Nonlinear physics:
from misbehaving springs to rogue waves

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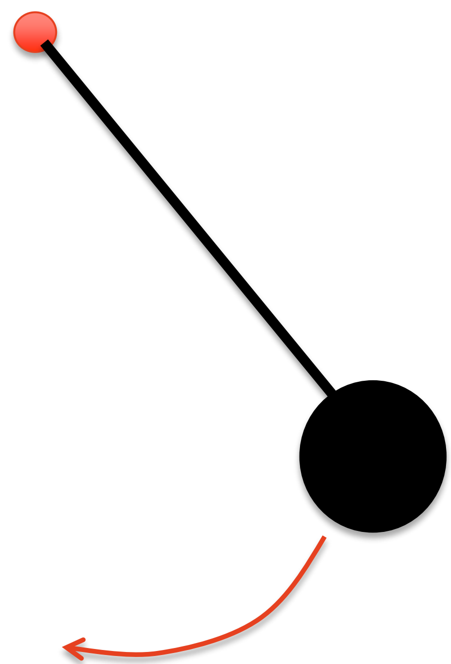
School of Physics

How to describe the motion of a pendulum?



Physical models use differential equations (Newton 1666)

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta$$

Frequency of response: $\omega_0^2 = \frac{g}{l}$

What is “nonlinearity”?

Nonlinearity refers to the model, not the solution

Low amplitude approximation

$$\frac{d^2\theta}{dt^2} = -K\theta$$

Linear model, as all terms are of powers 1 (or 0) in the dependent variable

More accurate model, able to treat large amplitudes

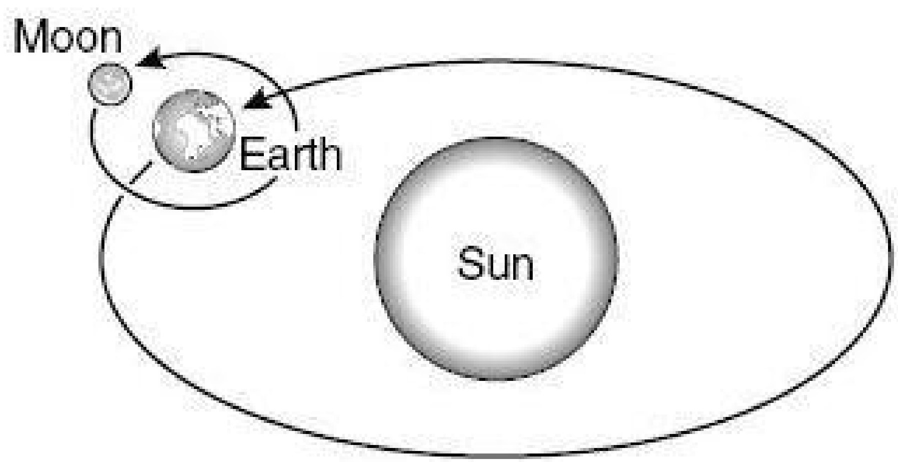
$$\frac{d^2\theta}{dt^2} = -Ksin(\theta)$$

Nonlinear model, as contains a term that can't be described by powers of only 1 or 0 in the dependent variable

The three body problem

Two-body, solved by Bernoulli.

Three-body: Predict the position of three gravitational bodies (Sun, Moon and Earth) as a function of time. Couldn't be done.



http://5010.mathed.usu.edu/Fall2017/JMitchell/poincare.html

$$F = \frac{-k}{r^2}\hat{\mathbf{r}}$$

The difficulty? Gravity is a nonlinear force

The longitude problem and the three body problem

The Longitude Prize (1714, ~$3.7 million):

The Discovery of the Longitude is of such Consequence to Great Britain for the safety of the Navy and Merchant Ships as well as for the improvement of Trade that for want thereof many Ships have been retarded in their voyages, and many lost..." [and there will be a Longitude Prize] "for such person or persons as shall discover the Longitude.

The Three-body Problem prize (1887, ~$20,000):

Given a system of arbitrarily many mass points that attract each according to Newton's law, under the assumption that no two points ever collide, try to find a representation of the coordinates of each point as a series in a variable that is some known function of time and for all of whose values the series converges uniformly.

Solution?

Euler could numerically integrate equations of motion and produce tables of lunar position, a first solution to the longitude problem.

Longitude problem finally solved by invention of stable timepieces.

Three-body problem not solved in the sense of the original problem.

Poincare solved a related problem, looking at qualitative behaviour (could the moon fly off to infinity?) rather than quantitative and was awarded the prize.

His work heralded the start of chaos theory.

Dynamics - A Capsule History

1666	Newton	Invention of calculus, explanation of planetary motion
1700s		Flowering of calculus and classical mechanics
1800s		Analytical studies of planetary motion
1890s	Poincaré	Geometric approach, nightmares of chaos
1920–1950		Nonlinear oscillators in physics and engineering, invention of radio, radar, laser
1920–1960	Birkhoff Kolmogorov Arnol'd Moser	Complex behavior in Hamiltonian mechanics
1963	Lorenz	Strange attractor in simple model of convection
1970s	Ruelle & Takens	Turbulence and chaos
	May	Chaos in logistic map
	Feigenbaum	Universality and renormalization, connection between chaos and phase transitions
		Experimental studies of chaos
	Winfree	Nonlinear oscillators in biology
	Mandelbrot	Fractals
1980s		Widespread interest in chaos, fractals, oscillators, and their applications

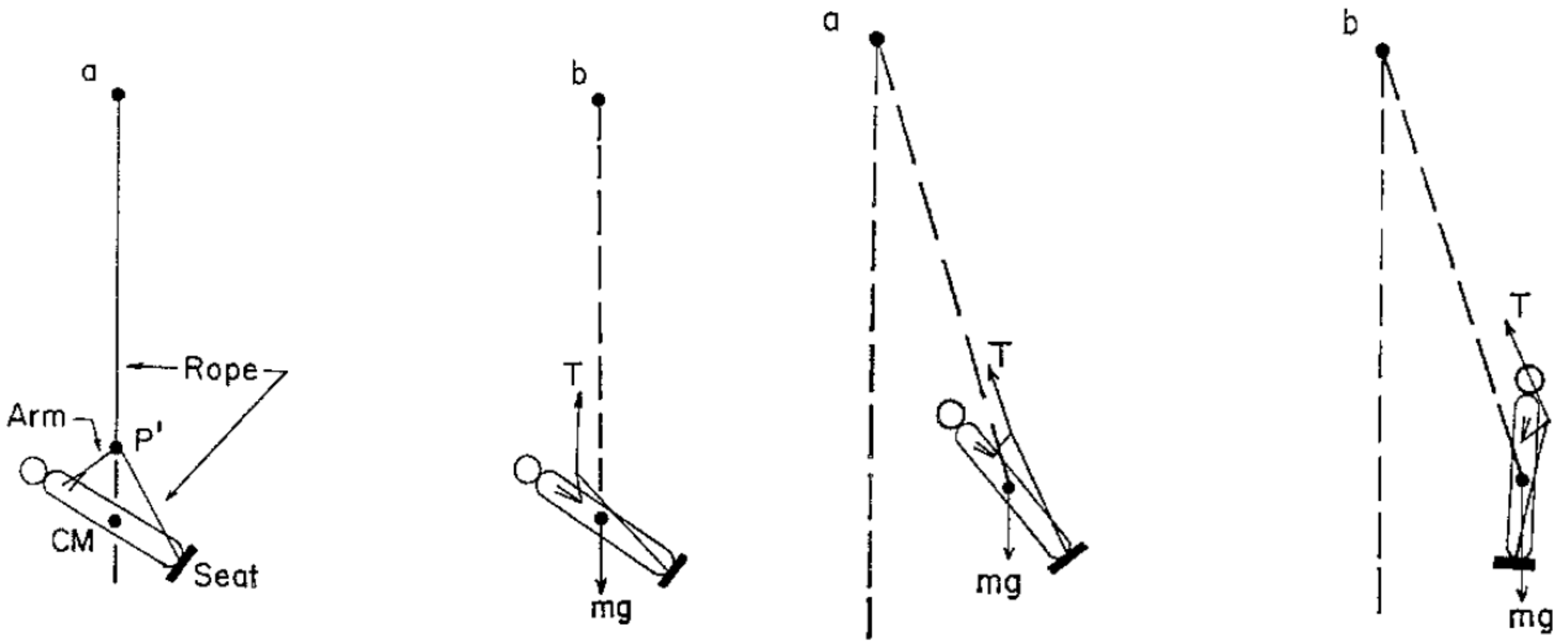
Strogatz, Nonlinear Dynamics.

