

## Lecture 4: Radiative Processes

### 4.1 Radiation Mechanisms

The coefficients of emission  $j_\nu$  and extinction  $k_\nu$  are macroscopic descriptions of the microphysical processes that result in emission, absorption, and scattering of photons. These microphysical processes are:

1. **bound-bound transitions**, which correspond to bound electrons being excited or de-excited to another bound state as a result of absorption or emission of a photon; this results in line radiation
2. **bound-free transitions** correspond to photoionisation and recombination, depending on whether a photon is absorbed or emitted and an electron is excited to a free state or a bound state, respectively
3. **free-free transitions** correspond to free electrons being excited or de-excited to another free state following absorption or emission of a photon; this results in continuum radiation
4. **scattering processes** can occur between atoms, free electrons and photons; they result in a redistribution of photon energies (i.e. scattering processes modify the radiation spectrum).

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### 4.2 The Einstein Coefficients

The macrophysical emission and extinction coefficients  $j_\nu$  and  $k_\nu$  can be described with a microphysical formalism through the use of the Einstein coefficients, which describe the probability of atomic transitions between two energy levels  $E_n$  and  $E_m$  (where  $E_m > E_n$ ) with corresponding populations  $N_n$  and  $N_m$  per unit volume and statistical weightings  $g_n$  and  $g_m$ . These coefficients enable all radiative transitions to be described in terms of just 3 fundamental processes:

(i) **spontaneous emission**: the transition probability per unit time for the emission of a photon of energy  $h\nu = E_m - E_n$  is defined as the **Einstein A-coefficient**,  $A_{mn}$ , and the emission rate per unit volume is  $N_m A_{mn}$ . In the absence of any other processes, the **mean lifetime** of particles in state 2 is  $\Delta t = A_{mn}^{-1}$  (in seconds). According to the Heisenberg uncertainty principle, there is a corresponding spread in energy  $\Delta E = h/(2\pi\Delta t)$ , implying  $\Delta\nu = 1/(2\pi\Delta t)$ . For a line transition, this natural broadening gives rise to a Lorentzian profile:

$$\phi(\nu) = \frac{\Gamma/4\pi^2}{(\nu - \nu_0)^2 + (\Gamma/4\pi)^2} \quad (1)$$

where  $\nu_0$  is the line centre frequency and  $\Gamma = 1/\Delta t$  is the *radiative damping constant*.

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The  $A_{mn}$  coefficient incorporates a summation over this emission probability distribution. In real sources, other processes (e.g. collisions) produce much broader line profiles. These are examined in the Intermediate Physics Stellar Astrophysics module.

(ii) true absorption: the transition probability per unit time for absorption of a photon of energy  $h\nu = E_m - E_n$  is given by  $B_{nm}\bar{J}$ , where  $B_{nm}$  is the **Einstein B-coefficient** for photoabsorption, and  $\bar{J}$  is the mean intensity, weighted over the line profile  $\phi(\nu)$ , viz.

$$\bar{J} = \int_0^\infty J_\nu \phi(\nu) d\nu$$

The rate of photoabsorption per unit volume is  $N_n B_{nm} \bar{J} [s^{-1} m^{-3}]$ .

(iii) stimulated emission: in some circumstances, a photon can induce a transition from a high atomic energy state  $E_m$  to a lower state  $E_n$  and a photon of energy  $h\nu = E_m - E_n$  is emitted in the process. The rate per unit volume of this stimulated emission is given by  $N_m B_{mn} \bar{J}$ , where  $B_{mn}$  is the **Einstein coefficient for stimulated emission**. An important property of stimulated emission is that it takes place into precisely the same photon state (direction and frequency). **The emitted photon is precisely coherent with the photon that stimulated the emission** (e.g. laser, maser emission).

### Summary of Einstein coefficients:

$A_{mn}$ = transition probability for spontaneous emission (radiative de-excitation) from state $m$ down to state $n$ per second per particle in state $m$ .
$B_{nm}\bar{J}$ = transition probability for true absorption (radiative excitation) from state $n$ up to state $m$ per second per particle in state $n$ .
$B_{mn}\bar{J}$ = transition probability for stimulated emission (induced de-excitation) from state $m$ down to state $n$ per second per particle in state $m$ .

Under the condition of **Radiative Equilibrium (RE)**, the number of radiative transitions per unit time per unit volume out of state  $n$  must equal the number of radiative transitions per unit time per unit volume into state  $n$ , i.e.

true absorption = spontaneous + stimulated emission

$$\implies \boxed{N_n B_{nm} \bar{J} = N_m A_{mn} + N_m B_{mn} \bar{J}} \quad \text{Radiative Equilibrium (RE)} \quad (2)$$

The Einstein coefficients are linked via the **detailed balance relations**, which connect any microphysical process to its inverse (i.e. emission and absorption):

$$g_n B_{nm} = g_m B_{mn} \quad A_{mn} = \frac{2h\nu^3}{c^2} B_{mn} \quad \text{Einstein-Milne relations} \quad (3)$$

Note that the Einstein coefficients and the Einstein-Milne relations are independent of  $T$ . Thus, they should be valid even under conditions where TE or LTE are not satisfied. This is verified from a full quantum mechanical treatment.

