Lecture 5: Stellar structure

Senior Astrophysics

2018-03-14
Outline

1. Stars
2. Simplifying assumptions
3. Stellar structure
4. Virial theorem
5. Website of the Week
6. Timescales
7. Structure again
8. Next lecture
Part 2: Stars

Stellar structure and evolution (7 lectures + 2 labs)

1. How stars work
2. How stars evolve
3. Stellar remnants
Motivation

http://heasarc.gsfc.nasa.gov/docs/RXTE_Live/class.html

Lecture 5: Stellar structure

Stars
What is a star?

- Held together by self-gravity
- Collapse is resisted by internal pressure
- Since stars continually radiate into space, there must be a continual energy source
Simplifying assumptions

- Stars are spherical and symmetrical
  
  *all physical quantities depend only on* \( r \)

- Ignore rotation

- Ignore outside gravitational influences

- Uniform initial composition
  
  *no initial dependence of composition on radius*

- Newtonian gravity
  
  *no relativistic effects*

- Stars change slowly with time
  
  *can neglect* \( \frac{d}{dt} \) *terms*
To describe an isolated, static, symmetric star, we need four equations:

1. **Conservation of mass**

2. **Conservation of energy** (at each radius, the change in the energy flux equals the local rate of energy release)

3. **Equation of hydrostatic equilibrium** (at each radius, forces due to pressure differences balance gravity)

4. **Equation of energy transport** (relation between the energy flux and the local gradient of temperature)
In addition, we need to describe

1. **Equation of state** (pressure of the gas as a function of its density and temperature)
2. **Opacity** (how transparent the gas is to radiation)
3. **Nuclear energy generation rate** as $f(r, T)$
Conservation of mass

- If \( m \) is the mass interior to radius \( r \), then \( m, r \) and \( \rho \) are not independent, because \( m(r) \) is determined by \( \rho(r) \).

- Consider a thin shell inside the star, radius \( r \) and thickness \( dr \)

  \[
  \text{Volume is } dV = 4\pi r^2 dr, \text{ so mass of shell is } \\
  dm = 4\pi r^2 dr \cdot \rho(r)
  \]

  or

  \[
  \frac{dm}{dr} = 4\pi r^2 \rho(r)
  \]

  the equation of mass conservation
Hydrostatic equilibrium

- Consider a small parcel of gas at a distance $r$ from the centre of the star, with density $\rho(r)$, area $A$ and thickness $dr$.
- Outward force: pressure on bottom face $P(r)A$
- Inward force: pressure on top face, plus gravity due to material interior to $r$:

$$P(r+dr)A + \frac{Gm(r)dm}{r^2} = P(r + dr)A + \frac{Gm(r)\rho A dr}{r^2}$$
In equilibrium forces balance, so

\[ P(r)A = P(r + dr)A + \frac{Gm(r)\rho A dr}{r^2} \]

i.e.

\[ \frac{P(r + dr) - P(r)}{dr} A dr = -\frac{Gm(r)}{r^2} \rho(r) A dr \]

or

\[ \frac{dP}{dr} = -\frac{Gm}{r^2} \rho \]

the equation of **hydrostatic equilibrium**
Estimate for central pressure

- We can use hydrostatic equilibrium to estimate \( P_c \): we approximate the pressure gradient as a constant

\[
\frac{dP}{dr} \sim -\frac{\Delta P}{\Delta R} = \frac{P_c}{R} = \frac{GM}{R^2 \rho}
\]

Now assume the star has constant density (!): so

\[\rho_c = \bar{\rho} = \frac{M}{V} \sim \frac{M}{\frac{4}{3} \pi R^3}\]

then so

\[P_c \sim \frac{3GM^2}{4\pi R^4}\]

For the Sun, we estimate \( P_c \sim 3 \times 10^{14} \text{ N m}^{-2} = 3 \times 10^9 \text{ atm.}\)
Gravity has a very important property which relates the gravitational energy of a star to its thermal energy.

Consider a particle in a circular orbit of radius $r$ around a mass $M$.

Potential energy of particle is

$$
\Omega = -\frac{GMm}{r}
$$

Velocity of particle is $v = \sqrt{\frac{GM}{r}}$ (Kepler)
So kinetic energy is

\[ K = \frac{1}{2}mv^2 = \frac{1}{2} GMm \]

i.e. \( 2K = -\Omega \) or \( 2K + \Omega = 0 \).

Total energy

\[ E = K + \Omega \]
\[ = \Omega - \frac{\Omega}{2} = \frac{\Omega}{2} < 0 \]

**Consequence**: when something loses energy in gravity it **speeds up**!
The virial theorem

- The virial theorem turns out to be true for a wide variety of systems, from clusters of galaxies to an ideal gas; thus for a star we also have

\[ \Omega + 2U = 0 \]

where \( U \) is the total internal (thermal) energy of the star and \( \Omega \) is the total gravitational energy.

- So a decrease in total energy \( E \) leads to a decrease in \( \Omega \) but an increase in \( U \) and hence \( T \), i.e. when a star loses energy, it heats up.

**Fundamental principle**: stars have a negative heat capacity: they heat up when their total energy decreases.

- This fact governs the fate of stars
NASA ADS
http://adsabs.harvard.edu/abstract_service.html
Querying the astronomical literature
There are three important timescales in the life of stars:

1. **dynamical timescale** — the time scale on which a star would expand or contract if the balance between pressure gradients and gravity was suddenly disrupted.

