Lecture 4: Acceleration Processes III

4.1 The Spectrum Due to 1st-Order Fermi Acceleration

In diffusive shock acceleration, all particles incident upon the shock must gain energy by varying amounts that depend on the number of times they cross the shock. It can be argued, qualitatively, that the steady-state distribution functions of the accelerated particles must necessarily be a featureless power law because there is no obvious momentum scale in the process. Only in the nonlinear regime, when the particle momenta are sufficiently high to cause a back reaction on the background plasma, does the particle rest mass become a crucial scale in the process. This limit is not considered here.

To calculate an explicit power law spectral index, we refer to the diffusion equation for particles (which is a Fokker-Planck equation in the limit of small differential changes in momenta, as is relevant here). Unlike the calculation for stochastic Fermi acceleration, however, we must include the effects of spatial diffusion, which is crucial to the diffusive shock acceleration mechanism.

Consider a plane shock located at a spatial coordinate $x = 0$. In the shock restframe, the fluid moves from upstream to downstream and we define this as the direction of increasing x. We want to calculate the steady state particle momentum distribution $f(p)$. In addition to the spatial diffusion resulting from scatterings, particles can also be convected (advected) with the fluid flow across the shockfront to the downstream flow. So terms involving spatial diffusion and convection (advection) need to be retained in the overall momentum diffusion equation, which can be written as:

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial x}\left(vf - D\frac{\partial f}{\partial x}\right) + \frac{1}{p^2}\frac{\partial}{\partial p}\left[p^2\left(-\frac{1}{3}p\frac{\partial v}{\partial x}f\right)\right] = 0 \quad \text{convection–diffusion equation} \tag{1}$$

Here, v is the fluid speed and D is the spatial diffusion coefficient. We can expand the last term and simplify, ignoring the term $\partial/\partial p(\partial v/\partial x)$ that describes the back reaction of the high-energy particles colliding with the scattering centres in the fluid. We get the following:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{\partial}{\partial x}\left(D\frac{\partial f}{\partial x}\right) = \frac{1}{3}p\frac{\partial v}{\partial x}\frac{\partial f}{\partial p} \tag{2}$$

The 2nd term on the LHS describes changes in the distribution function as particles convect along with the fluid in the downstream direction. The term involving D describes how f changes with location as a result of spatial diffusion associated with scattering. The term on the RHS describes changes in the particle distribution function resulting directly from changes in the particle momenta. This is the term associated with particle acceleration. To calculate a steady state solution for $f(p)$, we set $\partial f/\partial t = 0$. We then also set the acceleration term to zero and find a general solution that satisfies the balance between convection and diffusion. This general solution is then modified by considering the acceleration term and requiring that the final solution be continuous across the shock boundary.

First balancing convection and diffusion only, we have

$$v\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(D\frac{\partial f}{\partial x}\right) \tag{3}$$

and we also write

$$v = \begin{cases} v_- & , \ x < 0 \quad \text{(upstream)} \\ v_+ & , \ x > 0 \quad \text{(downstream)} \end{cases}$$

This differential equation has the following general solution in the upstream region, where convection opposes diffusion:

$$f(x,p) = f_-(p) + [f_0(p) - f_-(p)]\exp\left[\int_0^x \frac{v_- \mathrm{d}x}{D(x,p)}\right] \qquad , \ x < 0 \tag{4}$$

with $f_-(p) = f(-\infty, p)$ and $f_0(p) = f(0,p)$. It is not possible to balance diffusion against convection behind the shock and the only possible steady state solution has f spatially constant:

$$f(x,p) = f_0(p) \qquad , \ x > 0 \tag{5}$$

Our solutions satisfy equation (3) for the balance between convection and diffusion. But we really want a solution to

$$v\frac{\partial f}{\partial x} - \frac{\partial}{\partial x}\left(D\frac{\partial f}{\partial x}\right) = \frac{1}{3}p\frac{\partial v}{\partial x}\frac{\partial f}{\partial p} \tag{6}$$

where the RHS term is the acceleration term. Note that it involves $\partial v/\partial x$, whereas our solution above contains a constant v_- in the upstream region. So we may try to find a solution by letting v vary spatially. Performing the derivatives and substituting into (6), we find the requirement for a continuous momentum distribution across the shock:

$$\left[-D\frac{\partial f}{\partial x} - \frac{1}{3}vp\frac{\partial f}{\partial p}\right]_{0_-}^{0_+} = 0 \tag{7}$$