Note also the units for each coefficient:  $A_{mn}$  is in units of  $[s^{-1}]$  (typical values for **allowed** transitions at optical frequencies,  $\nu \sim 10^{15}$  Hz, are  $10^8 s^{-1}$ ; transitions with low probabilities, or long lifetimes, are referred to as **forbidden**). This implies that  $B_{mn}$  has units of  $[m^2 J^{-1}]$ . This links it to a physical quantity such as cross-section per energy. We will return to this in the next Lecture.

#### Relation between Einstein coefficients and coefficients for emission and absorption:

This relation is easiest to derive for monochromatic line transitions, under the assumption that the emission is distributed with the same line profile as the absorption. For emission, we know (from Lec. 1) that the total energy emitted in a volume  $dV$ , solid angle  $d\Omega$ , frequency range  $d\nu$  and time interval  $dt$  is  $j_\nu dV d\Omega d\nu dt$ . Since each atom contributes an energy  $h\nu$  distributed over  $4\pi$  in solid angle and  $\phi(\nu)$  in frequency for each transition, this total energy is equivalent to  $(h\nu/4\pi)\phi(\nu)N_m A_{mn} dV d\Omega d\nu dt$ . A similar argument follows for absorption, giving

$$j_\nu = \frac{h\nu_{mn}}{4\pi} N_m A_{mn} \phi(\nu) \quad k_\nu = \frac{h\nu_{nm}}{4\pi} (N_n B_{nm} - N_m B_{mn}) \phi(\nu) \quad (4)$$

where  $k_\nu$  now represents an **absorption coefficient corrected for stimulated emission** (i.e. it is formally equivalent to negative absorption). We can simplify  $k_\nu$  further by using the Einstein-Milne relations (3):

$$k_\nu = \frac{h\nu_{nm}}{4\pi} N_n B_{nm} \left( 1 - \frac{N_m g_n}{N_n g_m} \right) \phi(\nu) \quad (5)$$

The RTE and source function in terms of Einstein coefficients:

From the definition of the RTE  $dI_\nu/ds = j_\nu - k_\nu I_\nu$ , substituting the relations (4) gives

$$\frac{dI_\nu}{ds} = \frac{h\nu}{4\pi} N_m A_{mn} \phi(\nu) - \frac{h\nu}{4\pi} N_n B_{nm} \left(1 - \frac{N_m g_n}{N_n g_m}\right) \phi(\nu) I_\nu \quad (6)$$

This now represents a radiative transfer equation for line (bound-bound) transitions. We can consider solutions in much the same way as we did formally in Lec. 2.

Using the Einstein-Milne relations (3), the source function  $S_\nu \equiv j_\nu/k_\nu$  can also be written as

$$S_\nu = \frac{2h\nu^3}{c^2} \left(\frac{N_n g_m}{N_m g_n} - 1\right)^{-1} \quad (7)$$

Note that this does not refer just to line transitions. Although it was derived from consideration of only line transitions, the expression is now general and applies to all radiative transitions. In the following section, we shall see that  $S_\nu$  simplifies to  $B_\nu$  when the population ratio  $N_n/N_m$  obeys LTE.

### 4.3 Thermodynamic and Statistical Equilibrium

The emission and extinction coefficients  $j_\nu$  and  $k_\nu$  (and hence,  $S_\nu$ ) can be most easily determined for a gas in LTE, when all the microscopic changes are in **detailed balance** (i.e. all transitions into an energy state = all transitions out of that energy state), and the densities of the microscopic states are prescribed by the Maxwell–Boltzmann–Saha relations: [statistical mechanics](#):

(i) Maxwell's relation for the velocity distribution of thermal particles of mass  $m$ :

$$N(v) = 4\pi v^2 N_0 \left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left(-\frac{mv^2}{2kT}\right) \quad (8)$$

(ii) Boltzmann's relation for the relative populations of the ground atomic level  $N_1$  and the  $n$ th level  $N_n$  (per unit volume), with corresponding energies  $E_1$  and  $E_n$  and statistical weights  $g_1$  and  $g_n$ :

$$\frac{N_1}{N_n} = \frac{g_1}{g_n} \exp\left(\frac{E_n - E_1}{kT}\right) \quad (9)$$

