align*}
\]
The star is now making energy in two places at the same time. Only in the core is it hot enough to burn helium, but just outside the core it’s now hot enough to burn hydrogen for the first time. The contraction of the core stops, and the envelope of the star contracts and grows hotter.
This makes the star move down and to the left in the HR diagram: the **horizontal branch**.
The cycle then repeats itself. Helium is fused to carbon and oxygen in the star’s core, and carbon and oxygen ash build up. Eventually the core runs out of helium, so fusion stops, and the core resumes its collapse. This heats up the layer of helium just outside the core, which begins fusing in a shell of its own, inside the hydrogen burning shell.
This extra energy forces the envelope to expand again, and the star begins a second ascent of the giant branch: the asymptotic giant branch.
We can predict what happens next: the cycle repeats again and again, each time starting to fuse heavier and heavier elements to stave off the next collapse. The core reaches high temperatures by converting gravitational energy into thermal energy.

However, if the star is not massive enough, it will never reach high enough temperatures to fuse carbon: this is true for the Sun.

So for low mass stars, this is the end of the line. We will now look at the fate of different mass stars: very low mass (< 0.4 solar masses), medium mass (0.4 – about 6 solar masses) and high mass (> 6 solar masses).
Very low mass stars

Very low mass stars (mass less than about 0.4 solar masses) are different in one important respect from heavier stars: their interiors are *fully convective*. The fused helium is stirred through the whole star, so it has the whole of its hydrogen mass to prolong its stay on the main sequence.

Because they don’t have a core, very low mass stars can never develop an inert helium core surrounded by unburnt hydrogen, so can never become a red giant. Instead, when the star has used up essentially all their hydrogen, fusion will stop, the star will contract and heat up, then gradually cool as a helium white dwarf.
Medium mass stars

Intermediate mass stars, with masses between 0.4 and about 6 solar masses, have another evolutionary path. These stars get as far as burning helium in their cores, but cannot get hot enough to ignite carbon.

Thus when the helium core is exhausted, they cannot resist the gravitational force, and the core finally collapses to form a white dwarf.
The evolution of a solar-like star is summarised in this picture.
However, in the final supergiant stage, the surface gravity is so low that, during the collapse, the star loses its outer layers in spectacular fashion. The entire outer envelope is blown off in a massive stellar wind, in a series of *thermal pulses*. As the envelope of the star falls towards the contracting core, some of the material ignites, causing the outer surfaces to bounce and vibrate. Eventually the outer 40% of the star’s mass will be "coughed" into space, floating outward in a concentric set of spherical bubbles.
The core, now exposed, continues to contract and heat: in only a few thousand years it has reached a temperature of 30,000 K, so it becomes very blue.
Naively, we might expect the gas to be ejected spherically; the nebula, however, would appear as a ring because we see more gas along the sides of the bubble. This is the Helix nebula (NGC 7293), the nearest planetary nebula.
This animation shows the birth of the Helix Nebula. The star ejects its outer envelope at low velocity, exposing the hot core of the star. This star has a fast low-density wind that blows a big cavity in the dispersed envelope. UV radiation from the star heats the gas, causing it to glow.
However, many planetary nebulae have strange and wonderful shapes, indicating that the mass ejection process is very complicated. In binary systems, for instance, the gas outflow is influenced by the interactions between the stars. This is an HST/X-ray picture of the Cat’s Eye nebula (NGC 6543).
The Butterfly nebula (M2-9) is very far from circular. The central object is a binary orbiting within a gaseous disk; the expelled envelope interacted with this disk to form the spectacular shape we see today. In fact, most planetary nebulae are bipolar.
Simulations suggest that the strange shapes of planetary might be due to asymmetric winds from the progenitor confining the gas ejected during the planetary nebula phase. A disk-shaped cloud of gas around the star’s equator forces the gas to escape towards the poles, forming a bipolar nebula.

Simulation of gas escaping from a star, while being constrained by a torus of gas around the star.
These simulations reproduce some of the known planetary nebulae with amazing accuracy.

Hubble image of the “Hourglass nebula”, MyCn 18, compared to the interacting wind model viewed from an angle of 40°.
The “Ant Nebula”, Menzel 3.
The remaining core of the star, consisting mostly of carbon and oxygen (ash from the helium burning), starts off very hot, but has no fuel source, i.e. no nuclear reactions are taking place to replace the energy lost by radiation. It will only cool and shrink: it has become a white dwarf.
The planetary nebula NGC 2440, with its central white dwarf, one of the hottest white dwarfs known.
The planetary nebula known as the “Skull Nebula”, NGC 246, with its white dwarf, the fainter member of the binary star system seen at the nebula’s centre.
Massive stars

High mass stars (greater than about 6 solar masses) have much more violent endings.

