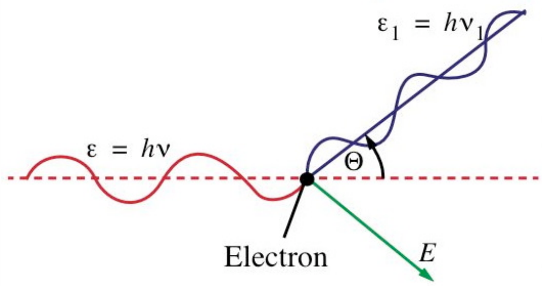
Lecture 10: Compton Scattering

Compton scattering is the scattering of photons off electrons. For low photon energies, it reduces to the classical case of Thomson scattering. For relativistic electrons, lower energy photons can be efficiently upscattered to energies reaching X-ray and γ-ray wavelengths. The photon upscattering process is referred to as *Comptonisation*. When referring to cooling of the electrons, the radiation process is called *inverse Compton scattering*. The emission (scattered) spectrum can be calculated analytically for single scatterings only. For multiple scatterings, numerical simulations are usually necessary.

10.1 Energy Transfer and Scattering Cross Section

In the classical (Thomson) limit, an electron will oscillate and emit radiation in response to incident electromagnetic waves. Quantum effects, however, modify the kinematics and interaction cross-section. Because a photon possesses momentum as well as energy, the recoil of the electron must be taken into account so the scattering cannot be elastic. It is easiest to treat the momentum transfer from a particle approach. Consider a photon of energy $\varepsilon = h\nu$ incident upon an electron initially at rest. The photon scatters through an angle Θ w.r.t. its initial propagation direction $\hat{\mathbf{k}}_i$. The energy of the photon and electron after the scattering event are $\varepsilon_1 = h\nu_1$ and E, respectively (see figure).



Initial and final 4-momenta of photons:

$$P_{\gamma i} = \frac{\varepsilon}{c}(1, \hat{\mathbf{k}}_i) \quad , \quad P_{\gamma f} = \frac{\varepsilon_1}{c}(1, \hat{\mathbf{k}}_f)$$

Initial and final 4-momenta of electron:

$$P_{ei} = (m_e c, \mathbf{0}) \quad , \quad P_{ef} = (E/c, \mathbf{p})$$

Conservation of 4-momentum requires

$$P_{ei} + P_{\gamma i} = P_{ef} + P_{\gamma f} \tag{1}$$

Rearranging and squaring gives

$$\begin{aligned} |P_{ef}|^2 &= |P_{ei} + P_{\gamma i} - P_{\gamma f}|^2 \\ &= |P_{\gamma i}|^2 + |P_{ei}|^2 + |P_{\gamma f}|^2 + 2P_{\gamma i}P_{ei} - 2P_{\gamma i}P_{\gamma f} - 2P_{\gamma f}P_{ei} \end{aligned} \tag{2}$$

Note that the modulus of a 4-vector A^μ is defined as $A^2 = A^\mu A_\mu = -(A^0)^2 + (A^1)^2 + (A^2)^2 + (A^3)^2$. This implies that the magnitudes of the 4-momenta for a photon and an electron are $P_\gamma^2 = 0$ and $P_e^2 = -m_\mathrm{e}^2 c^2$, respectively. So in the above expression, we have $|P_{ef}|^2 = -m_\mathrm{e}^2 c^2 = |P_{ei}|^2$ and $|P_{\gamma i}|^2 = 0 = |P_{\gamma f}|^2$, which leaves behind only those terms with a factor of 2 in front. These terms are: $P_{\gamma i}P_{ei} = -\varepsilon m_\mathrm{e}$, $P_{\gamma i}P_{\gamma f} = -\frac{\varepsilon\varepsilon_1}{c^2} + \frac{\varepsilon\varepsilon_1}{c^2}\hat{\mathbf{k}}_i \cdot \hat{\mathbf{k}}_f$, and $P_{\gamma f}P_{ei} = -\varepsilon_1 m_\mathrm{e}$. Substituting these in, rearranging and using $\hat{\mathbf{k}}_i \cdot \hat{\mathbf{k}}_f = \cos\Theta$ gives the following expression for the energy of the scattered photon:

$$\varepsilon_1 = \frac{\varepsilon}{1 + \frac{\varepsilon}{m_\mathrm{e}c^2}(1 - \cos\Theta)} \tag{3}$$

In terms of wavelength, $\lambda = hc/\varepsilon$, we have a change

$$\Delta\lambda = \lambda_1 - \lambda = \lambda_\mathrm{C}(1 - \cos\Theta) \tag{4}$$

where

$$\lambda_\mathrm{C} \equiv \frac{h}{m_\mathrm{e}c} \simeq 0.0243\,\text{Å} \qquad \text{Compton wavelength} \tag{5}$$

Thus, the wavelength change is of order λ_C. For long wavelengths ($\lambda \gg \lambda_\mathrm{C}$) or equivalently, $\varepsilon \ll m_\mathrm{e}c^2$, the scattering is approximately elastic (i.e. $\varepsilon_1 \simeq \varepsilon$). This is the Thomson regime.