Introducing “driving”: how does a swing work?

https://www.smythstoys.com/

Resonance

Driving by oscillating at same frequency as natural frequency of the swing (resonance)

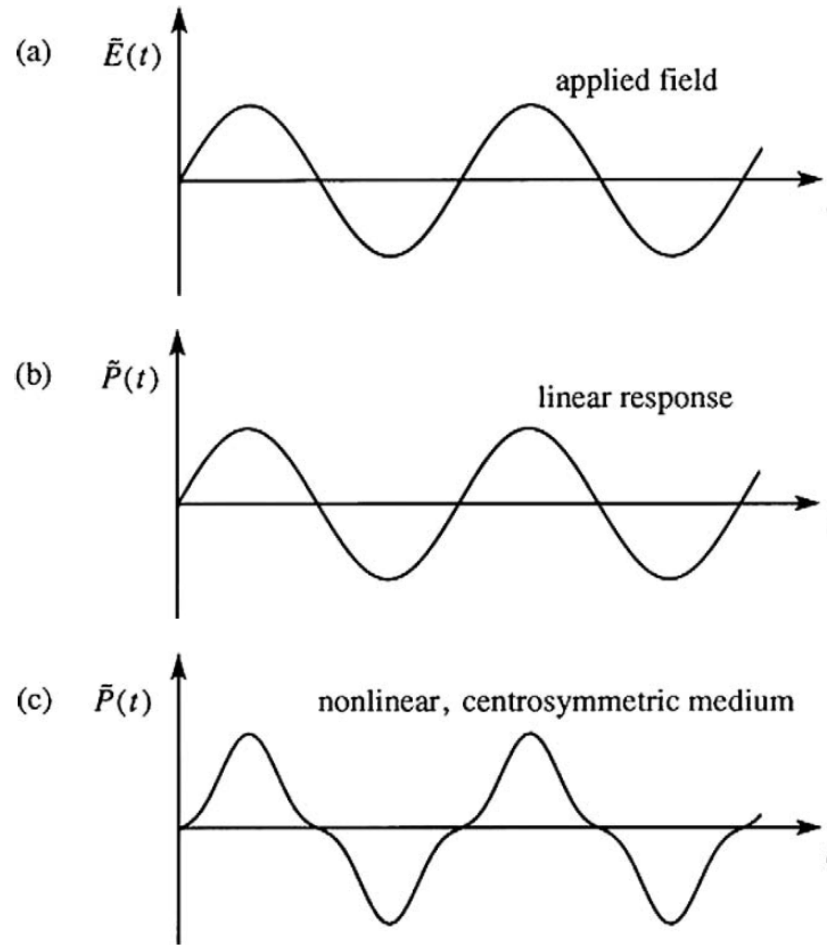


Gore, Am. J. Phys. **38** 378 (1970).

Modern aside: nonlinear optics

Invention of laser (1960) opened up the possibility of strong coherent light waves.

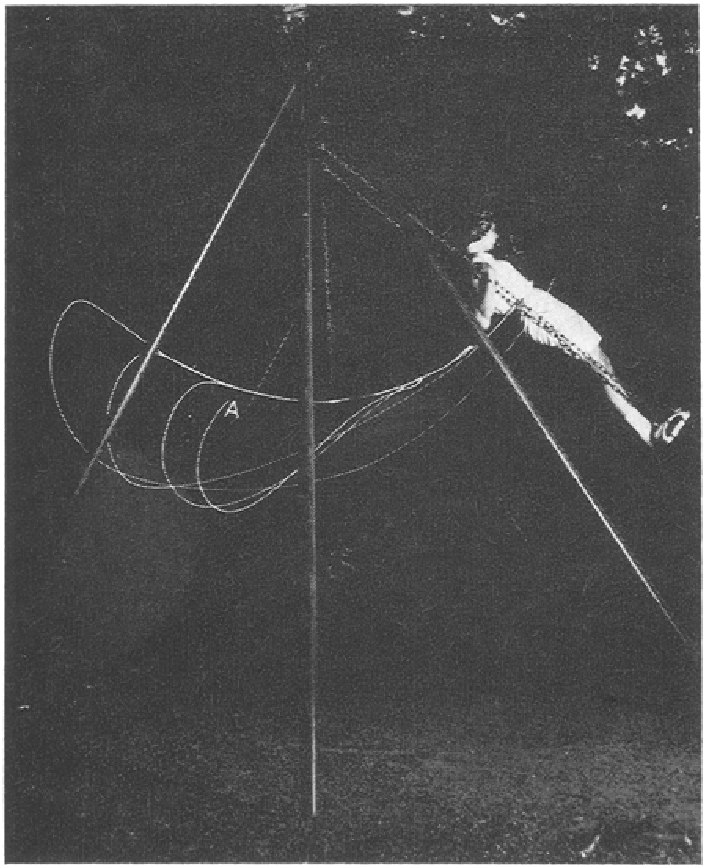
An electron in a material is driven by these waves. Because the amplitude of the driving is large, the electron response is nonlinear.