The two solutions (4) and (5) can now be joined at the shock using (7) to obtain a differential equation for the transmitted distribution function $f_+(p)$:

$$\frac{\mathrm{d}f}{\mathrm{d}\ln p} = \frac{3r}{r-1}(f_- - f_+)$$

which has a solution

$$f_+(p) = qp^{-q}\int_0^p \mathrm{d}p' f_-(p')\, p'^{(q-1)} \tag{8}$$

where

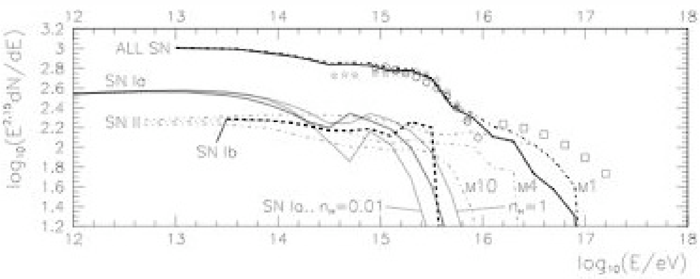
$$q = \frac{3r}{r-1} \qquad \text{and} \qquad r = \frac{v_1}{v_2}$$

This solution indicates that the downstream distribution $f_-(p)$ acts as an injection spectrum. If we insert a monoenergetic spectrum, i.e. $f_-(p) \propto \delta(p - p_0)$, into (8), we get a power law at $p > p_0$ of the form $f_+(p) \propto p^{-q}$.

The energy spectrum for an ideal, monatomic gas and a strong shock is

$$\boxed{n(\varepsilon) \propto p^2 f(p) \propto \varepsilon^{-2}} \tag{9}$$

Unlike stochastic Fermi acceleration, this result is not sensitive to assumptions about escape losses. Furthermore, different assumptions about the transport equation produce essentially the same spectrum, so the result is robust. One problem, however, is that diffusive shock acceleration is more efficient for ions than for electrons, yet it is the radiation emitted by accelerated electrons that we observe and the inferred electron energies are high. Some form of pre-acceleration of electrons is therefore required. Also, it is not entirely clear whether diffusive shock acceleration can account for the highest energy cosmic rays, which are believed to be accelerated at the shocks associated with supernova explosions.



Simulated spectra of protons accelerated at supernova shocks (solid curves), compared to the observed cosmic ray spectrum (symbols). (Hillas 2005, J. Phys. G: Nucl. Part. Phys., 31, R95)

4.2 MHD Shocks

Magnetic fields can change the nature of shocks and can change the behaviour of incoming particles and hence, their rate of acceleration via the Fermi mechanism. Consider again the restframe of a planar shock. Suppose the upstream region contains a magnetic field $\mathbf{B}_1$ that makes an angle ϕ_1 with the shock normal. In the downstream region, the field is $\mathbf{B}_2$ and makes an angle ϕ_2 with the shock normal. If we define the shock normal as being in the x-direction and the other coordinate as the z-direction, then the magnetic field will have components that are normal and tangential to the shock. The fluid velocity in the downstream region can also have a tangential component in this case. We now determine how the shock jump conditions are modified by the presence of the magnetic field.

The field has no effect on the continuity equation, but does introduce extra terms in the momentum and energy equations. The momentum equation becomes

$$\nabla \cdot (\rho \mathbf{v})\mathbf{v} = -\nabla p + \mathbf{J} \times \mathbf{B} \tag{10}$$

where $\mathbf{J}$ is the current density. In the MHD limit, the average changes in the fields occur sufficiently slowly over the time and length scales of interest that the magnetic field behaves as part of the fluid. In particular, Ampere's law in the MHD limit is $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ and using a

vector identity, we get

$$\mathbf{J} \times \mathbf{B} = \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B} = -\nabla \left(\frac{B^2}{2\mu_0}\right) + (\mathbf{B} \cdot \nabla)\frac{\mathbf{B}}{\mu_0}$$

Now we can separate the momentum equation into normal and tangential components:

$$\begin{aligned} \frac{\partial}{\partial x}\left[\frac{1}{2}\rho v_x^2 + p + \frac{B_z^2}{2\mu_0}\right] &= 0 \\ \frac{\partial}{\partial x}\left[\frac{1}{2}\rho v_x v_z - \frac{B_x B_z}{\mu_0}\right] &= 0 \end{aligned} \tag{11}$$

The energy equation now has a contribution from the Poynting flux:

$$\nabla \cdot \left[\left(\frac{1}{2}\rho v^2 + \frac{\gamma}{\gamma - 1}p\right)\mathbf{v} + \frac{\mathbf{E} \times \mathbf{B}}{\mu_0}\right] = 0 \tag{12}$$

We can use Ohm's law

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \qquad \text{Ohm's law}$$

for MHD in the infinitely conducting limit to get $\mathbf{E} = -\mathbf{v} \times \mathbf{B}$ so that

$$\frac{\mathbf{E} \times \mathbf{B}}{\mu_0} = \frac{1}{\mu_0}\left[B^2 \mathbf{v} - (\mathbf{v} \cdot \mathbf{B})\mathbf{B}\right]$$

