Quantum effects on the dispersion of ion acoustic waves

A. Mushtaq and D. B. Melrose
1School of Physics, University of Sydney, Sydney, New South Wales 2006, Australia
2Theoretical Plasma Physics Division, PINSTECH, Nilore, 44000 Islamabad, Pakistan

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The longitudinal response function for an isotropic, nonrelativistic, thermal plasma with the quantum recoil included exactly is used to generalize the dispersion relation for ion acoustic waves with and the absorption coefficient for Landau damping to include the quantum recoil. The results are compared to recent treatment of the dispersion relation derived using a fluid theory with the quantum effects included through the Bohm potential. © 2009 American Institute of Physics.

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I. INTRODUCTION

There is currently a great deal of interest in collective quantum effects in plasmas.1,2 The regime of interest is when the de Broglie wavelength of the charge carriers is comparable to the dimensions of the system, when quantum effects must be taken into account. The main line of research starts from a set of fluid equations for the electrons, derived either from an N-body description, a density matrix description, or a Madelung description of the wave function(s).3,4 The intrinsicively quantum term in the quantum fluid equations is the Bohm potential. As in classical fluid mechanics, the set of equations is closed by a suitable assumption concerning the thermodynamical relation between quantities. This quantum fluid description has been shown to have applications in many different settings,1,2,5 motivated by recent experimental progress and techniques.6 The fluid approach should be regarded as an approximations to a more rigorous kinetic approach, and the limits of validity of the fluid approach can be identified by deriving the fluid equations from kinetic theory. Our objective in this paper is to make an initial step in setting up the link between the quantum fluid equations and a fully relativistic quantum treatment of kinetic theory,7 which we refer to as quantum plasma dynamics (QPD). Our specific objectives relate to two recent results where the quantum fluid equations appear to give different results from kinetic theory. These relate to the dispersion relation for ion acoustic waves (IAWs) and their Landau damping.

Using the one dimensional (1D) quantum fluid formulation, Haas et al.8 derived the dispersion relation for IAWs in the dimensionless form

\[
\frac{\omega^2}{k^2} = \frac{1 + \frac{H^2 k^2}{4}}{1 + \frac{\omega_p \omega_A}{\omega_p}} k = k c_s / \omega_p, \quad \text{where} \quad c_s^2 = G V_e^2 (m_e / m_i), \quad H = \hbar \omega_p / m_e V_e \text{G} \quad \text{with} \quad G \to 1 \quad \text{for a nondegenerate distribution with electron thermal speed} \ V_e, \quad \text{and} \quad G \to v_F^2 / V_e^2 \text{for a completely degenerate speed with Fermi speed} \ v_F. \text{ The simplest useful approximation to Eq. (1) is in the long-wavelength limit when the denominator in Eq. (1) is approximated by unity. Then, in the nondegenerate case, Eq. (1) gives}
\]

\[
\omega^2 = \omega_q^2 (k) + \Delta_{\omega_i}^2
\]

with \ \Delta_{\omega_i}^2 = \hbar^2 k^4 / 4 m_i m_e \quad \text{and} \quad \omega_q^2 = \sqrt{T_e / m_i} \quad \text{is the ion sound speed. The term} \ \Delta_{\omega_i}^2 \text{arises from the Bohm term in QFT, and in its absence, Eq. (2) is the conventional dispersion relation for long-wavelength IAWs. To be more specific, after linearizing and Fourier transforming, the fluid equation for the electrons includes a pressure term} \ \sim V_e^2 \quad \text{and the Bohm term} \ \sim \hbar^2 k^4 / 4 m_e; \quad \text{there is a factor} \ m_e / m_i \text{in the derivation of Eq. (2) that changes} \ V_e^2 \text{into} \ V_i^2 \text{and the Bohm term into} \ \Delta_{\omega_i}^2. \text{ Our objective here is to derive the dispersion relation analogous to Eq. (2) using the kinetic theory for a nondegenerate distribution.}

We are aware of no specific results on quantum effects in Landau damping of IAWs. Recently, Zhu et al.9 derived a quantum correction to Landau damping for electron plasma waves by using the Vlasov equation including the Bohm potential term as a quantum force term. The same procedure may be applied to treat Landau damping of IAWs. However, the inclusion of a pseudoforce term associated with the Bohm potential in the Vlasov equation leads to a hybrid theory whose range of validity is not known. An exact result including all quantum effects in Landau damping is available using QPD. Another objective here is to apply this theory to Landau damping of IAWs and to compare and contrast the results from fluid theory with the exact theory.