Back to the swing

It turns out there is another way to drive a swing…

Parametric driving of a swing



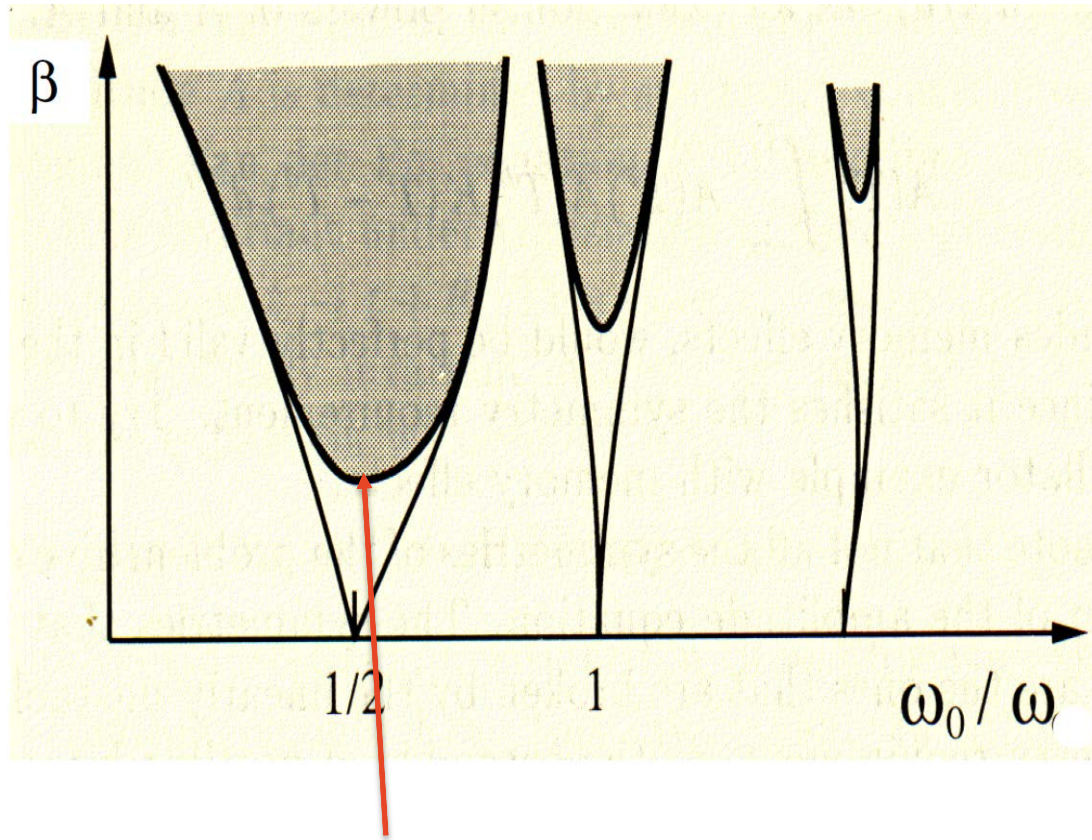
Tea and Falk, Am. J. Phys. **36** 1165 (1968).

Can also drive the swing by standing up and down, so that effectively the length of the pendulum is changing with time. As a parameter is changing (the length) this is known as "parametric driving"

$$\frac{d^2\theta}{dt^2} = -(\omega_0^2 + \beta cos(\omega t))\theta$$

Parametric resonance

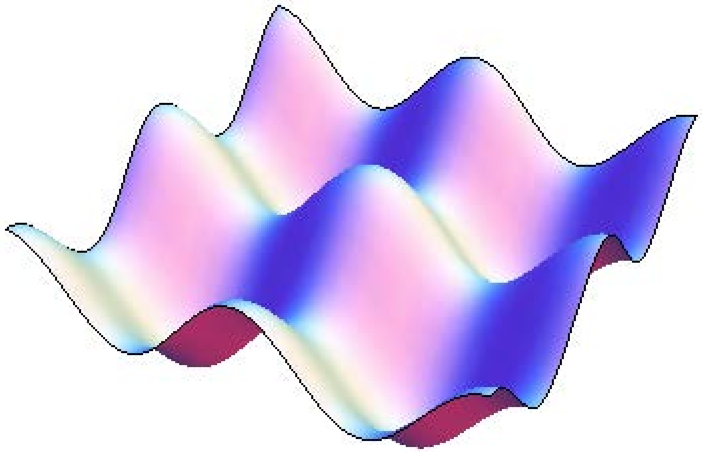
Standing up and down at twice the natural frequency of the swing leads to a growth in the amplitude. This is known as parametric resonance.



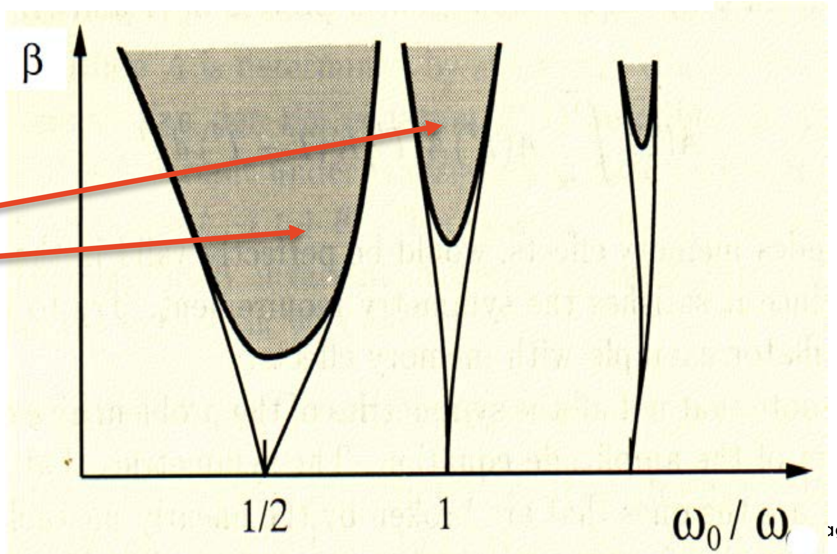
Driving at twice natural frequency leads to growth

Modern aside: band-gap theory

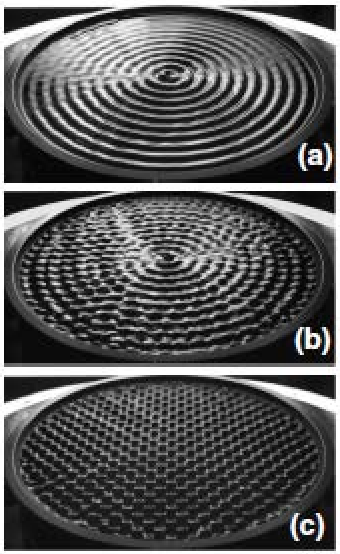
There is a close parallel between parametric resonance due to driving, and the appearance of bandgaps in crystalline materials (underpinning the semiconductor industry).



“bandgaps”: no bounded solutions



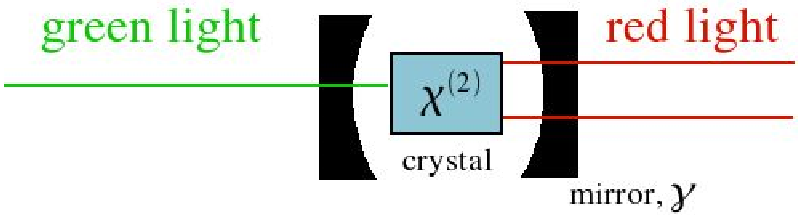
Other examples of parametric resonance?



Driven fluid surfaces

Shats *et al*. Phys. Rev. Lett. (2012)

Second-harmonic generation, optical parametric amplification



http://www.raphaelpooser.com/research/OPOs.html

What happens if we have more than one oscillator?

How do we describe the motion we observe?

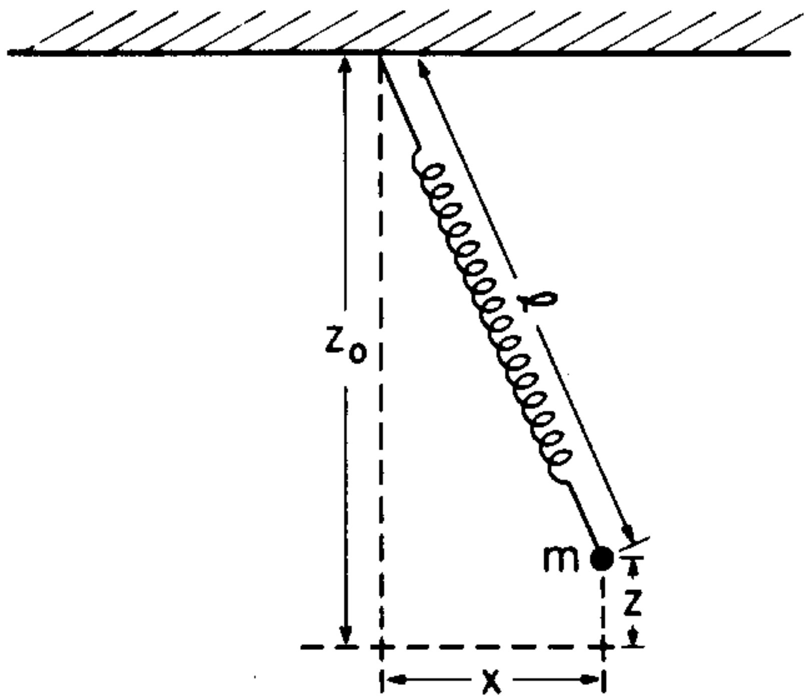
Normal modes

Same number of normal modes as degrees of freedom (dimensions the oscillator or oscillators can move in)

Each normal mode has a particular frequency, at which all the masses undergo periodic motion (some masses may be stationary)

Turning to the spring pendulum (i.e. a mass on a spring, free to move laterally)

– What do we expect to happen?
– Are there any normal modes?