The Klein-Nishina cross section

In addition to the effects of photon momentum, quantum corrections also modify the cross section for Compton scattering. The exact expression for the differential cross section for Compton scattering is derived from quantum electrodynamics and is known as the *Klein-Nishina* formula:

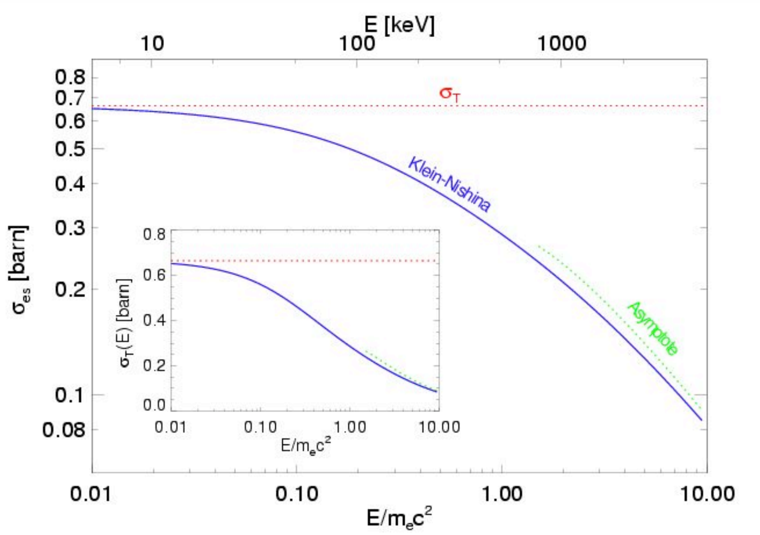
$$\frac{d\sigma_{\rm KN}}{d\Omega} = \frac{1}{2} r_0^2 \frac{\varepsilon_1^2}{\varepsilon^2} \left(\frac{\varepsilon}{\varepsilon_1} + \frac{\varepsilon_1}{\varepsilon} - \sin^2\Theta \right) \tag{6}$$

where $r_0 = e^2/(4\pi\epsilon_0 m_{\rm e} c) = 2.82 \times 10^{-15}$ m is the classical electron radius (defined in Lec. 8). This reduces to the classical differential Thomson cross section in the limit $\varepsilon_1 \sim \varepsilon$, *viz.* $d\sigma_{\rm T}/d\Omega = \frac{1}{2} r_0^2 (1 + \cos^2\Theta)$. The *total* cross section is obtained by integrating over solid angle, $\sigma_{\rm KN} = 2\pi \int_{-1}^{+1} (d\sigma_{\rm KN}/d\Omega)\, d\cos\Theta$:

$$\sigma_{\rm KN} = \sigma_{\rm T} \frac{3}{4} \left\{ \frac{1+x}{x^3} \left[\frac{2x(1+x)}{1+2x} - \ln(1+2x) \right] + \frac{1}{2x} \ln(1+2x) - \frac{1+3x}{(1+2x)^3} \right\} \tag{7}$$

where $x \equiv h\nu/m_{\rm e}c^2$.

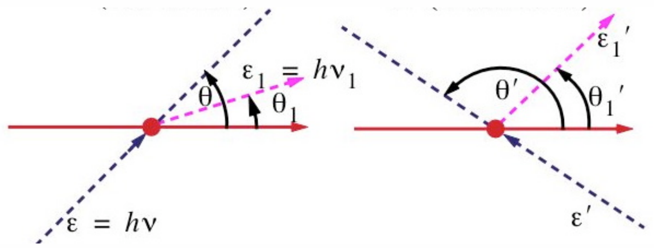
The overall effect of $\sigma_{\rm KN}$ is to reduce the scattering cross section relative to $\sigma_{\rm T}$ at high photon energies. Thus, *Compton scattering becomes less efficient at high energies*. The decline is shown in the plot below.