Massive stars can reach temperatures in their core high enough to begin fusing carbon, producing neon and oxygen. When the carbon is exhausted in the core, it contracts and carbon ignites in a shell. This pattern of core ignition and shell ignition continues with fuel after fuel, until the star develops a layered structure.
After carbon fuses, oxygen, neon and magnesium fuse to make silicon and sulphur, and then the silicon fuses to make iron.

The fusion of these nuclear fuels goes faster and faster as the atomic number increases, both because there are fewer atoms to fuse, and less energy is released each time.
These elements collect and burn in concentric shells, like an onion.

Once the silicon in the core has fused to make iron, the star can no longer support itself against collapse.
The table shows the time taken for each fusion process for a 25 solar mass star.

<table>
<thead>
<tr>
<th>Fuel</th>
<th>Temperature (K)</th>
<th>Time (y)</th>
<th>Sample reaction</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hydrogen</td>
<td>40 million</td>
<td>7,000,000</td>
<td>$^4_1\text{H} \rightarrow ^4\text{He}$</td>
</tr>
<tr>
<td>Helium</td>
<td>200 million</td>
<td>500,000</td>
<td>$^4^4\text{He} \rightarrow ^{12}\text{C}$</td>
</tr>
<tr>
<td>Carbon</td>
<td>600 million</td>
<td>600</td>
<td>$^{12}_2\text{C} \rightarrow ^{20}\text{Ne} + ^4\text{He}$</td>
</tr>
<tr>
<td>Neon</td>
<td>1.2 billion</td>
<td>1</td>
<td>$^{20}\text{Ne} + ^4\text{He} \rightarrow ^{16}\text{O} + ^2^4\text{He}$</td>
</tr>
<tr>
<td>Oxygen</td>
<td>1.5 billion</td>
<td>0.5</td>
<td>$^{16}_2\text{O} \rightarrow ^{28}\text{Si} + ^4\text{He}$</td>
</tr>
<tr>
<td>Silicon</td>
<td>2.7 billion</td>
<td>1 day</td>
<td>$^{28}_2^2\text{Si} \rightarrow ^{56}\text{Fe}$</td>
</tr>
</tbody>
</table>
Why iron?

Why does the sequence of nuclear burning end with iron? Iron is the most "tightly bound" of the elements. If an element is to burn by fusion then energy must be given off when that element is combined with another. The new nucleus is more tightly bound than the two separate bits; it would take energy to break the new nucleus apart to the original parts.
But with iron we have the situation where to add any other element costs energy, that is the new nucleus is less tightly bound. Indeed, up at the high end of the periodic table we have elements like uranium and plutonium that give up considerable energy when split apart in the process of nuclear fission. Iron then represents the energy minimum and once a star has converted its matter into iron then it has no further recourse. The star has reached a dead end.
Abundances of the elements

This sequence explains the first half of the cosmic abundance diagram: elements up to iron are formed in stars, and the relative abundances are well explained by what we know of fusion processes.
Nuclear chemist William Harkins (c. 1931) noted that

"elements of low atomic weight are more abundant than those of high atomic weight and that, on the average, the elements with even atomic numbers are about ten times more abundant than those with odd atomic numbers of similar value."
But what of elements heavier than iron? How are they formed?

(We’ll find out next week...)
During the later stages of burning, more and more energy is being lost to neutrinos: most of the star’s energy, by the time carbon burning starts.