Looking at small amplitude oscillations around the equilibrium point

– Take equilibrium to be $x = 0$, $z = 0$:

$$\ddot{x} + \frac{g}{z_0} x - \lambda x z = 0$$

$$\ddot{z} + \frac{k}{m} z - \lambda \frac{x^2}{2} = 0$$

Is this a nonlinear system?

As expected, in the linear case, two normal modes

Pendulum equation

$$\ddot{x} + \frac{g}{z_0}x - \cancel{\lambda xz} = 0$$

$$\ddot{z} + \frac{k}{m}z - \lambda \cancel{\frac{x^2}{2}} = 0$$

Spring equation

$$\omega_p = \sqrt{g / z_0}, \qquad \omega_s = \sqrt{k / m}$$

coefficients of linear terms determine the frequencies of the pendulum and spring oscillations respectively

What is the equation of motion for the pendulum mode, if we start purely in the spring mode?

Spring motion: $z = a\cos(\omega_s t)$

Equation of motion for the pendulum mode:

$$\ddot{x} + \left(\omega_p^2 - \lambda a\cos(\omega_s t)\right)x = 0$$

Does it look familiar?

$$\ddot{x} + \left(\omega_p^2 - \lambda a \cos(\omega_s t)\right) x = 0$$

Parametric resonance condition

Let’s set the parametric resonance condition:

$$\omega_s = 2\omega_p$$

$$\ddot{x} + \frac{g}{z_0} x - \lambda xz = 0$$

$$\ddot{z} + \frac{k}{m} z - \lambda \frac{x^2}{2} = 0$$

$$\omega_s = 2\omega_p \Rightarrow \sqrt{k / m} = 2\sqrt{g / z_0}$$

$$\therefore z_0 = 4gm / k = 4(z_0 - l_0)$$

$$\therefore z_0 = 4l_0 / 3$$

See e.g. Olsson Am. J. Phys. (1976) or Cayton Am. J. Phys. (1977).

What do we expect will happen?

The misbehaving spring

Nonlinearity couples the two linear modes!

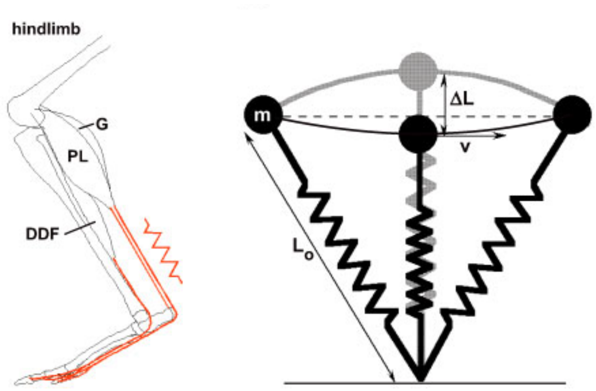
Recap:

In a linear system the modes are separate, if we are in a normal mode we will always stay in a normal mode.

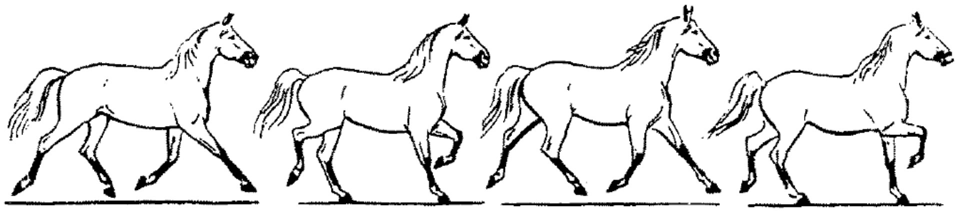
In the presence of nonlinearity, energy may be exchanged between these modes.

Aside: applications for a spring pendulum...

Masses on the ends of springs form the basis of theory in a wide range of contexts...

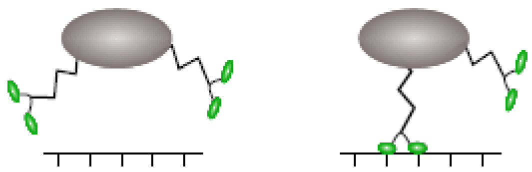


Biewener, J. Exp. Zoo. (2006)



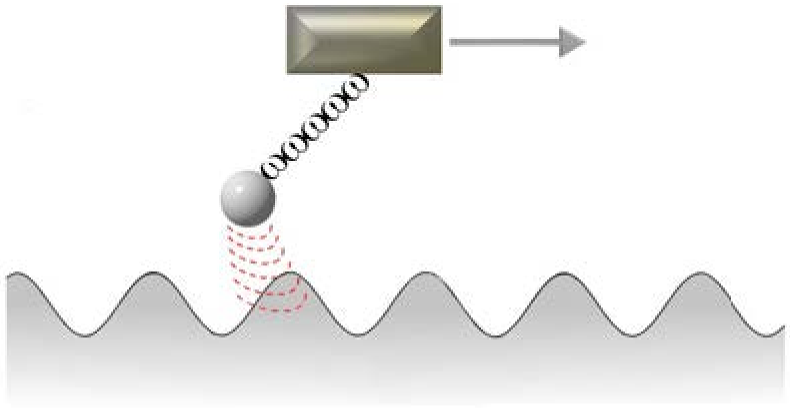
Collins and Stewart, J. Nonlin. Sci. (1993)

Understanding animal/robot gaits



Berger *et al.*, Cell. and Mol. Bioengineering (2013)

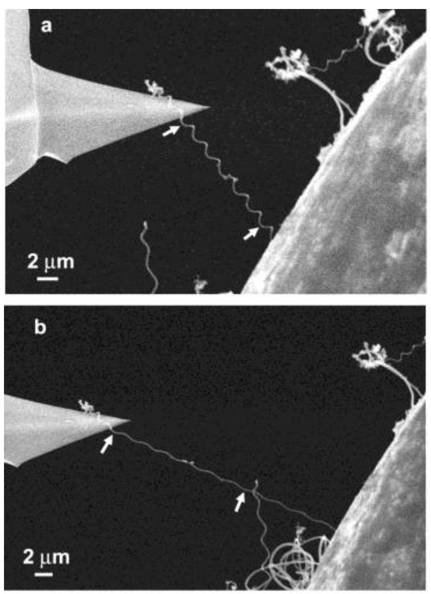
Understanding molecular motor dynamics



The Tomlinson model for describing stick-slip frictional dynamics

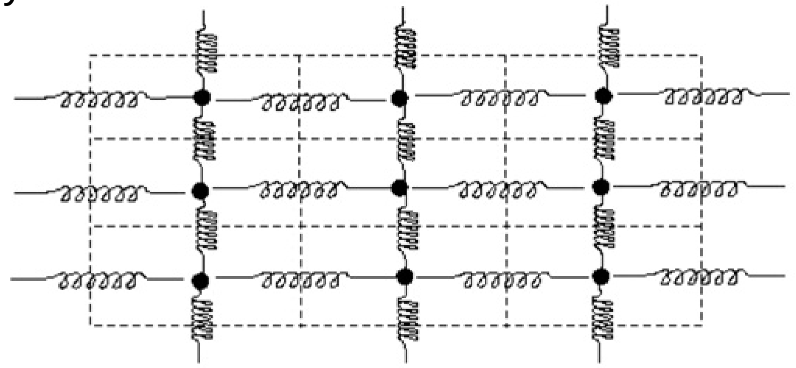
Vanossi and Braun, J. Phys. Condens. Matt. (2007)

