Lecture 2: Radiative Energy Transport

2.1 Emission and Extinction Coefficients

The energy in a beam of radiation propagating through a medium will be modified by emission, absorption, and scattering. These effects are described by emission and extinction coefficients.

The emission coefficient:

The ***emission coefficient***, j_ν (units $\mathrm{W\,sr^{-1}\,Hz^{-1}\,m^{-3}}$) is defined such that the energy added to a beam of radiation by an elemental volume $dV = dA\,ds$ of matter intercepting the beam is

$$dE = j_\nu \, dA\, ds\, d\Omega\, d\nu\, dt$$

In general, j_ν also contains a scattering component. Note also that sometimes j_ν is used to represent a mass emission coefficient (in units of $\mathrm{W\,sr^{-1}\,Hz^{-1}\,kg^{-1}}$).

The extinction coefficient:

Here, we will consider the combined effects of absoprtion and scattering of radiation in the definition of the ***extinction coefficient***, k_ν (in units m^{-1}). It is defined such that the energy removed from a beam of radiation by an elementary volume of cross-section dA and length ds is

$$dE = k_\nu \, I_\nu \, dA\, ds\, d\Omega\, d\nu\, dt$$

Note that a mass extinction coefficient, κ_ν, is also sometimes used, where $\rho\kappa_\nu = k_\nu$ (i.e. units of κ_ν are $\mathrm{m^2\,kg^{-1}}$.

The quantity $1/k_\nu$ is known as the ***mean free path*** of a photon. It is the mean distance travelled by a photon in a medium before it is absorbed and/or scattered (i.e. units are m).

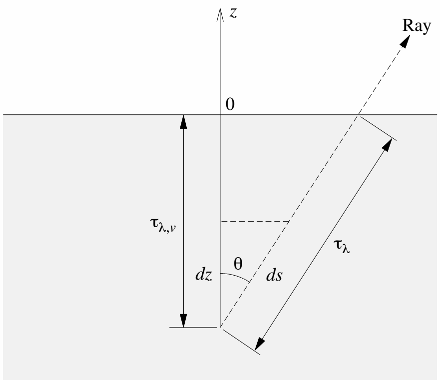
Note that both j_ν and k_ν are properties of the *medium*, not of the radiation field (generally, they are functions of the temperature, T, and mass density, ρ, of the medium).

optical depth:

The ***optical depth*** of a medium is a dimensionless quantity that gives a measure of how much radiation is attenuated as it passes through it along a geometrical path s.

$$\tau_\nu = \int_{s_1}^{s_2} k_\nu(z)\, ds = \frac{1}{\mu}\int_{z_1}^{z_2} k_\nu(z)\, dz \qquad (1)$$

$ds = dz/\cos\theta = dz/\mu$ for plane-parallel geometry



Note that for applications to stellar atmospheres, the optical depth is defined such that it decreases with height (increasing z), as shown in this figure, i.e.

$$d\tau = -k_\nu dz/\mu$$

for (plane-parallel) stellar atmospheres.

2.2 The Equation of Radiative Transfer

The net change in the intensity of a radiation beam as a result of its interaction with matter is obtained by combining the effects of emission and extinction:

net change in radiative energy = emission - extinction

$$\Longrightarrow dI_\nu\, dA\, d\Omega\, d\nu\, dt = j_\nu\, dA\, ds\, d\Omega\, d\nu\, dt - k_\nu\, I_\nu\, dA\, ds\, d\Omega\, d\nu\, dt$$

This implies

$$\boxed{\frac{dI_\nu}{ds} = j_\nu - k_\nu I_\nu} \qquad (2)$$

and using the definition (1) of optical depth gives

$$\frac{dI_\nu}{d\tau_\nu} = \frac{j_\nu}{k_\nu} - I_\nu$$

We now introduce the definition of the ***source function***, $\boxed{S_\nu \equiv j_\nu/k_\nu}$, so that

$$\boxed{\frac{dI_\nu}{d\tau_\nu} = S_\nu - I_\nu \qquad \text{equation of radiative transfer}} \qquad (3)$$

2.3 Simple solutions to the RTE

1. No material present

$$j_\nu = 0 = k_\nu \quad \Longrightarrow \quad \frac{dI_\nu}{ds} = 0 \quad \Longrightarrow \quad I_\nu = const$$

2. Uniform medium in thermodynamic equilibrium

In thermodynamic equilibrium, the radiation field does not change with time or position, so $dI_\nu/d\tau_\nu = 0$. i.e.

rate of extinction = rate of emission

Therefore, $I_\nu = S_\nu$ and

$$\boxed{j_\nu = k_\nu I_\nu = k_\nu B_\nu = \text{rate of thermal emission} \qquad \text{Kirchoff's law}} \tag{4}$$

3. Emission only

$$k_\nu = 0 \quad \Longrightarrow \quad \mu\frac{dI_\nu}{dz} = j_\nu \quad \Longrightarrow \quad I_\nu = I_\nu(0) + \frac{1}{\mu}\int_0^z j_\nu dz$$

where $I_\nu(0)$ is the incident specific intensity.

