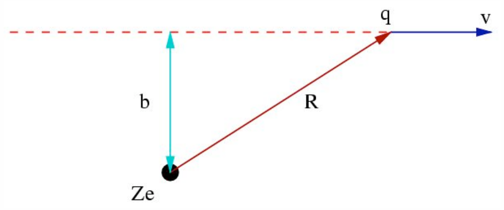
Lecture 6: Bremsstrahlung Radiation

When a high speed electron encounters the Coulomb field of another charge, it emits bremsstrahlung radiation, also known as free-free emission. The word *bremsstrahlung* means braking radiation because the electron rapidly decelerates when the other charge is a massive ion. The derivation can be done classically using the dipole approximation for nonrelativistic particles, with quantum corrections added as “Gaunt factors” to the classical formulas. The quantum corrections become important when photon energies become comparable to energies of the emitting particles. We only need to consider electron-ion bremsstrahlung because for collisions between like charges (e.g. electron-electron), the dipole approximation predicts zero radiation and a higher order calculation is required. This also means that less radiation is emitted for collisions between like particles. In electron-ion bremsstrahlung, the electrons are the primary emitters because their accleration is $\sim m_{\rm p}/m_{\rm e}$ times greater.

6.1 Emission from Single Speed Electrons

Consider an electron moving with velocity $\mathbf{v}$ past an ion of charge Ze with impact parameter b. We will assume *small-angle scattering* so there is negligible deviation in the electron’s trajectory from a straight line (see figure).



Let $\mathbf{R}$ be the position vector of the electron from the ion. Then the dipole moment is $\mathbf{d} = -e\mathbf{R}$ and its second time derivative is $\ddot{\mathbf{d}} = -e\dot{\mathbf{v}}$. We want an emission spectrum for the bremsstrahlung radiation using eqn. (17) in Lec. 5, *viz.*

$$\frac{dW}{d\omega} = \frac{\mu_0 e^2}{3\pi c} |\dot{\mathbf{v}}(\omega)|^2 \tag{1}$$

and using

$$\dot{\mathbf{v}}(\omega) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \dot{\mathbf{v}}(t) \exp(i\omega t)\, dt \tag{2}$$

Consider first that the electron is interacting with the ion only over a finite *collision time* $\tau \simeq b/v$. When $\omega\tau \gg 1$ the exponential in the integrand oscillates rapidly and the resulting integral is small. When $\omega\tau \ll 1$, on the other hand, the exponential term is approximately unity and the resulting integral is just $\dot{v} = dv/dt \approx \Delta v$ over the time interval $dt \approx \tau$. So we have

$$|\dot{\mathbf{v}}(\omega)| \simeq \begin{cases} \frac{1}{(2\pi)^{1/2}}\Delta v & \omega\tau \ll 1 \\ 0 & \omega\tau \gg 1 \end{cases} \tag{3}$$

So our radiation spectrum goes as

$$\frac{dW}{d\omega} \simeq \begin{cases} \frac{\mu_0 e^2}{6\pi^2 c}|\Delta\mathbf{v}|^2 & b \ll v/\omega \\ 0 & b \gg v/\omega \end{cases} \tag{4}$$

Now we can work out Δv by noting that the total Coloumb force on the electron is $Ze^2/(4\pi\epsilon_0 R^2)$ in the $-\hat{\mathbf{R}}$ direction. The perpendicular component of acceleration is the strongest and will thus make the dominant contribution to the radiation spectrum, so we can write

$$\Delta v \simeq \frac{1}{4\pi\epsilon_0}\frac{Ze^2}{m_{\rm e}}\int \frac{b\,dt}{(b^2+v^2t^2)^{3/2}} = \frac{1}{4\pi\epsilon_0}\frac{Ze^2}{m_{\rm e}}\frac{2}{bv} \tag{5}$$

(the integral turns out to be elementary). Substituting this into the expression for the radiation spectrum gives

$$\frac{dW}{d\omega} \simeq \begin{cases} \frac{8Z^2e^6}{3\pi c^3(4\pi\epsilon_0)^3 m_{\rm e}^2 b^2 v^2} & b \ll v/\omega \\ 0 & b \gg v/\omega \end{cases} \tag{6}$$

This is the spectrum for small angle scatterings by a single electron off a single ion. Next we want to generalise this to the case of a realistic plasma in which we have many electrons interacting with many ions.