And springs appear widely in material models



...naturally for e.g. carbon micro- and nano-coils

...or to provide a conceptual simplification, e.g. in the study of material elasticity



Chen, Nano Lett. (2003)

Holecek and Moravec, Int. J. Sol. Struct. (2006)

When nonlinearity and engineering collide: Tacoma narrows bridge disaster (1940)

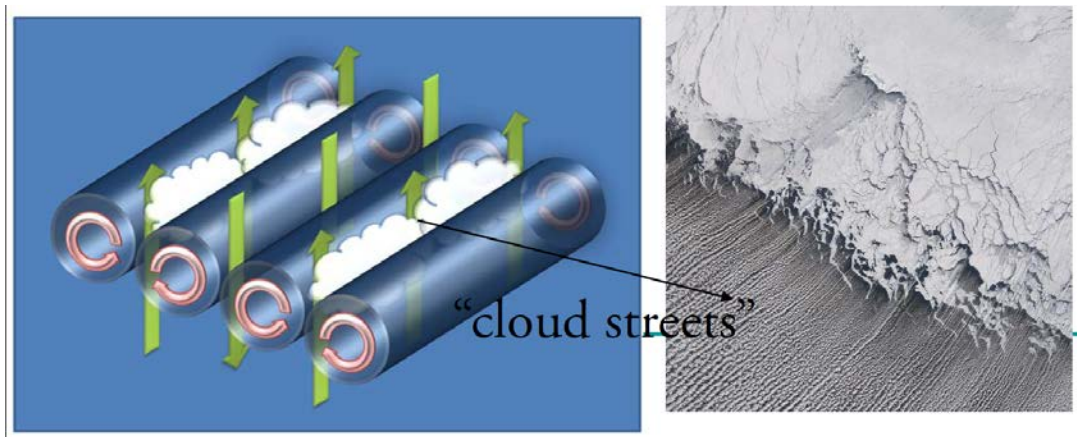
Most things we use in everyday life operate in the linear regime (where nothing strange can happen).

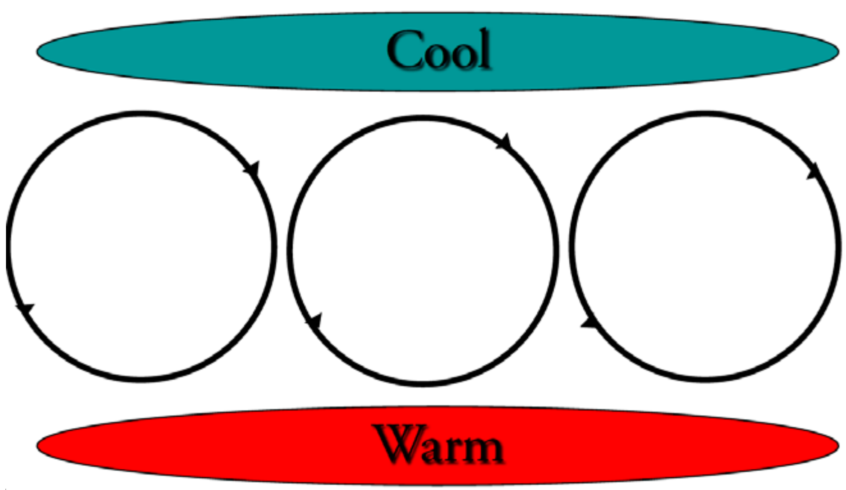
If nonlinearity begins to play a role then things can go bad very quickly.

https://www.youtube.com/watch?v=XggxeuFDaDU

Another turning point in dynamical systems: Lorenz and a model of the weather (1961)

A simple model to describe the formation of cloud streets





$$\dot{x} = \sigma(y - x),$$
$$\dot{y} = rx - y - xz,$$
$$\dot{z} = xy - bz.$$

Lorenz wanted to look in more detail at numerical results from the previous day…

He plugged in a value from the previous day, and restarted the simulation.

After some time, completely different results appeared.

Tracking down the problem...

After much hair pulling Lorenz determined the reason for the discrepancy.

Tracking down the problem…

He had taken as a start point values output from the earlier run of the simulation.

One such initial condition was 0.506

He realised though that the computer was using numbers at higher precision.

The full number stored in the computer was 0.506127

Lorenz had discovered that chaotic systems are very sensitive to initial conditions.

The “butterfly effect”

“Does the flap of a butterflies wings in Brazil set off a tornado in Texas?”

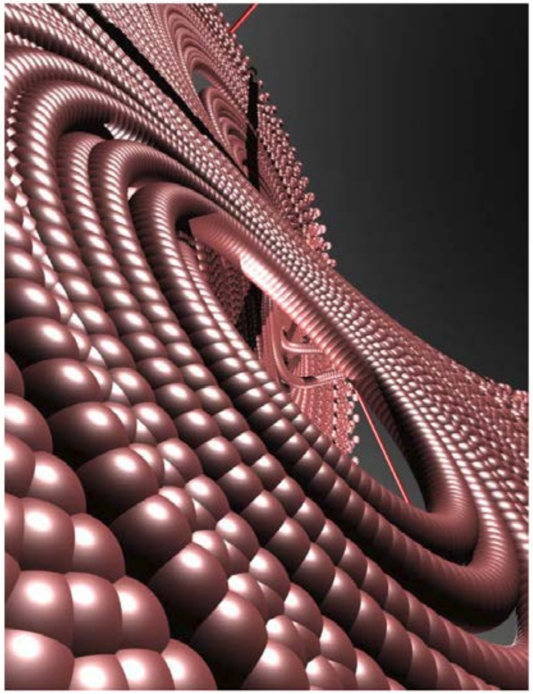
Lorenz’s original quote:

