

Lecture 16: Accretion Disks I – Structure

Because its binding energy increases towards small r , accreting gas gets hotter as this energy is converted into internal energy. To determine how hot the gas gets, we need to look at the structure of the accretion disk. This is done by assuming the inflow proceeds in a fluid-like fashion so we can use the gasdynamic conservation equations. The strategy is as follows:

1. write down continuity, momentum and energy eqns
2. adopt cylindrical coord system (r, ϕ, z) , assuming steady-state $(\partial/\partial t = 0)$ and axisymmetry $(\partial/\partial \phi = 0)$
3. neglect any z dependence in fluid flow, i.e. $v = v(r)$ and assume symmetry across midplane $z = 0$, so that $\rho(r, +z) = \rho(r, -z)$
4. integrate eqns vertically over disk height $z = h$, assuming a geometrically thin disk ($|z| \leq h \ll r$ everywhere)

The radiative flux term is in the energy equation, so solving for that and then equating to a blackbody gives an estimate of the disk temperature at each radius.

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16.1 Fluid equations

(i) continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad (1)$$

(ii) momentum equation (Euler equation):

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} = -\rho \nabla \phi_G - \nabla p + \mathbf{f}^{\text{visc}} \quad (2)$$

where $\phi_G = -GM(r^2 + z^2)^{-1/2}$ is the gravitational potential and \mathbf{f}^{visc} is the viscous force per unit volume responsible for momentum transport along velocity gradients by random (thermal and turbulent) motions as well as by organised bulk motions.

- *Bulk viscosity* depends on velocity gradients along the direction of bulk fluid flow.
- *Kinematic viscosity* depends on *shearing motions* – velocity gradients orthogonal to the direction of fluid flow.

Accretion disks are *differentially rotating flows* (i.e. radially varying azimuthal velocity), so we only consider kinematic viscosity:

$$\mathbf{f}^{\text{visc}} = \nabla \cdot (2\rho\nu\underline{\underline{s}}) \quad (3)$$

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where

$$\nu = \text{coefficient of kinematic viscosity} \quad , \quad \underline{\underline{s}} = \text{fluid shear tensor}$$

Using the continuity equation, we can also re-write the momentum equation as:

$$\frac{\partial(\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v} \mathbf{v}) = -\rho \nabla \phi_G - \nabla p + \nabla \cdot (2\rho \nu \underline{\underline{s}}) \quad (4)$$

(iii) energy equation:

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \rho v^2 + \rho \phi_G + \rho \varepsilon_{\text{int}} \right) + \nabla \cdot \left[\left(\frac{1}{2} \rho v^2 + \rho \phi_G + \rho w \right) \mathbf{v} + \mathbf{F}^{\text{rad}} - 2\rho \nu \underline{\underline{s}} \cdot \mathbf{v} \right] = 0 \quad (5)$$

where \mathbf{F}^{rad} = radiative energy flux and

$$\left. \begin{aligned} \varepsilon_{\text{int}} &= \text{internal thermal energy} = \frac{1}{\gamma-1} \frac{p}{\rho} \\ w &= \varepsilon_{\text{int}} + p/\rho = \text{specific enthalpy} = \frac{\gamma}{\gamma-1} \frac{p}{\rho} \end{aligned} \right\} \text{adiabatic gas}$$

For an ideal gas with adiabatic index $\gamma = \frac{5}{3}$, the adiabatic equation of state implies

$$w = \frac{5}{2} \frac{p}{\rho} = \frac{5}{2} \frac{kT}{\mu m_p}, \text{ where } \mu \text{ is molecular mass.}$$

16.2 Disk equations

We now adopt a cylindrical polar coordinate system to the coordinate-free gasdynamic equations above and we apply the simplifying assumptions appropriate for modelling accretion disks: time stationarity, axisymmetry, reflection symmetry, $\mathbf{v} = \mathbf{v}(r) = (v_r, v_\phi, 0)$, with $|v_r| \ll v_\phi$, and $|z| \leq h \ll r$, where $h = h(r)$ is the disk scaleheight. We also need to determine a functional form for the shear viscosity. The shear tensor $\underline{\underline{s}}$ depends on velocity gradients orthogonal to the fluid flow and we assume \mathbf{v} has no azimuthal or vertical gradients. In accretion disks, the fluid flow is dominated by the rotation velocity $v_\phi = r\Omega$, which varies with r and so the shear viscosity will appear in the azimuthal component of the momentum equation. We will use a specific model for the shear viscosity when we consider angular momentum conservation. Terms involving divergence etc are calculated according to the rules of differential calculus in curvilinear coordinates (see e.g. Landau & Lifshitz, *Fluid Mechanics*).