Radiation spectrum for an electron-ion plasma

Let the ion and electron number densities in the plasma be $n_{\rm i}$ and $n_{\rm e}$. Then the flux of electrons incident on an ion is $n_{\rm e}v$ for a fixed electron speed v. The element of area about an ion over which an electron encounter occurs is approximately $2\pi b\,db$. So the emission per unit time per unit volume per unit frequency range is

$$\frac{dW}{d\omega dV dt} = n_{\rm e}n_{\rm i}2\pi v\int_{b_{\rm min}}^{\infty}\frac{dW(b)}{d\omega}b\,db \tag{7}$$

where $b_{\rm min}$ is a minimum impact parameter to be chosen. Now it is difficult to see how the solution for $dW/d\omega$ obtained above in the asymptotic limits $b \ll v/\omega$ and $b \gg v/\omega$ can be used to solve this integral over a full range of impact parameters. However, it turns out that the solution can be well approximated by using the non-zero asymptotic solution for $dW/d\omega$ because the integral is just logarithmic in b:

$$\frac{dW}{d\omega dV dt} \simeq \frac{16 n_{\rm e}n_{\rm i}Z^2e^6}{3c^3(4\pi\epsilon_0)^3m_{\rm e}^2 v}\ln\left(\frac{b_{\rm max}}{b_{\rm min}}\right) \tag{8}$$

where $b_{\rm max} \sim v/\omega$ is some value beyond which the $b \ll v/\omega$ limit no longer applies and the contribution to the integral becomes negligible. We can set $b_{\rm max} = v/\omega$, even though it is

uncertain because it is inside the logarithm. An appropriate value of $b_{\rm min}$ can be chosen to correspond to the break down of the small-angle scattering approximation. So when $\Delta v \sim v$, (5) implies $b_{\rm min} \simeq 2Ze^2/(4\pi\epsilon_0 m_{\rm e}v^2)$.

The exact expression for the radiation spectrum can be obtained with a full quantum treatment. For convenience, a quantum correction is added to the classical formula. This correction term is known as the *Gaunt factor*,

$$G_{\rm ff}(v,\omega) = \frac{\sqrt{3}}{\pi}\ln\left(\frac{b_{\rm max}}{b_{\rm min}}\right) \qquad \text{free-free Gaunt factor} \tag{9}$$

giving

$$\frac{dW}{d\omega dV dt} \simeq \frac{16\pi n_{\rm e}n_{\rm i}Z^2e^6}{3^{3/2}c^3(4\pi\epsilon_0)^3m_{\rm e}^2 v}G_{\rm ff}(v,\omega) \tag{10}$$

This is now the bremsstrahlung radiation emitted per unit time per unit volume per unit frequency by single-speed electrons interacting with many ions. Next we compute the volume emissivity for a thermal distribution of electron speeds.

6.2 Emission from a Thermal Distribution of Electrons

In a thermal plasma, the velocity distribution of the electrons (and ions) is Maxwellian, which is isotropic. The number of thermal particles with velocity $\mathbf{v}$ in the range $d^3\mathbf{v}$ is

$$dn(\mathbf{v}) = f(\mathbf{v})d^3\mathbf{v} = f(v)4\pi v^2\,dv \propto \exp\left(-\frac{mv^2}{2kT}\right) v^2\,dv$$

We need to average the single-speed radiation spectrum over this distribution function for all electron speeds satisfying $\frac{1}{2}m_\mathrm{e}v^2 \gtrsim h\omega/2\pi$, i.e.

$$\frac{dW(T,\omega)}{d\omega dV dt} = \frac{\int_{v_\mathrm{min}}^{\infty} \frac{dW(v,\omega)}{d\omega dV dt} v^2 \exp(-mv^2/2kT)\,dv}{\int_0^\infty v^2 \exp(-mv^2/2kT)\,dv} \tag{11}$$

Using $d\omega = 2\pi\,d\nu$, the final result is the expression for the free-free volume emissivity:

$$\boxed{j_\nu^\mathrm{ff} \equiv \frac{dW}{dV dt d\nu} = \frac{1}{(4\pi\epsilon_0)^3}\frac{32\pi e^6}{3m_\mathrm{e}^{3/2}c^3}\left(\frac{2\pi}{3kT}\right)^{1/2} Z^2 n_\mathrm{e} n_\mathrm{i} \exp\left(-\frac{h\nu}{kT}\right)\bar{G}_\mathrm{ff}} \tag{12}$$

where $\bar{G}_\mathrm{ff}$ is now the *velocity averaged Gaunt factor*. Its value is typically of order unity.

Points to note:

1. The only frequency dependence is in the exponential term $\exp(-h\nu/kT)$. So the spectrum declines exponentially at frequencies $h\nu \gg kT$, but is approximately flat for $h\nu \ll kT$.
2. The emissivity is also proportional to $T^{-1/2}$. So for $h\nu \ll kT$, the spectrum is lower for higher T. But for $h\nu \gg kT$, the exponential cutoff extends to higher frequencies for higher T, so there is more high-energy emission.
3. The units of emissivity are $\mathrm{W\,m^{-3}\,Hz^{-1}}$. So to calculate the total radiative power (i.e. luminosity) in bremsstrahlung emission from a real astrophysical source, we simply integrate j_ν^ff over an appropriate source volume and over the frequency bandwidth that we are interested in.