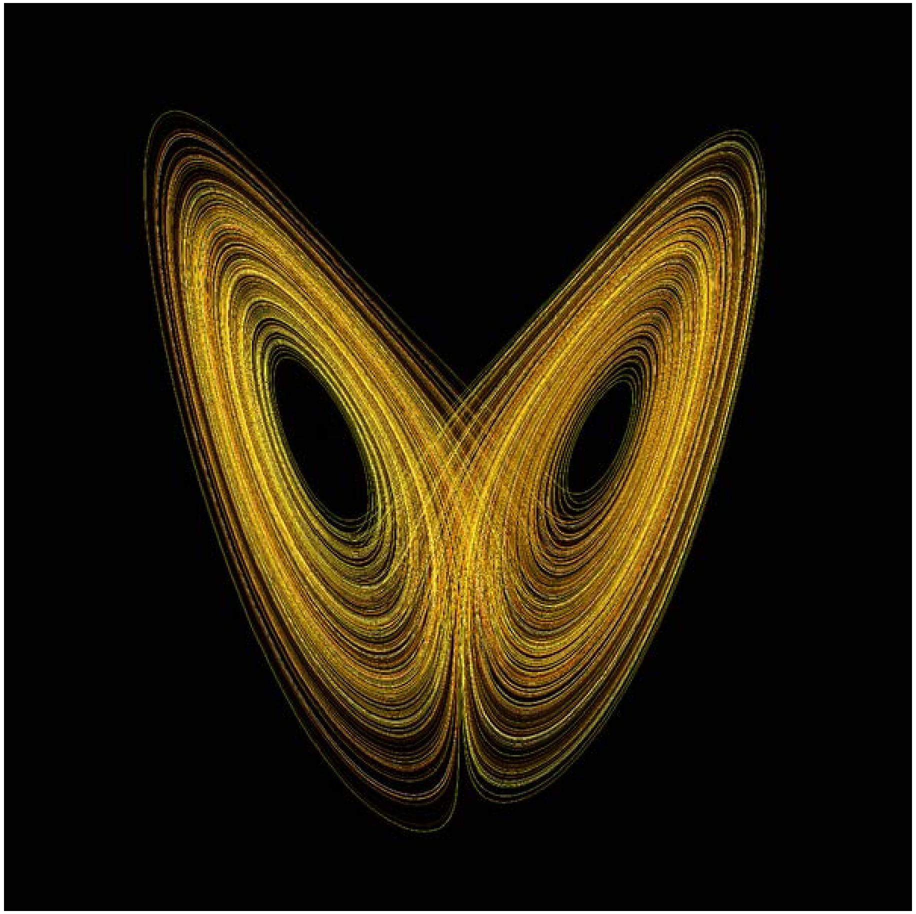
“…one flap of a seagull’s wings could change the course of weather forever”

This is why weather prediction is hard. After Lorenz’s result it looked impossible.

The Lorenz strange attractor

While the dynamics are chaotic, they converge on a strange shape in the 3D space, known as a strange attractor.This attractor has fractal like properties.



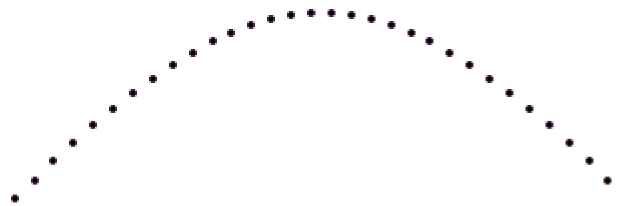


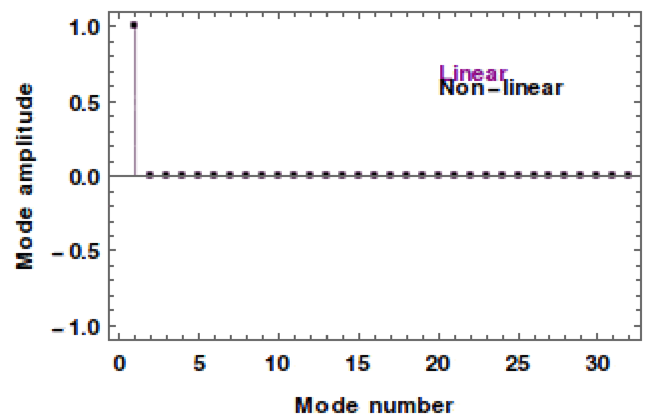
A serendipitous discovery in another numerical simulation…
Fermi-Pasta-Ulam-Tsingou (1953)

Testing one of the earliest computers, Fermi decided to test a fundamental theorem of thermodynamics: the equipartition theorem (every degree of freedom has the same energy at equilibrium)

He reasoned that a small amount of nonlinearity would lead to mode mixing and equipartition.

At first it looked like this was indeed happening…



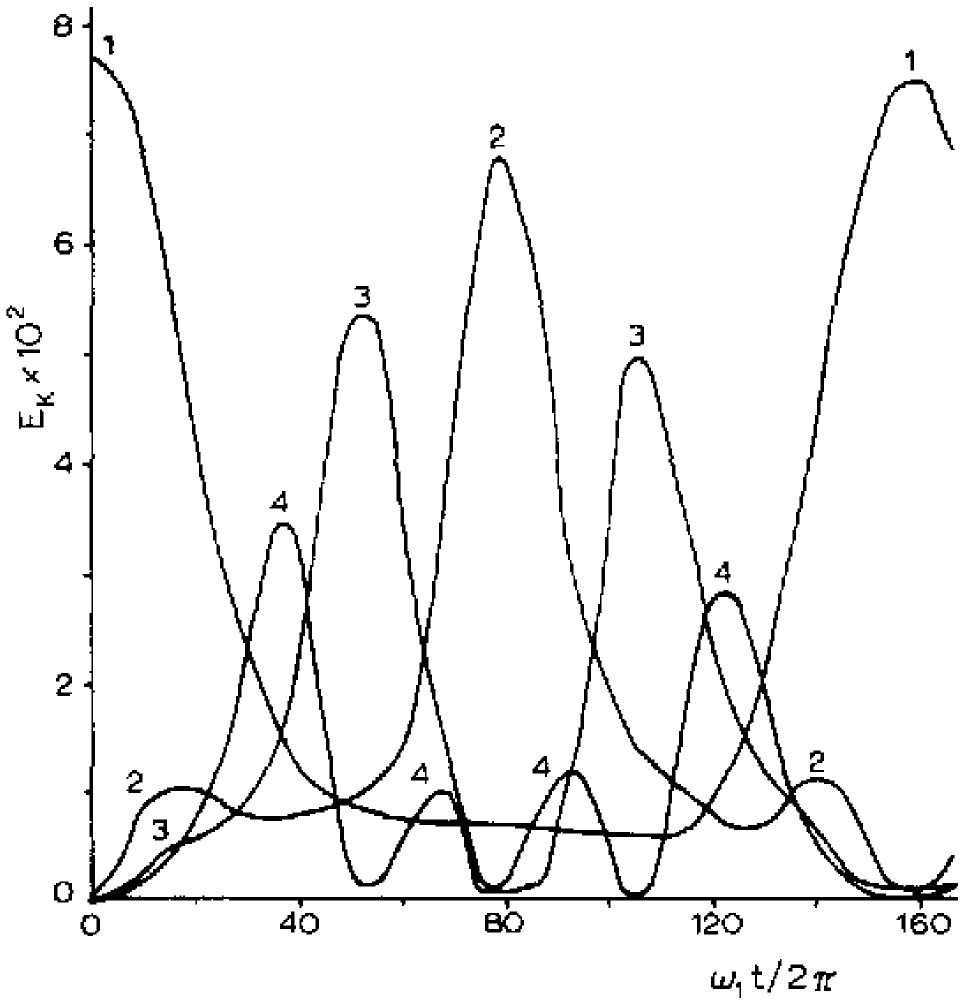


https://en.wikipedia.org/wiki/Fermi%E2%80%93Pasta%E2%80%93Ulam%E2%80%93Tsingou_problem

But the computer was left running overnight

And all the energy came back to the fundamental mode.

This became known as FPU recurrence, and confounded scientists for decades.