1. Mass transfer.

The continuity equation (1) gives

$$\frac{1}{r} \frac{\partial}{\partial r} (r \rho v_r) = 0 \quad (6)$$

and integrating vertically gives

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r v_r \int_{-h}^{+h} \rho \, dz \right) = \frac{1}{r} \frac{\partial}{\partial r} (r v_r \Sigma) = 0 \quad (7)$$

where

$$\Sigma \equiv \int_{-h}^{+h} \rho \, dz = 2 \int_0^h \rho \, dz \quad \text{is the surface mass density} \quad (8)$$

Multiplying (7) by an annulus area $2\pi r dr$ gives the net mass flux across the annulus.

So we can define

$$\boxed{\dot{M}_a \equiv 2\pi r \Sigma (-v_r)} \quad \text{mass accretion rate} \quad (9)$$

which takes into account that an *inflow* has $v_r = -|v_r|$. Thus, the continuity equation implies

$$\frac{\partial \dot{M}_a}{\partial r} = 0 \implies \dot{M}_a = \text{constant with radius} \quad (10)$$

2. Radial momentum transport.

The radial component of the momentum equation (4) gives

$$\frac{1}{r} \left[\frac{\partial}{\partial r} (r \rho v_r^2) - \rho v_\phi^2 \right] = -\rho \frac{\partial \phi_G}{\partial r} - \frac{\partial p}{\partial r} \quad (11)$$

The dominant terms are the $\rho v_\phi^2/r$ and gravitational terms, so ignoring the other radial gradient terms yields

$$\frac{\rho v_\phi^2}{r} \simeq \frac{GM\rho}{r^2} \implies \boxed{v_\phi^2 \simeq \frac{GM}{r} \equiv v_K^2} \quad \text{Keplerian rotation} \quad (12)$$

So radial momentum is conserved when Keplerian rotation prevails, i.e. Keplerian rotational motion provides support against gravity. Note that this is differential rotation: each infinitesimal disk annulus rotates with a different $v_\phi(r)$, which increases towards small r . This means there is a velocity shear between each annulus in the fluid flow. The Keplerian angular velocity is defined as $\Omega_K = v_K/r = (GM/r^3)^{1/2}$.

3. Angular momentum transport.

The azimuthal component of the momentum equation (4) gives

$$\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \rho v_r v_\phi) = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 2\rho \nu s_{r\phi}) \tag{13}$$

where $s_{r\phi} = \frac{1}{2} r \frac{\partial \Omega}{\partial r}$ is the $r\phi$ component of the shear tensor \underline{s} . Integrating this equation vertically and using (9) gives

$$\frac{d}{dr} \left[\dot{M}_a v_\phi r + 2\pi r \Sigma \nu r^2 \Omega' \right] = 0 \tag{14}$$

where $\Omega' = \partial \Omega / \partial r = -\frac{3}{2} \frac{\Omega}{r}$ and where it has been assumed that the coefficient of kinematic viscosity ν does not vary with height. The first term describes the rate of change of angular momentum associated with the mass flux through an annulus of thickness dr . We know that \dot{M}_a is constant w.r.t. r and that $v_\phi \simeq v_K \sim r^{-1/2}$, so $d(v_\phi r)/dr > 0$ and angular momentum increases with r for a Keplerian flow. The second term describes the viscous torque necessary to transport angular momentum from small to large r , thus allowing accretion to proceed.

HOW A DISK SPIRALS AND RADIATES

Material in the inner parts of an accretion disk takes less time to complete an orbit than does material in the outer parts (right). Bits of material that are closer to the center of the disk slide past material that is slightly farther out. In an accretion disk around a star or black hole, large-scale blobs of gas collide violently in a turbulent flow (bottom). This process transports angular momentum outward, causing some of the gas to lose rotational support and spiral inward (far right). And because the collisions make the material very hot, the disk radiates large amounts of visible, ultraviolet and x-ray radiation.

Rotational velocity

Angular momentum Mass transport

The result of all these collisions is that angular momentum is transferred to the outer reaches of the disk while the gas whirls inward to the central star or black hole.

Inner blob
Outer blob

Radiation

New path of inner blob
New path of outer blob

Two blobs of gas in slightly different orbits collide with each other because the inner blob is moving a bit faster than the outer one.

The collision transfers energy and angular momentum from the inner to the outer blob. The heated gas generates radiation.

Deprived of energy, the inner blob falls to a closer orbit and gains speed. The outer blob is flung to a farther orbit, slowing it down.

(From “A Universe of Disks”, O. Blaes, 2004, Scientific American, Vol. 291, No. 4).

