

Lecture 6: Radiative Processes III

6.1 LTE Continuum Transition Rates

Continuum transition rates are somewhat more complicated to calculate than line transition rates since the energy of the free electron must be taken into account. The oscillator strength can be used, but now it involves an integral over the free electron velocity distribution (a Maxwellian under LTE). What is usually calculated instead is the absorption cross-section. For bound-free and free-bound transitions, the energy of the photon absorbed or emitted must balance the ionisation potential $\chi_n = E_i - E_n$ of the energy level involved and also the kinetic energy of the free electron:

$$h\nu = \chi_n + \frac{1}{2}m_e v^2 \quad (1)$$

For free-free transitions, both electron and photon have continuum energies, so the above restriction is no longer relevant, although an integration over the electron energy distribution is still necessary.

The photoionisation (bound-free) absorption cross-section

Recall from Lec. 5 that the absorption coefficient can be written in terms of an absorption cross-section: $\sigma_\nu^{nm} = k_\nu^{nm} / N_n$. When the upper energy state m lies in the continuum, the process is referred to as **photoionisation** and the cross-section for absorption of a photon and ionisation from energy level n is approximately given by

$$\sigma_\nu^{\text{bf}} \simeq \left(\frac{1}{4\pi\epsilon_0} \right)^5 \frac{64\pi^4 Z^4 m_e e^{10}}{3^{3/2} c h^6 \nu^3 n^5} \simeq 2.81 \times 10^{25} \frac{Z^4}{\nu^3 n^5} \text{ m}^2 \quad (2)$$

where Z is the atomic number. This expression is only valid for $h\nu > \chi_n$ (c.f. eqn. 1). For $h\nu < \chi_n$, photoionisation cannot proceed. This means that the frequency dependence of σ_ν^{bf} is quite sharp near $h\nu \gtrsim \chi_n$ and then falls off as ν^{-3} . In real spectra, the abrupt absorption features near frequencies corresponding to ionisation energies from different n levels are referred to as **absorption edges**.

Recombination (free-bound) radiation

The inverse of photoionisation is **recombination** and corresponds to a free-bound transition – i.e. capture of a free electron to a bound energy state n and the emission of a photon. The volume emissivity for this process under LTE is

$$j_\nu^{\text{fb}} = \frac{2h\nu^3}{c^2} N_e N_i \left(\frac{h^2}{2\pi m_e kT} \right)^{3/2} \sigma_\nu^{\text{bf}} \exp \left[-\frac{(h\nu - \chi_n)}{kT} \right], \quad h\nu > \chi_n \quad (3)$$

Note that because $\sigma_\nu^{\text{bf}} \propto \nu^{-3}$ (from eqn. 2), the only frequency dependence is in the exponential factor. So, j_ν^{fb} rises sharply near $h\nu \gtrsim \chi_n$ and then declines exponentially.

Qualitative derivation of j_ν^{fb} (ADV only):

Note that even though LTE has been assumed, the expression for j_ν^{fb} does not exactly equal $k_\nu^{\text{bf}} B_\nu$, as required by Kirchoff's law. This is because the expression for σ_ν^{bf} given by eqn. (2) is an approximation only. An exact expression for σ_ν^{bf} (and hence, k_ν^{bf}) is obtained from a full quantum mechanical treatment. One then must balance the total rates of emission and absorption, according to the radiative equilibrium relation (c.f. eqn. 2 in Lec. 4). The derivation of the exact expression for the free-bound emission coefficient given by (3) involves the following expression:

$$j_\nu^{\text{fb}} = \frac{\chi_n}{4\pi} N_e N_n \frac{\partial}{\partial \nu} \int v \sigma_\nu^{\text{fb}}(v) f(v) dv = \frac{\chi_n}{4\pi} N_e N_n \frac{\partial}{\partial \nu} \langle v \sigma_\nu^{\text{fb}} \rangle$$