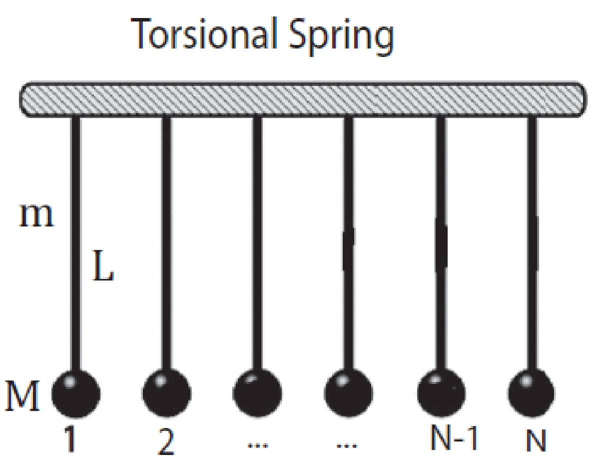


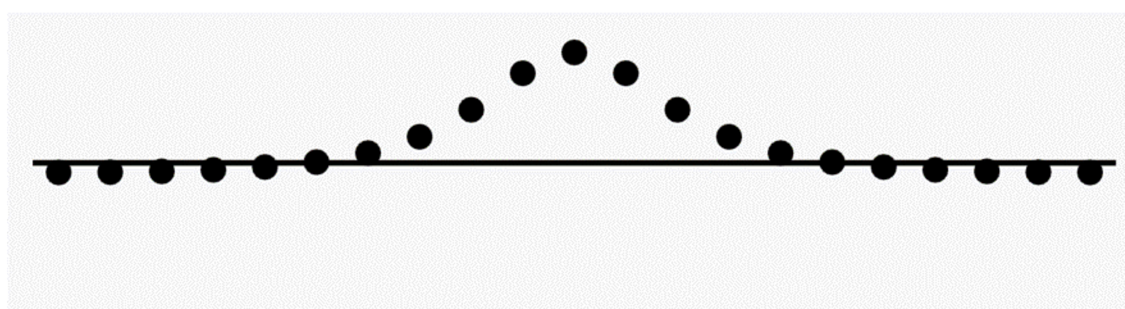
A new effect: energy localisation, breathers and solitons.

The interplay of nonlinearity and linear properties (such as diffraction) could be exactly balanced.

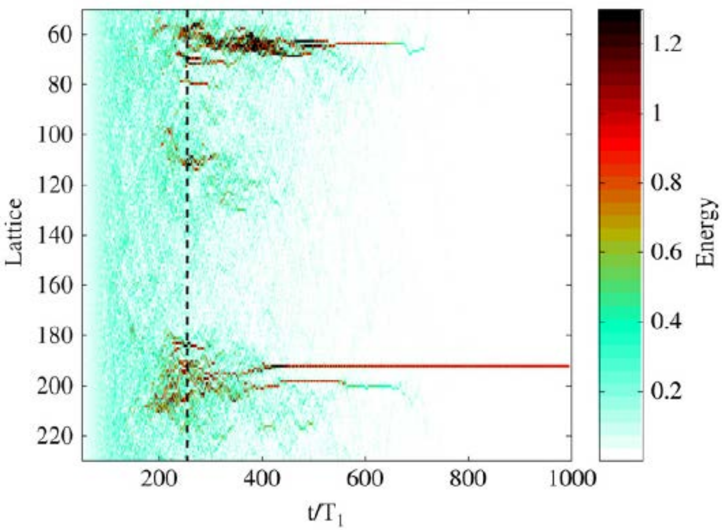
A new nonlinear-only solution had been found. In waves, this is known as a soliton (or solitary wave).

In lattices, there is a close relative known as the discrete breather.



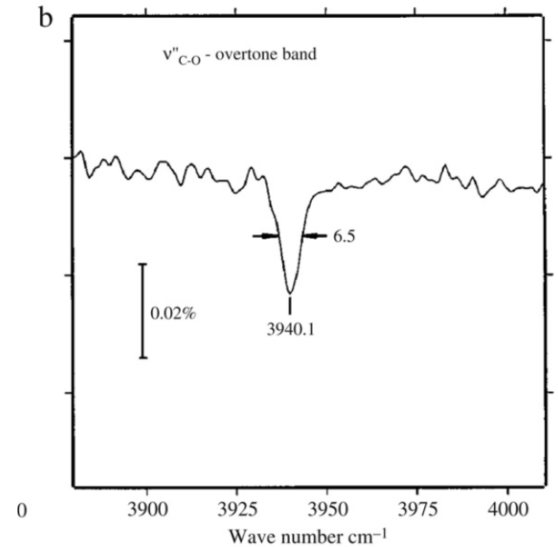


Discrete breathers



Maniardis and Flach, Euro. Phys. Lett. (2006)

Discrete breather appearing in spectrum of CO deposited on crystal surface (2D array of anharmonic oscillators)

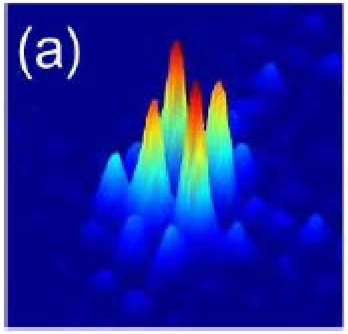


Jacob, Phys. Rev. Lett. (1996)

Discrete breather formation in driven MEMS cantilever array

Optical localised excitation in nonlinear waveguide array

Neshev *et al.* Phys. Rev. Lett. (2004)



John Scott-Russell and the wave of translation

Little did FPUT know, but they had just observed something that had been seen 120 years earlier:

> "I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped—not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour [14 km/h], preserving its original figure some thirty feet [9 m] long and a foot to a foot and a half [30−45 cm] in height. Its height gradually diminished, and after a chase of one or two miles [2–3 km] I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation."
>
> **John Scott-Russell**

A recreation of the original observation (1995)

https://www.researchgate.net/figure/Recreation-of-a-solitary-wave-on-the-Scott-Russell-Aqueduct-on-the-Union-Canal_fig7_303697433

Solitons and nonlinearity in photonics

Applications span: solitons for data transmission in telecommunications; soliton lasers; all-optical signal processing; optical sensing...and more.

https://phys.org/news/2019-07-laser-solitons-theory-topology-potential.html

But what about other types of waves?

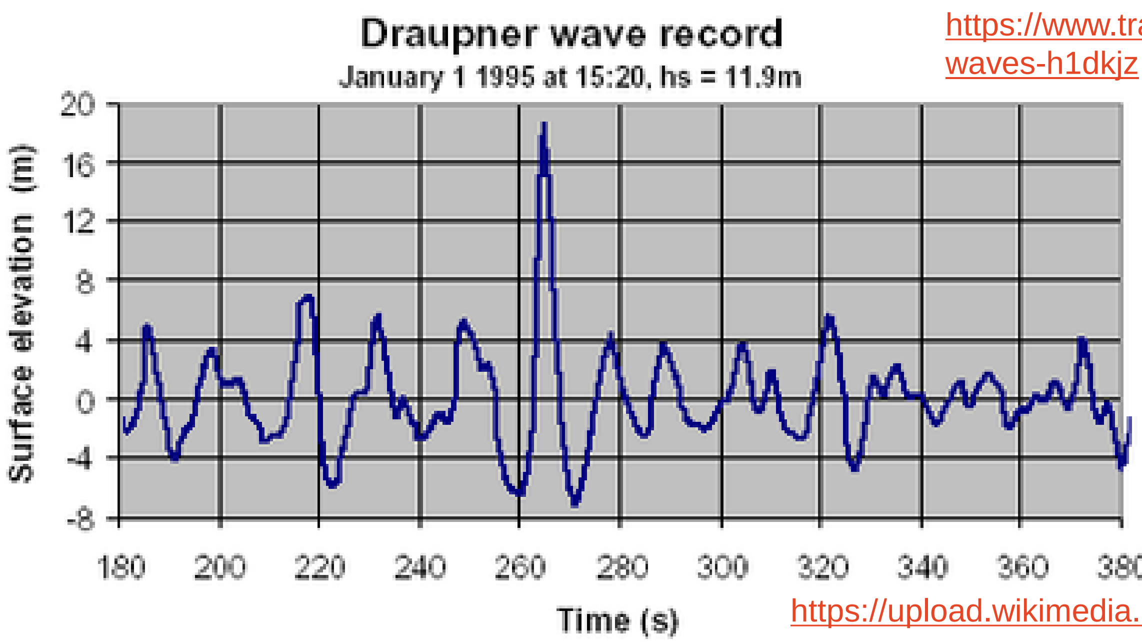
Surely ocean waves can be nonlinear?