4. Absorption only

$$j_\nu = 0 \quad \Longrightarrow \quad \mu\frac{dI_\nu}{dz} = -k_\nu I_\nu$$

and for a finite slab, the emergent radiation is

$$I_\nu(\mu) = I_\nu(0)\exp\left(-\frac{1}{\mu}\int_0^z k_\nu\, dz\right) = I_\nu(0)\exp\left(-\tau_\nu\right)$$

2.4 Formal solution to the RTE

The radiative transfer equation

$$\frac{dI_\nu}{d\tau_\nu} = S_\nu - I_\nu$$

is a 1st-order differential equation, so the formal solution is obtained by multiplying through by an integrating factor e^{τ_ν}, giving

$$\frac{d}{d\tau_\nu}\left(e^{\tau_\nu} I_\nu\right) = S_\nu e^{\tau_\nu}$$

$$\Longrightarrow I_\nu(\tau_2)e^{\tau_2} - I_\nu(\tau_1)e^{\tau_1} = \int_{\tau_1}^{\tau_2} S_\nu(\tau_\nu)e^{\tau_\nu}\, d\tau_\nu$$

where $\tau_1 = \tau_\nu(s_1)$ and $\tau_2 = \tau_\nu(s_2)$. After rearranging, this then gives the formal solution

$$\boxed{I_\nu(\tau_2) = I_\nu(\tau_1)e^{-(\tau_2-\tau_1)} + \int_{\tau_1}^{\tau_2} S_\nu(\tau)e^{-(\tau_2-\tau)}\, d\tau} \qquad (5)$$

Example: Radiation transport through a homogeneous medium for which S_ν is constant:

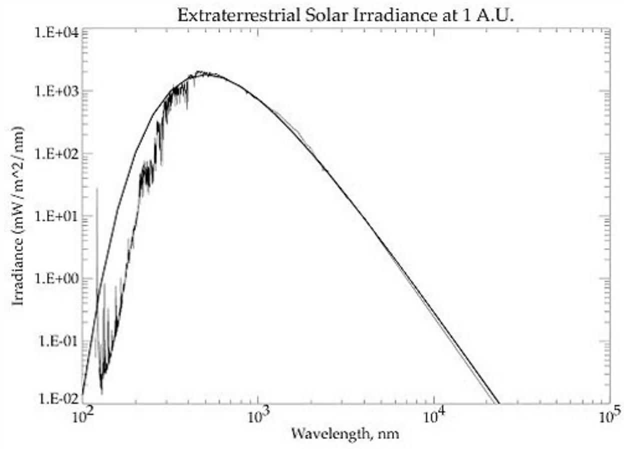
$$\begin{aligned} I_\nu(\tau_2) &= I_\nu(\tau_1)e^{-(\tau_2-\tau_1)} + S_\nu\left[1 - e^{-(\tau_2-\tau_1)}\right] \qquad (6)\\ &\simeq \begin{cases} S_\nu & \text{optically thick } (\Delta\tau \gg 1) \\ I_\nu(\tau_1) & \text{optically thin } (\Delta\tau \ll 1) \end{cases} \end{aligned}$$

Thus, ***in the limit of large optical depths, the specific intensity of the radiation field approaches the source function of the medium.*** i.e. the radiation field comes into equilibrium with the medium.

The exponential decay of a beam of photons has the interpretation that the probability of a photon travelling an optical depth $\Delta\tau = \tau_2 - \tau_1$ before being absorbed or scattered is just $e^{-\Delta\tau}$.

Local Thermodynamic Equilibrium (LTE) vs strict Thermodynamic Equilibrium (TE)

In real astrophysical sources, S_ν is a sensitive function of the temperature, T, and density, ρ, of the medium. In stellar and planetary atmospheres, for example, both these quantities vary significantly between the outer surface layer and the innermost layer. Strict TE prevails *only* when $S_\nu = B_\nu$ *everywhere*. A thermal source emitting a spectrum that is not quite Planckian at all frequencies (i.e. exhibits lines and edges) is referred to as being in *local thermodynamic equilibrium* (LTE).



Example: The solar spectrum is produced in the photosphere, where the *average* temperature is $T \simeq 5800\,\mathrm{K}$. At this T, a blackbody peaks at a frequency $\nu_{\rm peak} \approx 2.82kT/h \simeq 3 \times 10^{14}\mathrm{Hz}$ and wavelength $\lambda_{\rm peak} \simeq 500\,\mathrm{nm}$ which is in the near infrared (NIR). (n.b. $\lambda_{\rm peak}$ does not equal $c/\nu_{\rm peak}$ – see assignment 1)

The cosmic microwave background (CMB) radiation is the best example of a real astrophysical source in strict TE. Generally, however, most real astrophysical sources are far from being in an ideal TE state. The example below shows the FIR spectrum of the star forming region in the Orion nebula. The underlying dotted/dashed curves are multiple-T blackbody fits to the observed spectrum (solid line).