The energy conservation relation then reads

$$\frac{\partial}{\partial x}\left[\left(\frac{1}{2}\rho v^2 + \frac{\gamma}{\gamma-1}p\right)v_x + \frac{B^2}{\mu_0}v_x - \frac{\mathbf{v}\cdot\mathbf{B}}{\mu_0}B_x\right] \quad (13)$$

In addition, we have Maxwell's relations, which require

$$\frac{\partial B_x}{\partial x} = 0 \quad , \quad \frac{\partial}{\partial x}\left(v_x B_z - B_x v_z\right) = 0$$

These boundary conditions now need to be combined. We define the following parameters:

$$v_{1\mathrm{A}} = \frac{B^2}{\rho_1 \mu_0} \qquad \text{Alfvèn velocity}$$
$$M_{\mathrm{A}} = \frac{v_1}{v_{1\mathrm{A}}} \qquad \text{Alfvèn Mach number} \quad (14)$$

The change in the magnetic field across the shock is then given by

$$B_{2x} = B_{1x} \quad , \quad B_{2z} = B_{1z}\left[\frac{r(M_{\mathrm{A}}^2 - \cos^2\phi_1)}{M_{\mathrm{A}}^2 - r\cos^2\phi_1}\right] \quad (15)$$

Combined with the other jump conditions derived in the unmagnetised case, these are referred to as the Rankine-Hugoniot jump conditions. A magnetised shock is referred to as *parallel* when $\phi_1 \simeq 0$ (i.e. $\mathbf{B}_1$ parallel to shock normal $\hat{\mathbf{x}}$, so $B_z = 0$) and *perpendicular* when $\phi_1 \simeq \pi/2$. Thus, there is no change in the magnetic field for a parallel shock, whereas for a perpendicular shock, the magnetic field increases across the shock.

The energy of a charged particle near an MHD shock is not modified by the magnetic field. However, the field controls the trajectory of the charged particle and can thus either enhance or diminish the Fermi acceleration process. Consider a perpendicular shock, for example, and suppose the particle energy is sufficiently large that its gyroradius is larger than the shock thickness (usually the case). As the particle gyrates around $\mathbf{B} \simeq B\hat{\mathbf{z}}$, it crosses the shock many times. The overall efficiency of Fermi acceleration is thus enhanced as a result of this higher probability of approaching the shockfront.

4.3 Direct Particle Acceleration by Electric Fields

The dynamics of charged particles are governed by electromagnetic fields, which pervade the entire Universe. The equation of motion of a particle of charge q, momentum $\mathbf{p} = \gamma m\mathbf{v}$ and Lorentz factor $\gamma = (1 - v^2/c^2)^{1/2}$ in a magnetic field $\mathbf{B}$ and electric field $\mathbf{E}$ is

$$\frac{d\mathbf{p}}{dt} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \tag{16}$$

In most (but not all) astrophysical situations, static E-fields cannot be sustained because ionised plasmas are very highly electrically conducting and the charged particles move freely to short out any component of $\mathbf{E}$ parallel to $\mathbf{B}$. Perpendicular to $\mathbf{B}$, particle motion is restricted by a B-field to a circular motion with *gyrofrequency*

$$\Omega = \frac{|q|B}{\gamma m} \tag{17}$$

The sense of gyration is right-handed for negative charges ($q/|q| = -1$).

The *gyroradius* is $R = v_\perp/\Omega = p_\perp/(|q|B)$, where $v_\perp = v \sin\alpha$ is the velocity component perpendicular to $\mathbf{B}$ and α is called the *pitch angle*. Combined with the parallel velocity component $v_{||} = v\cos\alpha$, the net motion of a charged particle in a magnetic field is a

spiralling motion.

From (16), the work done on a charged particle is $W = \mathbf{v} \cdot d\mathbf{p}/dt = q\mathbf{v} \cdot \mathbf{E}$. Thus, since magnetic fields do no work and static electric fields cannot be sustained, acceleration of charged particles to high energies can only be attributed to the time-varying E-field induced by a time-varying B-field, *viz.* Faraday's law: $\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t$

Although diffusive shock acceleration appears to be an efficient and natural particle acceleration mechanism, whether it can account for the highest energy cosmic rays is still an open question. Let's see what the requirements are for electric field acceleration. We can do the following order-of-magnitude calculation: $\frac{E}{L} \sim \frac{B}{L/c} \Rightarrow E \sim Bc$ where L is a characteristic length scale over which the field varies. The total energy that can be given to a particle is

$$\varepsilon = \gamma mc^2 \sim \int qE\mathrm{d}l \sim qBcL$$

We find $\varepsilon \sim 5\,\mathrm{J} \sim 10^{19}\,\mathrm{eV}$ for $B \sim 10^6\,\mathrm{T}$ and $L \sim 100\,\mathrm{km}$. These physical parameters are precisely what we find in neutron stars.