https://www.youtube.com/watch?v=NTgAqH9MEeg

Rogue waves

Stories abound of immense waves striking ships seemingly out of nowhere

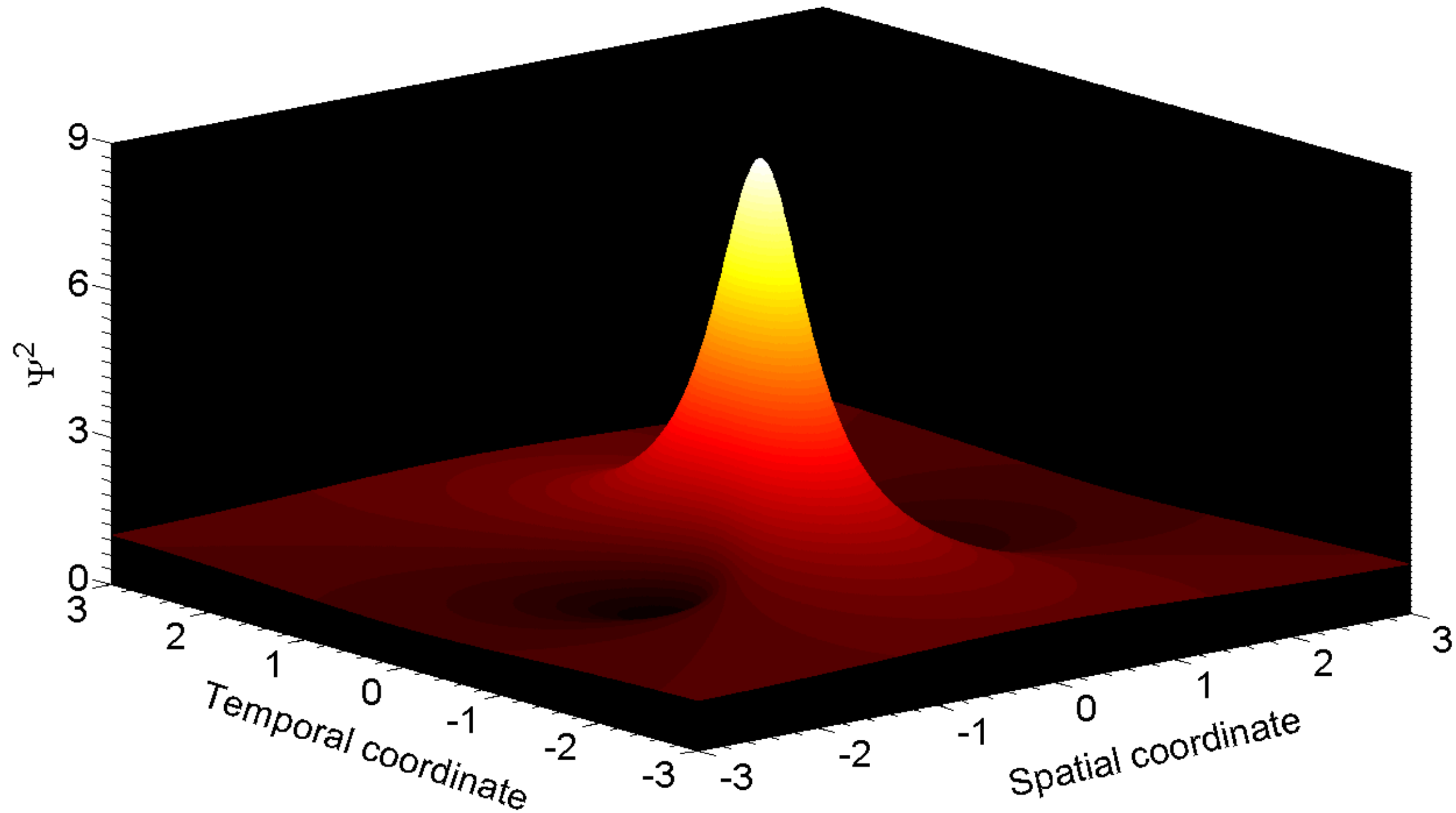
https://www.traveller.com.au/whats-the-story-behind-rogue-waves-h1dkjz



Hard evidence was obtained in 1995: a wave much higher than the average wave height struck an oil platform.

https://upload.wikimedia.org/wikipedia/en/e/e6/Draupner_close-up.png

Another type of localised nonlinear object: the peregrine soliton



https://en.wikipedia.org/wiki/Peregrine_soliton#/media/File:Soliton_de_Peregrine.png

Strogatz, Nonlinear Dynamics.

A dynamical perspective of the world

Number of variables →

	$n = 1$	$n = 2$	$n \geq 3$	$n >> 1$	Continuum
Linear	*Growth, decay, or equilibrium* Exponential growth RC circuit Radioactive decay	*Oscillations* Linear oscillator Mass and spring RLC circuit 2-body problem (Kepler, Newton)	Civil engineering, structures Electrical engineering	*Collective phenomena* Coupled harmonic oscillators Solid-state physics Molecular dynamics Equilibrium statistical mechanics	*Waves and patterns* Elasticity Wave equations Electromagnetism (Maxwell) Quantum mechanics (Schrödinger, Heisenberg, Dirac) Heat and diffusion Acoustics Viscous fluids
				The frontier	
Nonlinear	Fixed points Bifurcations Overdamped systems, relaxational dynamics Logistic equation for single species	Pendulum Anharmonic oscillators Limit cycles Biological oscillators (neurons, heart cells) Predator-prey cycles Nonlinear electronics (van der Pol, Josephson)	*Chaos* Strange attractors (Lorenz) 3-body problem (Poincaré) Chemical kinetics Iterated maps (Feigenbaum) Fractals (Mandelbrot) Forced nonlinear oscillators (Levinson, Smale) Practical uses of chaos Quantum chaos ?	Coupled nonlinear oscillators Lasers, nonlinear optics Nonequilibrium statistical mechanics Nonlinear solid-state physics (semiconductors) Josephson arrays Heart cell synchronization Neural networks Immune system Ecosystems Economics	*Spatio-temporal complexity* Nonlinear waves (shocks, solitons) Plasmas Earthquakes General relativity (Einstein) Quantum field theory Reaction-diffusion, biological and chemical waves Fibrillation Epilepsy Turbulent fluids (Navier-Stokes) Life

Nonlinearity ↓

Some ongoing work in the School

Pure-quartic solitons: returning to the effect of linear properties.

Highly-constrained systems: plasmonics, light at an interface, nanophotonics.

Mission to Alpha-Centauri!

Thank you