For $x \gg 1$, the asymptotic solution is $\sigma_{\rm KN} \sim \frac{3}{8} \sigma_{\rm T} x^{-1} \left(\ln 2x + \frac{1}{2} \right)$

Scattering from electrons in motion

In general, electrons will not be at rest, but will be moving, sometimes with relativistic velocities. Whenever a moving electron has energy greater than that of an incident photon, the energy transfer is from electron to photon. This is *inverse Compton scattering*. The results for scattering by a stationary electron are extended to a moving electron using a Lorentz transformation. Let K be the observer's frame and K' be the rest frame of an electron. The relative velocity βc defines the Lorentz factor $\gamma = (1-\beta^2)^{-1/2}$. A scattering event as seen in each frame is shown in the figure below. In K, the electron's velocity is in the x direction and all angles in both frames are measured from this axis.



observer's frame K electron rest frame K'

In K', all the previous formulas for scattering from stationary electrons are valid. Transforming the photon's initial energy into K':

$$\varepsilon' = \varepsilon\gamma(1-\beta\cos\Theta) \tag{8}$$

and transforming the scattered photon energy back into K:

$$\varepsilon_1 = \varepsilon_1'\gamma(1+\beta\cos\Theta_1') \tag{9}$$

Thus, in transforming to the electron rest frame, the photon picks up a factor γ and in transforming back to the lab frame, it picks up an additional factor γ. Hence, the photon energy can increase by a factor γ^2 in the lab frame, implying that Compton scattering by relativistic electrons can be quite efficient.

A maximum gain of $\sim\gamma^2$ is only possible for scatterings that are in the Thomson regime in the rest frame (i.e. $\varepsilon_1' \simeq \varepsilon'$) and which have $\Theta, \Theta_1' \gtrsim \pi/2$. The condition for Thomson scattering in the rest frame is

$$\varepsilon' \ll m_e c^2 \Longrightarrow \gamma h\nu \ll m_e c^2 \tag{10}$$

10.2 Single Scattering Power

We want to obtain an expression for the average inverse Compton power due to an isotropic distribution of photons scattering off electrons. As before, the procedure is to derive all quantities in the electron rest frame, calculate the scattering in the Thomson limit in the rest frame (i.e. let $\varepsilon_1' \simeq \varepsilon'$) and then transform everything back into the lab (observer) frame. Let $n(\varepsilon)d\varepsilon$ be the number density of photons having energy in the range $\varepsilon + d\varepsilon$. The total power emitted (i.e. scattered) in the electron's rest frame is given by

$$\frac{dE_1'}{dt'} = c\sigma_{\rm T} \int \varepsilon_1' n'(\varepsilon')\, d\varepsilon' \tag{11}$$

Now we know that the emitted power is an invariant. Another invariant is the quantity $n(\varepsilon)d\varepsilon/\varepsilon$. So

$$\frac{dE_1}{dt} = c\sigma_{\rm T} \int \varepsilon' n' d\varepsilon' = c\sigma_{\rm T} \int \varepsilon'^2 \frac{n d\varepsilon}{\varepsilon} \tag{12}$$

Now we substitute $\varepsilon' = \varepsilon\gamma(1 - \beta\cos\Theta)$ from eqn. (8) to get

$$\frac{dE_1}{dt} = c\sigma_{\rm T}\gamma^2 \int (1 - \beta\cos\Theta)^2 \varepsilon n\, d\varepsilon \tag{13}$$

which now only contains quantities in frame K. For an isotropic distribution of photons, we have

$$\langle (1 - \beta\cos\Theta)^2 \rangle = 1 + \frac{1}{3}\beta^2$$

giving

$$\frac{dE_1}{dt} = c\sigma_{\rm T}\gamma^2 \left(1 + \frac{1}{3}\beta^2\right) U_\gamma \tag{14}$$

where $U_\gamma = \int \varepsilon n d\varepsilon$ is the initial photon energy density. Now dE_1/dt is the rate at which the electron loses energy. The nett power converted into increased radiation is this minus the rate at which the initial photon energy distribution decreases, $d\varepsilon/dt = \sigma_{\rm T} c U_\gamma$. So

$$\frac{dE_{\rm rad}}{dt} = \frac{dE_1}{dt} - \frac{d\varepsilon}{dt} = c\sigma_{\rm T} U_{\rm ph} \left[\gamma^2 \left(1 + \frac{1}{3}\beta^2\right) - 1\right] \tag{15}$$

which gives the following for the inverse Compton power for a single electron:

$$\boxed{P_{\rm ic} = \frac{4}{3}\sigma_{\rm T} c\gamma^2\beta^2 U_\gamma \qquad \text{inverse Compton power}} \tag{16}$$

This has used $\gamma^2 - 1 = \gamma^2\beta^2$.