In Sec. II we use QPD result for the response of an arbitrary relativistic quantum Fermi gas to write down the longitudinal response function, \ K^4(\omega, k), \text{ for an electron-ion gas including all quantum effects. In Sec. III we evaluate} \ K^4(\omega, k) \text{ for the specific case of Maxwellian distributions of electrons and ions and use the real part of} \ K^4(\omega, k) \text{ to derive the dispersion relation for IAWs. In Sec. IV, we use the imaginary part of} \ K^4(\omega, k) \text{ to treat Landau damping of IAWs. In Sec. V we compare these results with the corresponding results from fluid theory.}
II. QUANTUM LONGITUDINAL RESPONSE FUNCTION

The dispersion equation for longitudinal waves in any isotropic plasma is \( K^L(\omega, k) = 0 \), where \( K^L \) is the longitudinal dielectric function. In an electron-ion plasma it is convenient to introduce the electron and ion susceptibilities such that the dispersion equation becomes

\[
K^L(\omega, k) = 1 + \chi_e^L(\omega, k) + \chi_i^L(\omega, k) = 0.
\] (3)

We derive a form for \( K^L \) that includes quantum effects by starting from the most general QPD form and making appropriate approximations to it. A QPD calculation leads to a response tensor that is written down in Appendix B in both 4-tensor and 3-tensor forms. The longitudinal dielectric function, \( K^L(\omega, k) \), is obtained by projecting the 3-tensor form onto \( k k_j / k^2 \).

This gives

\[
K^L(\omega, k) = 1 - \sum_{\alpha=e,i} \frac{e^2}{e_0 m_\alpha} \int \frac{d^3 p}{\gamma} \frac{[1 - (k \cdot v)^2 / c^2]^2 f_\alpha(p)}{(\omega - k \cdot v)^2 - [h(\omega^2 - k^2 c^2)/2m_\alpha e^2]^2}.
\] (4)

We treat the ions in the same way as the electrons, assuming \( f_\alpha(p) \) to be a Maxwellian distribution.

The most important quantum effect for the present discussion is the term \( [h(\omega^2 - k^2 c^2)/2m_\alpha e^2]^2 \) in the denominator of Eq. (4). This term is associated with the quantum recoil. In the absence of this recoil term, Eq. (4) is identical to the expression derived using the relativistic, classical kinetic theory. A subtle point is that a nonrelativistic treatment of the quantum recoil is different from the nonrelativistic limit of a relativistically correct treatment of the recoil; specifically, the term \( \omega^2 c^2 \) in Eq. (4) is absent in a strictly nonrelativistic treatment.

In the strictly nonrelativistic limit, Eq. (4) reduces to

\[
K^L(\omega, k) = 1 - \sum_{\alpha=e,i} \frac{e^2}{e_0 m_\alpha} \int \frac{d^3 p}{\gamma} \frac{f_\alpha(p)}{(\omega - k \cdot v)^2 - \Delta_\alpha^2},
\] (5)

where the quantum recoil is included through

\[
\Delta_\alpha = \frac{\hbar k^2}{2m_\alpha}.
\] (6)

III. QUANTUM RECOIL FOR THERMAL PARTICLES

The longitudinal response function (5) is evaluated in Appendix A for a Maxwellian distribution. The electron and ion susceptibilities are given by setting \( \alpha = e,i \) in

\[
\chi^L_{\alpha}(\omega, k) = -\frac{\omega_{\text{pe}}}{\sqrt{2} |k| V_\alpha} \frac{1}{2\Delta_\alpha} \left[ \phi(y_{-\alpha}) - \frac{\phi(y_{+\alpha})}{y_{+\alpha} - y_{-\alpha}} \right],
\] (7)

where \( \phi(y) \) is the plasma dispersion function, defined by Eq. (A2), with argument \( y_{\pm\alpha} = (\omega \pm \Delta_\alpha) / \sqrt{2} k V_\alpha \).

The dispersion relation for IAWs follows from Eq. (3) under the assumption that the phase speed is intermediate between the electron and the ion thermal speeds. In a fluid approach, this corresponds to the electrons behaving isothermally and the ions behaving adiabatically. In a kinetic approach this corresponds to making the approximations \( y_{\pm e} \ll 1 \) and \( y_{\pm i} \gg 1 \). Using the approximation \( \phi(y) \approx y^2 - \frac{c_i^2}{c_e^2} y^4 + \cdots \) in Eq. (7), the electron longitudinal susceptibility becomes

\[
\chi_e^L(\omega, k) = \frac{1}{|k|^2 \lambda_{De}^2} \left[ 1 - \frac{(3\omega^2 + \Delta_\alpha^2)}{3|k|^2 V_e^2} \right],
\] (8)

where \( \lambda_{De} = V_e / \omega_{pe} \) is the electron Debye length. For ions, the approximation \( y_{\pm i} \gg 1 \) leads to

\[
\chi_i^L(\omega, k) = -\frac{\omega_{pi}}{\omega^2 - \Delta_i^2} \left[ 1 + \frac{(3\omega^2 + \Delta_i^2)|k|^2 V_i^2}{(\omega^2 - \Delta_i^2)^2} \right].
\] (9)