<table>
<thead>
<tr>
<th>Fuel</th>
<th>Temperature (K)</th>
<th>Time (y)</th>
<th>photon luminosity (J/s)</th>
<th>neutrino luminosity (J/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hydrogen</td>
<td>40 million</td>
<td>7,000,000</td>
<td>2.7x10^{31}</td>
<td>—</td>
</tr>
<tr>
<td>Helium</td>
<td>200 million</td>
<td>500,000</td>
<td>5.3x10^{31}</td>
<td>&lt;1.0x10^{29}</td>
</tr>
<tr>
<td>Carbon</td>
<td>600 million</td>
<td>600</td>
<td>4.3x10^{31}</td>
<td>7.4x10^{32}</td>
</tr>
<tr>
<td>Neon</td>
<td>1.2 billion</td>
<td>1</td>
<td>4.4x10^{31}</td>
<td>1.2x10^{36}</td>
</tr>
<tr>
<td>Oxygen</td>
<td>1.5 billion</td>
<td>0.5</td>
<td>4.4x10^{31}</td>
<td>7.4x10^{36}</td>
</tr>
<tr>
<td>Silicon</td>
<td>2.7 billion</td>
<td>1 day</td>
<td>4.4x10^{31}</td>
<td>3.1x10^{38}</td>
</tr>
</tbody>
</table>
While all this fusion is taking place, the star is zig-zagging across the HR diagram.
By this stage, the iron core exceeds 1.4 solar masses. When silicon fusion finishes, the core begins to collapse for the final time. And this time, there is *nothing* to stop it. The end result is the destruction of the star in a supernova explosion: we’ll look at this in detail next week.
Mass loss

One major complication in understanding the evolution of massive stars is that they lose large amounts of mass at many stages of their life.

Hot stars emit a continuous outflow of matter from their surfaces as a stellar wind. Our own Sun has a solar wind which reaches speeds of 400–700 km/s with a mass loss rate of about $10^{-14}$ solar masses per year. Over a ten billion year lifespan, at this rate the Sun will lose about 0.01% of its mass to the solar wind.
By contrast, the winds from hot stars can be a billion times stronger, losing up to $10^{-5}$ solar masses per year at speeds of up to 3000 km/s. This means that even during the much shorter lives of the stars (a few million years), they can lose on the order of half or more of their mass.

The “Pistol Nebula” and its central star, which may have weighed up to 200 times the mass of the Sun before shedding much of its mass in violent eruptions.
The Bubble Nebula, NGC 7635, is being pushed out by the stellar wind of massive central star BD+602522, which has a mass about 40 times the mass of the Sun. The bubble is about 3 parsecs (10 ly) across.
This has substantial implications for the evolution of the star. Reducing the mass of the star reduces the pressure and temperature in the interior, which can reduce the mass of the core.

And it is the mass of the core when fusion stops which governs whether the star explodes as a supernova, and what kind of remnant it leaves behind.
Very massive stars, with initial masses greater than 40 solar masses, become *luminous blue variables* after leaving the main sequence. During this phase, which lasts for perhaps 40,000 years, the stars are highly variable and losing mass through strong winds. At minimum brightness they appear as blue B-type supergiants; during outburst they are much redder. Every few centuries they have sudden giant eruptions, ejecting large amounts of mass.

*Radio images of two luminous blue variables, AG Carinae and Henize 3–519, showing rings of emission from mass lost during major eruptions.*
Eta Carinae is the most famous luminous blue variable. During the 1840s it brightened by 4 magnitudes, becoming one of the brightest stars in the sky. Hubble images show two huge bubbles of gas, remnant of the expulsion of about a solar mass of material.
Wolf-Rayet stars represent the most extreme stage of mass loss in the life of a massive star: the star has lost so much mass that they are actually exposing the underlying layers which have already undergone nuclear fusion.

HST image of the Wolf-Rayet star WR124, showing the star surrounded by hot clumps of gas being ejected at high speed.
The amount of mass lost in these stages of a star’s life essentially determines what its ultimate fate is. We’ll discuss the endpoints of stellar evolution in a few lectures, but we note here that most of the uncertainty in what mass of star leads to what kind remnant is because of the uncertainty in how much mass is lost by the time the star ends its life.
The final fate of the Earth

Mass loss is also vital for determining the final fate of the Earth. As the Sun ages, it will expand and become a red giant.
The final radius the Sun reaches is important. However, as the Sun loses mass, its gravitational pull on the planets weakens, and their orbits expand.

Will they expand enough to keep ahead of the expanding photosphere, or will they be engulfed?
Recent calculations* suggest that although the Sun’s outer surface may not quite reach Earth, tidal interaction between the Earth and the giant Sun will drag the Earth inwards, to be engulfed by the Sun.

This will take place just before the Sun reaches the tip of the RGB, around $7.59 \pm 0.05$ Gyr from now.

In any case, the Earth is likely to become uninhabitable long before that point is reached.

We’ve built up a wonderful picture for stellar evolution: but the timescales are so long that we can never observe a star evolving. How certain are we about all this?