Emitted power for a distribution of electrons

For a nonthermal power law distribution of electrons $N(\gamma) = K_e \gamma^{-p}$, we can obtain the total power per unit volume from

$$P_{ic,tot} = \int_{\gamma_1}^{\gamma_2} P_{ic} N(\gamma) d\gamma$$

This gives, for $\beta \simeq 1$,

$$\boxed{P_{ic,tot} = \frac{4}{3}\sigma_T c U_\gamma K_e (3-p)^{-1}(\gamma_2^{3-p} - \gamma_1^{3-p}) \qquad \text{nonthermal power-law electrons}} \quad (17)$$

For a thermal distribution of electrons, $\gamma = 1$ and $\langle \beta^2 \rangle = 3kT_e/m_e c^2$. The total power needs to be derived from the single electron power in the more general case where energy transfer in the electron rest frame is not neglected. The result is

$$\boxed{P_{ic,tot} = \sigma_T c U_\gamma N_e \frac{4kT_e}{m_e c^2} \qquad \text{thermal electrons}} \quad (18)$$

where N_e is the total electron number density.

10.3 Single Scattering Spectra

The spectrum resulting from single scattering events between a distribution of photons and a distribution of relativistic electrons depends on both the specified distributions. The spectrum can be calculated for a scattering event with a single photon energy and single electron energy and the nett spectrum is obtained by averaging over the electron and incident photon distributions. The derivation for the spectrum due to inverse Compton scattering is different from the derivations for true emission processes. The treatment deals with intensity based on photon number and the full details are omitted. The relevant expression is that for the total scattered power per unit volume per energy due to a nonthermal power law distribution of electrons (i.e. volume emissivity per unit energy rather than frequency):

$$j_{\varepsilon_1} = \frac{3}{16\pi}\sigma_T c \varepsilon_1 K_e \int d\varepsilon \, \frac{n(\varepsilon)}{\varepsilon} \int_{\gamma_1}^{\gamma_2} d\gamma \, \gamma^{-(p+2)} f\left(\frac{\varepsilon_1}{4\gamma^2 \varepsilon}\right) \quad (19)$$

where the function f is defined by $f(x) = 2x \ln x + x + 1 - 2x^2$.

For sufficiently large limits on the γ integral, we have

$$\boxed{j_{\varepsilon_1}^{\rm ic} = \frac{3}{\pi}2^{p-2}\frac{p^2+4p+11}{(p+1)(p+3)^2(p+5)}\sigma_{\rm T}c\varepsilon_1^{-(p-1)/2}K_{\rm e}\int \varepsilon^{(p-1)/2}n(\varepsilon)\,d\varepsilon} \tag{20}$$

Thus, inverse Compton scattering also predicts a power law spectrum with a spectral index

$$\alpha = \frac{1}{2}(p-1) \tag{21}$$

identical to the case of synchrotron emission. The power law spectrum is independent of the incident photon distribution.

Scattering of blackbody photons

The above derivation implies that if the incident photon distribution is a blackbody spectrum, the resulting spectrum after a single scattering by nonthermal electrons should be a power law. For a blackbody, we have

$$n(\varepsilon) = \frac{8\pi}{(hc)^3}\varepsilon^2\left[\exp\left(\frac{\varepsilon}{kT}\right)-1\right]^{-1} \tag{22}$$

Inserting this into the expression j_{ε_1} above, and solving the integrals gives

$$j_{\varepsilon_1}^{\rm ic,bb} = \frac{\sigma_{\rm T}}{h^3c^2}f_{\rm bb}(p)(kT)^{(p+5)/2}K_{\rm e}\varepsilon_1^{-(p-1)/2} \tag{23}$$

where

$$f_{\rm bb}(p) = 3\frac{2^{p+1}(p^2+4p+11)}{(p+1)(p+3)^2(p+5)}\gamma\left(\frac{p+5}{2}\right)\zeta\left(\frac{p+5}{2}\right)$$

where ζ is the Riemann zeta function.

Synchrotron Self-Comptonisation

A particularly interesting case of inverse Compton scattering is that in which the seed photons are synchrotron photons emitted by the scattering electrons. In this case, the incident photon spectrum is the synchrotron power law spectrum, which can be written as

$$n(\varepsilon) = \frac{U_\gamma(\varepsilon_0)}{\varepsilon_0}\left(\frac{\varepsilon}{\varepsilon_0}\right)^{-(p-1)/2} \quad , \qquad \varepsilon_{\rm min} \lesssim \varepsilon_{\rm max} \tag{24}$$

where ε_0 is some fiducial seed photon energy. The solution for the synchrotron self-Compton volume emissivity is

$$\boxed{j^{\rm ssc}_{\nu_1} = f(p)\sigma_{\rm T} c K_{\rm e} U_{\gamma\nu_0} \ln\left(\frac{\varepsilon_{\rm max}}{\varepsilon_{\rm min}}\right)\left(\frac{\nu_1}{\nu_0}\right)^{-(p-1)/2}} \tag{25}$$

where the the relation $j_{\nu_1} = h j_{\varepsilon_1}$ has been used and where

$$f(p) = \frac{3}{\pi} 2^{p-2} \frac{p^2+4p+11}{(p+1)(p+3)^2(p+5)}$$

The term $\ln(\varepsilon_{\rm max}/\varepsilon_{\rm min})$ is known as the *Compton logarithm*.