2. **thermal timescale** — how long a star would take to radiate away its thermal energy if nuclear reactions stopped.

3. **nuclear timescale** — how long a star would take to exhaust its nuclear fuel at the current rate.
Dynamical timescale:

- the timescale on which a star would expand or contract if it were not in equilibrium; also called the **free-fall timescale**

$$\tau_{\text{dyn}} \equiv \frac{\text{characteristic radius}}{\text{characteristic velocity}} = \frac{R}{v_{\text{esc}}}$$

Escape velocity from the surface of the star:

$$v_{\text{esc}} = \sqrt{\frac{2GM}{R}}$$

so

$$\tau_{\text{dyn}} = \sqrt{\frac{R^3}{2GM}}$$

For the sun, $$\tau_{\text{dyn}} \simeq 1100 \text{ s}$$
Thermal timescale:
- the timescale for the star to radiate away its energy if nuclear reactions were switched off: also called the Kelvin-Helmholtz timescale
- Total gravitational energy available

\[ E_{\text{grav}} \sim \frac{GM^2}{R} \]

- If the star radiates energy at \( L \) (J/s), then it can keep up this rate for

\[ \tau_{\text{th}} \sim \frac{E_{\text{grav}}}{L} \sim \frac{GM^2}{RL} \]

For the sun, \( \tau_{\text{th}} \sim 3 \times 10^7 \text{ y} \ll \text{age of Earth.} \)
Nuclear timescale: times to exhaust nuclear fuel at current rate.

\[ \tau_{\text{nuc}} \sim \frac{\eta M_c c^2}{L} \]

where \( \eta \) is an efficiency factor for nuclear fusion: \( \eta \sim 0.7\% \) (see next lecture), and \( M_c \) is the mass of the core.

For the sun, \( \tau_{\text{nuc}} \sim 10^{10} \) y
For stars,

\[ \tau_{\text{dyn}} \ll \tau_{\text{th}} \ll \tau_{\text{nuc}} \]

- \( \tau_{\text{dyn}} \) = timescale of collapsing star, e.g. supernova
- \( \tau_{\text{th}} \) = timescale of star before nuclear fusion starts, e.g. pre-main sequence lifetime
- \( \tau_{\text{nuc}} \) = timescale of star during nuclear fusion, i.e. main-sequence lifetime
Most stars, most of the time, are in hydrostatic and thermal equilibrium, with slow changes in structure and composition occurring on the (long) timescale $\tau_{\text{nuc}}$ as fusion occurs.

If something happens to a star faster than one of these timescales, then it will NOT be in equilibrium. e.g. sudden addition of energy (nearby supernova?), sudden loss of mass (binary interactions)
We have derived two of the equations which define the structure of stars:

\[
\frac{dP}{dr} = -\frac{GM}{r^2} \rho \quad \text{hydrostatic equilibrium}
\]

\[
\frac{dM}{dr} = 4\pi r^2 \rho \quad \text{mass conservation}
\]

We need two more equations.
Equation of energy generation

- Assume the star is in thermal equilibrium, so at each radius $T$ does not change with time.
- Rate of energy generation/unit mass = $\varepsilon$. Then

\[ \text{Shell mass } dm = 4\pi r^2 \rho dr \]

\[ \text{Luminosity at } r: L(r) \]

\[ \text{Luminosity at } r + dr: L(r) + dL \]

\[ \frac{dL}{dr} = 4\pi r^2 \rho \varepsilon \]

so

\[ dL = 4\pi r^2 \rho dr \times \varepsilon \]

the equation of energy generation
Equation of energy transport

- Fourth equation describes how energy is transported through the layers of the star, i.e. how the gas affects the radiation as it travels through.
- Depends on local density, opacity and temperature gradient.
- Will not derive here, but quote result:

\[
\frac{dT}{dr} = -\frac{3}{4ac} \frac{\kappa \rho}{T^3} \frac{L}{4\pi r^2}
\]

the equation of energy transport. \(a\) is called the radiation constant and has the value

\[
a = \frac{4\sigma}{c} = 7.566 \times 10^{-16} \text{ Jm}^{-3}K^{-4}
\]

where \(\sigma = 5.670 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}\) is the Stefan-Boltzmann constant. Comes from considering radiation pressure.
Equations of stellar structure

- This gives us four equations in four unknowns — $m(r)$, $L(r)$, $\rho(r)$ and $T(r)$ — so enough for a solution, provided we know $P(\rho, T)$, $\kappa$ and $\varepsilon$.

- Also need **boundary conditions**:
  - centre of star ($r \to 0$): $M \to 0$, $L \to 0$
  - surface of star ($r \to R_*$): $T \to T_s$, $P \to 0$, $\rho \to 0$

- The calculation of full stellar models is a **very** hard problem, and must be done numerically, since in general $\kappa$ and (especially) $\varepsilon$ are strong functions of density and temperature.
Stellar models

- Some progress can be made by making simplifying assumptions, e.g. if pressure is only a function of density, then the first two equations can be solved independently from the equations involving temperature.
- We will be investigating numerical models of stars in Lab 2.
Next lecture

- is our first computer lab. This will be held in

  **SNH Learning Studio 4003**

where we will be exploring the Saha-Boltzmann equation.

- Review your 2nd year Matlab notes, and perhaps bring them to the lab for reference.
- **Please read** the exercises before the lab. There’s a lot in there (mostly repeat of lecture material), so don’t want to lose time.
- If you can’t make Friday’s session, there’s also a lab on **now** (10–11 am Wednesday).