Integrating radially and assuming the viscous torque vanishes at the inner boundary r_i :

$$\dot{M}_a [v_\phi(r)r - v_\phi(r_i)r_i] = 3\pi r^2 \Sigma \nu \Omega(r) \implies \dot{M}_a = 3\pi \Sigma \nu \left[1 - \left(\frac{r}{r_i} \right)^{-1/2} \right]^{-1} \quad (15)$$

This prescribes the magnitude of the viscosity needed to account for a given mass accretion rate. On dimensional grounds, we expect $\nu \sim v\lambda$, where v is a characteristic speed and λ is a characteristic mean-free-path between the particle collisions that mediate viscous transport. In standard accretion disk theory (Shakura & Sunyaev, 1973), the following prescription is used:

$$\nu = \alpha c_s h \quad \text{alpha prescription} \quad (16)$$

where $c_s = (kT/\mu m_p)^{1/2}$ is the sound speed and $\alpha \lesssim 1$ is a constant that absorbs all the unknown details of the flow. This α prescription is motivated by turbulent motions, which typically have subsonic turnover velocities and maximum eddy sizes limited by the disk thickness h .

There are 2 serious problems with standard α -disks:

1. there is no clear understanding of the nature of the viscosity; all the unknown physics has been shifted from ν to α
2. when typical physical parameters are inserted into the right hand side of (15), the implied mass accretion rates severely underpredict the values inferred from the observed radiative power of accreting sources

accretion cannot be attributed to ordinary kinematic viscosity

Numerical simulations have demonstrated and confirmed the longstanding belief that *magnetic turbulence* can transport angular momentum efficiently enough to account for the values of \dot{M}_a inferred from observations of accreting sources. This is because unlike kinematic viscosity, magnetic field lines can transport stresses between two spatially distinct regions (i.e. between fluid elements that are connected by field lines but are not adjacent to each other). This greatly enhances the overall rate of momentum (and energy) transport.

4. Energy transport.

In the energy equation (5), we take into account that the radiative flux \mathbf{F}^{rad} has both radial and vertical components. Indeed, it is the z -component that describes the potentially observable radiation emitted from the disk surface. So energy conservation implies

$$\frac{1}{r} \frac{\partial}{\partial r} \left[\left(\frac{1}{2} \rho v^2 + \rho \phi_G \right) r v_r + r F_r^{\text{rad}} - 2 \rho \nu r s_{r\phi} v_\phi \right] + \frac{\partial F_z^{\text{rad}}}{\partial z} = 0 \quad (17)$$

where the enthalpy term has been neglected because it is $\sim kT/\mu m_p$ which is much smaller than the bulk velocity. Now we integrate vertically and note that the term involving F_r^{rad} is $\sim h/r \ll 1$ times smaller than the F_z^{rad} term. So we have

$$\frac{1}{r} \frac{d}{dr} \left[\left(\frac{1}{2} v^2 + \phi_G \right) \Sigma r v_r - \Sigma \nu r^2 \Omega' v_\phi \right] + 2 F^{\text{rad}} = 0 \quad (18)$$

where F^{rad} now denotes the disk surface flux. We now use the relations

$\dot{M}_a = 2\pi r \Sigma (-v_r) = \text{constant}$ from the continuity equation (9),

$v_\phi \simeq v_K = r \Omega_K = (GM/r)^{1/2}$ from radial momentum conservation (12), and

$\dot{M}_a v_K r [1 - (r/r_i)^{-1/2}] = 3\pi r^2 \Sigma \nu \Omega_K$ from angular momentum conservation (15):

$$\begin{aligned} F^{\text{rad}} &= -\frac{1}{2r} \frac{d}{dr} \left[\left(\frac{1}{2} v_K^2 - \frac{GM}{r} \right) \Sigma r v_r + \frac{3}{2} \Sigma \nu r \Omega_K v_K \right] \\ &= -\frac{1}{4\pi r} \frac{d}{dr} \left\{ \frac{1}{2} \frac{GM \dot{M}_a}{r} + \frac{GM \dot{M}_a}{r} \left[1 - \left(\frac{r}{r_i} \right)^{-1/2} \right] \right\} \\ &\simeq -\frac{1}{4\pi r} \frac{d}{dr} \left[\frac{3}{2} \frac{GM \dot{M}_a}{r} - \frac{GM \dot{M}_a}{r_i} \left(\frac{r}{r_i} \right)^{-3/2} \right] \end{aligned} \quad (19)$$

Performing the differentiation and rearranging gives

$$\boxed{F^{\text{rad}}(r) \simeq \frac{3GM \dot{M}_a}{8\pi r^3} \left[1 - \left(\frac{r}{r_i} \right)^{-1/2} \right]} \quad \text{radiative disk flux} \quad (20)$$

This is the radial profile of the radiative flux predicted by standard accretion disk theory.