The best evidence comes from studies of star clusters. Clusters of stars are ubiquitous in the Galaxy.
Generally, we divide clusters into two classes: **open clusters**, consisting of 100–1000 young stars, all born from the same molecular cloud;

*The Jewel Box Cluster (NGC 4755, or Kappa Crucis)*
...and *globular clusters*, which are extremely old groups of 10,000–1,000,000 stars, born in the very earliest stages of the Galaxy’s evolution.

*The dense globular cluster M80 (NGC 6093)*
Since all stars in a cluster form at the same time, they are the same age. Their HR diagram, therefore, becomes a powerful test of stellar evolution theory. Stars of different masses, evolving at different rates, will appear at different places in the diagram.
This is the HR diagram for the globular cluster M13, an old globular cluster. We can clearly see the main sequence, the red giant branch, and the horizontal branch. There is one more notable feature: a gap in the horizontal branch. Stars are rarely found in this region, and those that are are unstable: they pulsate in brightness.

This region is known as the instability strip.
The most famous of these variable stars are the **Cepheid variables**, which are giants of mass greater than 2 solar masses, which pulsate with periods of several days. The pressure-temperature thermostat is out of sync, so the stars expand and contract in a regular fashion. Most importantly, the higher the luminosity of the star, the longer the period, so Cepheids can be used as **standard candles** for determining the scale of the Galaxy and the Universe.
Aside: In fact, the more we look, the more it seems that globular clusters aren’t quite that simple...

Figure 3. Evidence of multiple populations in ω Cen: Upper left panel: distribution in colour of RGB stars from Lee et al. (1999). Central upper panel: distribution in [Fe/H] of main sequence stars, adapted from Stanford et al. (2007). Upper right panel: distinct RGBs, from Ferraro et al. (2004). Lower panel: collection of CMDs showing the various populations, both for evolved and main sequence stars, from Bedin et al. (2004).
We can summarise the final fates of the different types of stars in the following diagram.
Next week we’ll look in detail at the explosions that end the lives of massive stars: supernova explosions.
Further reading

• “Cosmic Catastrophes: Exploding Stars, Black Holes, and Mapping the Universe” by J. Craig Wheeler (Cambridge UP, 2007) – the first edition from 2007 was subtitled “Supernovae, Gamma-ray Bursts and Adventures in Hyperspace” – has a good discussion of how stars evolve; most of the book is taken up with how they die, so I’ll be referring to this book again in the next couple of lectures. Note: This book is not easy; he covers an awful lot of material. His diagrams are great, however, and it’s worth sticking with.

• There’s a nice biography of Henrietta Leavitt, the discoverer of the Cepheid period-luminosity relation: “Miss Leavitt’s Stars: The untold story of the woman who discovered how to measured the universe” by George Johnson (Atlas Books, 2005), which mostly goes to show how little we know of her.

• The Australia Telescope has several very nice pages on their Outreach site. “Post-Main Sequence Stars”, http://outreach.atnf.csiro.au/education/senior/astrophysics/stellarevolution_postmain.html is a good summary of the material we discussed in this lecture. The “HR Diagram Activities” page http://outreach.atnf.csiro.au/education/senior/astrophysics/stellarevolution_hractivity.html has links to several sites which allow you to explore stellar evolution in the Hertzsprung-Russell diagram.

• Bruce Balick has a lovely page about planetary nebulae and their formation, at http://www.astro.washington.edu/balick/WFPC2/index.html
Sources for images used:

- Faint young sun paradox: from http://zebu.uoregon.edu/~imamura/122/lecture-1/lecture-1.html
- Cosmic abundance of the elements: from Astro 105: The Milky Way, Lecture XII http://www.pas.rochester.edu/~afrank/A105/LectureXII/LectureXII.html
- Pistol nebula: from Hubblesite http://hubblesite.org/newscenter/newsdesk/archive/releases/1997/33/
- Sun becoming a red giant: from Science@NASA, Sizzling comets circle a dying star http://science.nasa.gov/headlines/y2001/ast11jul_1.htm
- Artist’s impression of planet around a red giant: image by Dirk Terrell http://www.boulder.swri.edu/~terrell/dtart_old.htm
- Hertzsprung-Russell diagrams of clusters as they age: from “Explorations: An Introduction to Astronomy” by Thomas Arny, Fig. 13.24 http://www.mhhe.com/physsci/astronomy/arny/instructor/graphics/ch13/1324.html