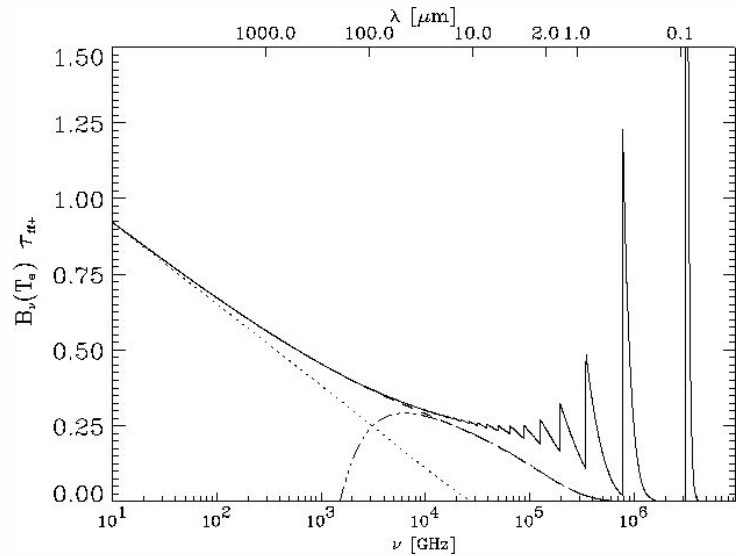
where $f(v)$ is a Maxwellian distribution of free electrons (with units $[v^{-1}]$). Now the integral over v needs to be performed over the delta function $\delta(h\nu - \chi_n - \frac{1}{2}m_e v^2)$, since the electron energies are restricted over this range. Saha's relation for the relative population of ionisation states and Maxwell's velocity distribution for the free electrons then imply that the

absorption and emission cross-sections are related by

$$\sigma_{\nu}^{\text{fb}}(\nu) = \frac{2h^2\nu^2}{m^2c^2v^2} \frac{g_n}{g_e g_i} \sigma_{\nu}^{\text{bf}}$$

Substituting this in and performing the integral then gives the expression for j_{ν}^{fb} given above.

This figure shows the free-bound (re-combination) emissivity (solid line) and the free-free (bremsstrahlung) emissivity (dashed line) for $T = 7 \times 10^3$ K. The dotted and dash-dash-dot lines are (incorrect) classical approximations.



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Bremsstrahlung (free-free) radiation

Recall that in the LS -coupling scheme, the rates of electronic transitions are determined by assuming that electrostatic (Coulomb) interactions dominate. For bound-bound transitions, Coulomb interactions between close-neighbour bound electrons are the most important. For free electrons, the distant Coulomb interaction with the ionised atom's electrostatic potential is relevant. The emission coefficient (volume emissivity) for free-free emission is

$$j_{\nu}^{\text{ff}} \simeq \sqrt{\frac{32\pi}{3}} \bar{G}_{\nu}^{\text{ff}} Z^2 N_i N_e \sigma_{\text{T}} r_0 m_e c^2 \left(\frac{kT}{m_e c^2} \right)^{-1/2} \exp\left(-\frac{h\nu}{kT}\right) \quad (4)$$

where $\bar{G}_{\nu}^{\text{ff}}$ is the free-free Gaunt factor (usually of order unity),

$r_0 = e^2/(4\pi\epsilon_0 m_e c^2) \simeq 2.8 \times 10^{-15}$ m is the classical electron radius and

$\sigma_{\text{T}} = (8/3)\pi r_0^2 \simeq 6.65 \times 10^{-29}$ m² is the Thomson cross-section. The above expression

for j_{ν}^{ff} is exact and so the absorption coefficient for the inverse process, **free-free absorption**, can be derived directly from Kirchoff's law.

Note that the frequency dependence of j_{ν}^{ff} is only in the exponential factor (neglecting the slow ν dependence in $\bar{G}_{\nu}^{\text{ff}}$). Thus, the emission spectrum is constant at frequencies $h\nu \ll kT$, and declines exponentially at $h\nu \gg kT$.

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6.2 Scattering Effects

Scattering is not an intrinsic radiative process, but rather it redistributes photon energies to produce a radiation spectrum that may be quite different from the one predicted by radiation processes alone. One of the most important type of scattering is electron scattering. The simplest treatment is to assume isotropic scattering. For nonrelativistic electrons, scattering off photons is approximately elastic, or **coherent**: the energy “absorbed” is approximately the energy “emitted”.