(iii) Saha's relation for the ratio of number densities of two different ionisation states of an atomic species:

$$\frac{N_{i+1}N_e}{N_i} = \frac{g_{i+1}g_e}{g_i} \left( \frac{2\pi m_e kT}{h^2} \right)^{3/2} \exp\left(\frac{-\chi_i}{kT}\right) \quad (10)$$

where  $N_i$  is the number density of atoms in the  $i$ th ionisation state,  $N_{i+1}$  is the number density in the next ionisation state,  $N_e$  is the number density of free electrons,  $\chi_i$  is the ionisation potential difference between the  $i$ th and  $i + 1$  states, and the factors  $g_i$ ,  $g_{i+1}$ , and  $g_e$  are the corresponding statistical weights (degeneracy) of the discrete energy states. For free (unbound) electrons,  $g_e = 2$ , owing to two spin states. Generally, one can use  $g_n = 2n^2$  for bound energy state. For the more complex atomic systems, a quantum treatment is usually necessary to calculate the statistical weightings.

**Example:** Calculate the fraction of Hydrogen atoms in the  $n = 2$  state with respect to the ground state for 3 different temperatures:  $T_1 = 6000$  K,  $T_2 = 10,000$  K,  $T_3 = 30,000$  K. (N.B. These are typical temperatures for G, A/B, and O/B stars, respectively).

Use the Boltzmann relation (9) to get the relevant expression

$$\frac{N_2}{N_1} = \frac{g_2}{g_1} \exp\left[-\frac{(E_2 - E_1)}{kT}\right]$$

Now use  $g_1 = 2$  and  $g_2 = 8$  and  $E_2 - E_1 = E_{21} = hc/\lambda_{21}$ , where  $\lambda_{21} = 1216 \text{ \AA}$  is the Lyman  $\alpha$  wavelength. Substituting these values in gives

$$\frac{N_2}{N_1}(T_1) \simeq 10^{-8}, \quad \frac{N_2}{N_1}(T_2) \simeq 3 \times 10^{-5}, \quad \frac{N_2}{N_1}(T_3) \simeq 8 \times 10^{-2}$$

So in the hottest stars (O-type), only 8% of H-atoms are in the  $n = 2$  state relative to the ground state. But at this temperature, most H-atoms would have ionised. So, we can also use the Saha relation to calculate the fraction  $N_{H^+}/N_H$  of ionised-to-neutral Hydrogen once we have a value for  $N_e$ . We get this from a typical electron pressure of  $P_e = N_e kT = 10 \text{ N m}^{-2}$ . We find  $N_{H^+}/N_H \approx 0$  (for  $T_1$ ), 5 (for  $T_2$ ),  $10^6$  (for  $T_3$ ).

### The LTE source function:

Recall the generalised expression for the source function  $S_\nu$  given by (7) derived from the Einstein coefficients. The Boltzmann relation implies

$N_n g_m / (g_n N_m) = \exp(E_m - E_n) / kT$ . Substituting this into (7) immediately gives

$$S_\nu = \frac{2h\nu^3}{c^2} \left[ \exp\left(\frac{h\nu}{kT}\right) - 1 \right]^{-1} \equiv B_\nu(T) \quad (11)$$

Thus, we arrive at the result that the source function equals the Planck function under LTE.

### The LTE absorption and emission coefficients:

Similarly, we can simplify the absorption coefficient under LTE conditions. Substituting the Boltzmann relation (9) into the expression (5) for the absorption coefficient yields

$$k_\nu = \frac{h\nu}{4\pi} N_n B_{nm} \left[ 1 - \exp\left(-\frac{h\nu}{kT}\right) \right] \phi(\nu) \quad (12)$$

Note that  $1 - \exp(-h\nu/kT)$  is the LTE correction factor due to stimulated emission. Under LTE, the emission and absorption coefficients are related via Kirchoff's law:  $j_\nu = k_\nu B_\nu(T)$ . Thus, it is only necessary to determine either  $k_\nu$  or  $j_\nu$ .

### Non-LTE and inverted populations (ADV only):

Any population that does not satisfy the Boltzmann relation, i.e. with

$$\frac{N_m g_n}{N_n g_m} \neq \exp\left(-\frac{h\nu}{kT}\right)$$

is usually referred to as **non-thermal**. By definition, such populations cannot be in LTE.

A specific example of when non-thermal populations become important is when stimulated emission is important. Recall from the absorption coefficient (5) that the correction factor for stimulated emission is  $1 - N_m g_n / (N_n g_m)$ . Under LTE,

$N_m g_n / (N_n g_m) = \exp(-h\nu/kT)$ , which is always less than unity (i.e. stimulated emission is negligible under LTE conditions). Under some conditions, however, it is possible to attain a state such that the higher energy levels are populated more than the lower energy levels, i.e.  $N_m g_n / (N_n g_m) > 1$ . This is referred to as an **inverted population** and gives a net negative absorption, which is formally equivalent to emission.