10.4 Multiple Scatterings: the Compton y Parameter

The spectrum resulting from repeated scatterings is usually calculated numerically using Monte Carlo techniques. Qualitatively, however, we can expect that the more scatterings that occur, the more the seed photon distribution becomes distorted. A useful parameter that measures the importance of scattering in a medium is the Compton y parameter:

$$\boxed{y \equiv \text{fractional energy change} \times \text{mean no. of scatterings}} \tag{26}$$

The mean number of scatterings is determined by the optical depth, $\tau = \sigma N_{\rm e} r$, where r is the size of the scattering region. A value of $\tau \sim 1$ means that on average, a photon will scatter once before escaping the region. Specifically, we have

$$\text{mean no. of scatterings} \simeq \max(\tau, \tau^2) \tag{27}$$

The scattering regimes are defined in terms of the y parameter as follows:

$$\begin{array}{ll} y \ll 1 & \text{negligible spectral changes} \\ y \lesssim 1 & \text{power law spectrum, with exponential cut-off} \\ y \gg 1 & \text{saturated Comptonisation} \end{array} \tag{28}$$

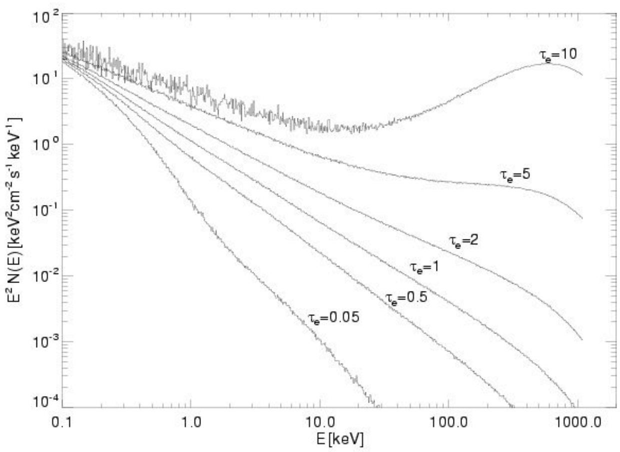
For $y \lesssim 1$, it is possible to obtain a power law scattered spectrum, *even if the scattering electrons have a thermal distribution*. For a thermal electron distribution, the Compton y parameter is defined as

$$\boxed{y = \frac{4kT_\mathrm{e}}{m_\mathrm{e}c^2}\left(1 + \frac{4kT_\mathrm{e}}{m_\mathrm{e}c^2}\right)\mathrm{max}(\tau_\mathrm{T}, \tau_\mathrm{T}^2)} \tag{29}$$

y parameter for thermal Comptonisation (30)

where $\tau_\mathrm{T} = \sigma_\mathrm{T} N_\mathrm{e} r$ is the Thomson optical depth of the scattering region of size r. In the saturated Comptonisation limit ($y \gg 1$), the incident photon spectrum is completely distorted beyond recognition. The resulting spectrum approaches a similar distribution to the scattering electrons, implying that the photons and electrons come into thermal equilibrium. An incident nonthermal (i.e. power law) photon spectrum, for example, will become thermalised by the scatterings and approach a blackbody spectrum at the temperature of the scattering electrons, so the spectrum will peak at $h\nu_1 \simeq 2.8kT_\mathrm{e}$.

Some example spectra of multiple Compton scatterings calculated from Monte Carlo simulations (see e.g. Sunyaev & Titarchuk, 1980, *Astron. Astrophys.*, 86, 121.):



Emergent spectra from a spherical region with varying optical depths containing electrons with $kT_\mathrm{e} = 0.7m_\mathrm{e}c^2$. The incident seed photons are injected at the centre with a blackbody spectrum at $kT \ll kT_\mathrm{e}$. The Compton y parameter thus ranges from $y \simeq 0.5$ for the $\tau_\mathrm{T} = 0.05$ spectrum, to $y \simeq 10^3$ for the $\tau_\mathrm{T} = 0.05$ spectrum.