For isotropic (j_ν^{scat} and k_ν^{scat} independent of direction Ω) and coherent scattering, the total volume rate of emission balances the total volume rate of absorption, giving:

$$\int j_\nu^{\text{scat}} d\Omega = \int k_\nu^{\text{scat}} I_\nu d\Omega \implies j_\nu^{\text{scat}} = k_\nu^{\text{scat}} \int I_\nu d\Omega = k_\nu^{\text{scat}} J_\nu \quad (5)$$

Thus, the source function for pure scattering is

$$S_\nu^{\text{scat}} \equiv \frac{j_\nu^{\text{scat}}}{k_\nu^{\text{scat}}} = J_\nu \quad (6)$$

Thus, the RTE for pure scattering is

$$\frac{dI_\nu}{ds} = -k_\nu^{\text{scat}} (I_\nu - J_\nu) \quad (7)$$

This equation is an integro-differential equation because S_ν depends on the solution I_ν in all directions through a given point. Therefore, it cannot be solved by the formal solution for pure radiative processes.

A useful approach is to treat the processes of absorption, emission and propagation of radiation in probabilistic terms for a single photon rather than the average behaviour of a large number of photons. e.g. the exponential decay of a beam of photons, $e^{-\tau_\nu}$, is interpreted as the probability that a photon travels an optical depth τ_ν before encountering an absorption or scattering event. The net effective optical depth is just

$$\tau_\nu = \int (k_\nu^{\text{abs}} + k_\nu^{\text{scat}}) ds \quad (8)$$

where the combined effect of pure absorption and scattering in $k_\nu^{\text{abs}} + k_\nu^{\text{scat}}$ is referred to as the **extinction coefficient**, defined previously in Lec. 2. In fact, a random walk argument shows that the mean number of scatterings is $\max(\tau_\nu, \tau_\nu^2)$.

In a homogeneous, thermal medium, the RTE is

$$\frac{dI_\nu}{ds} = -k_\nu^{\text{abs}}(I_\nu - B_\nu) - k_\nu^{\text{scat}}(I_\nu - J_\nu) = -(k_\nu^{\text{abs}} + k_\nu^{\text{scat}})(I_\nu - S_\nu)$$

where the source function is

$$S_\nu = \frac{k_\nu B_\nu + k_\nu^{\text{scat}} J_\nu}{k_\nu + k_\nu^{\text{scat}}}$$

and in LTE, $J_\nu = B_\nu$, implying $S_\nu = B_\nu$. So, even when scattering is important, the Planck function is still recovered under LTE. This is reached in the limit of large optical depths (i.e. many scatterings).

6.3 Collisional Processes

Electronic transitions can also proceed via collisions between atoms and free electrons. When T is sufficiently high that a substantial fraction of atoms are in an ionised state and there are many free electrons available, collisional transition rates exceed radiative transition rates for densities above a **critical density**, N_{crit} . Collisional transition rates (per unit time) are parameterised by the coefficients C_{nm} , analogous to the Einstein coefficients. The collisional de-excitation probability is

$$C_{mn} = N_e \langle \sigma_{mn} v \rangle \quad (9)$$

and under LTE conditions, the collisional excitation probability is related via

$$C_{nm} = \frac{N_m}{N_n} C_{mn} = \frac{g_m}{g_n} \exp\left(-\frac{h\nu}{kT}\right) C_{mn} \quad (10)$$

Including the collisional rates in the statistical equilibrium equation (c.f. eqn. 2 in Lec. 4 for radiative rates only) yields

$$N_n(B_{nm}\bar{J} + C_{nm}) = N_m(A_{mn} + B_{mn}\bar{J} + C_{mn}) \quad (11)$$

The critical density is defined as the density of free electrons at which the collisional and radiative rates are equal, i.e. $A_{mn}(1 + B_{mn}\bar{J}/A_{mn}) = C_{mn}$. Using the Einstein-Milne relation $A_{mn} = (2h\nu^3/c^2)B_{mn}$ gives

$$N_{\text{crit}} = \frac{A_{mn}}{\langle\sigma_{mn}v\rangle} \left(1 + \frac{c^2}{2h\nu^3}\bar{J}\right) \quad (12)$$

Collisional rates can become extremely important when transitions cannot proceed radiatively, i.e. when they are forbidden or semi-forbidden. In many such cases the collisional rates are comparable to the allowed radiative transition rates, thus allowing the forbidden transitions to proceed collisionally. This is the case for the HI 21 cm (forbidden) line transition, which has a radiative rate $A \sim 10^{-7} \text{ yr}^{-1}$ (!) but a collisional rate $C \sim 400 \text{ yr}^{-1}$ at densities that are typical of the interstellar medium ($N \sim 10^6 \text{ m}^{-3}$).