The dispersion relation for IAWs follows by solving \( K^L(\omega, k) = 0 \), which gives

\[
1 + \frac{1}{|k|^2 \lambda_{De}^2} \left[ 1 - \frac{(3\omega^2 + \Delta_\alpha^2)}{3|k|^2 V_e^2} \right] - \frac{\omega_{pi}}{\omega^2 - \Delta_i^2} \times \left[ 1 + \frac{(3\omega^2 + \Delta_i^2)|k|^2 V_i^2}{(\omega^2 - \Delta_i^2)^2} \right] = 0.
\] (10)

An approximate dispersion relation is obtained by assuming that the quantum recoil terms are small with \( |k|^2 V_e^2 \gg \omega^2 \).

This gives the dispersion relation

\[
\omega^2 = \frac{\omega_{pe}^2(k)}{1 + \frac{\Delta_\alpha^2}{3|k|^2 V_e^2 (1 + |k|^2 \lambda_{De})}} + \Delta_i^2.
\] (11)

Further approximation to Eq. (11) follows by neglecting the quantum recoil term for the ions, \( \Delta_i \rightarrow 0 \), and making the long-wavelength approximation. This gives

\[
\omega^2 = \omega_{pe}^2(k) + \frac{1}{3} \Delta_i^2.
\] (12)

The dispersion relation (11) is different from relation (1), and the approximate form (12) is different from the analogous form (2) by a factor of 1/3 in the final term. In both these approximate forms, \( \Delta_i^2 \) arises from quantum corrections to the contribution of the electrons, multiplied by a factor \( m_i / m_e \). In the fluid approach, the quantum correction is due to the Bohm term, which describes quantum mechanical diffusion in the coordinate space. On Fourier transforming, the Bohm term gives a term \( \Delta_i^2 \) in the dispersion relation for Langmuir waves and the term \( \Delta_e^2 \) in Eq. (12) after multiplying by \( m_i / m_e \). This derivation from QFT, there is no restriction on \( k \) in deriving the term \( \Delta_i^2 \). In contrast, in the derivation from kinetic theory, the derivation of the dispersion relation for Langmuir waves, and hence of the quantum term \( \Delta_i^2 \), requires \( y_{\pm i}^2 \gg 1 \), which is effectively \( k \ll 1 / \lambda_{De} \) for Langmuir waves. This limit on the dispersion relation for Langmuir waves is well known in kinetic theory, and it also applies to the quantum correction. The derivation of Eq. (12) for IAWs requires a different approximation, \( y_{\pm i}^2 \ll 1 \). This requires \( k \ll m_i V_i / \hbar \), that is, that the wavelength be much greater than the de Broglie length for a thermal electron. No such limit is implied by the derivation using QFT. Thus, the kinetic approach implies limitations that are not evident in the QFT treatment.

We can find no simple physical explanation for the difference of a factor of 3 in the quantum corrections in Eqs. (2)
and (12). We note that there is a factor of 3 difference (in nonquantum theory) in the dispersive term in the dispersion relation for Langmuir waves derived using the fluid theory and using the kinetic theory. In both cases, these factors arise from an expansion of the plasma dispersion function, \( \phi(y) \), defined by Eq. (A2). In the case of IAWs, the second term in the expansion \( \phi(y) = y^2 - 4y^4/3 + \cdots \) for small \( y = \omega/\sqrt{2kV_c} \) gives the factor 1/3 in Eq. (12), and for Langmuir waves, the third term in the expansion \( \phi(y) = 1 + 1/2y^2 + 3/4y^4 + \cdots \) for large \( y \) gives the dispersive term. This factor of 3 does not appear naturally in the fluid theory, and neither do they have simple interpretations in the kinetic theory.

\[ \text{Im} K^L(\omega, k) = \sqrt{\frac{\pi}{2}} \sum \frac{\omega_{pe}^2}{|k|V_a^2} 1 \frac{1}{2\Delta_n} \times [\exp(-y_{<,a}^2) - \exp(-y_{>,a}^2)]. \]  

The imaginary part of response function for electrons with \( y_{<,a}^2 \ll 1 \) is

\[ \text{Im} K^L(\omega, k) = \frac{\pi}{2} \frac{\omega_{pe}^2}{|k|^3V_c^3} \left( \frac{\sinh H_e}{H_e} \right), \]  

where we use \( [\exp(-y_{<,a}^2) - \exp(-y_{>,a}^2)] = 2 \sinh H_e \), with \( H_e = [\hbar \omega / 2m_eV_c^2] \). The imaginary part of the response functions with \( y_{>,a}^2 \gg 1 \) [Eq. (A8)] is

\[ \text{Im} K^L(\omega, k) = \frac{\pi}{2} \frac{\omega_{pe}^2}{|k|^3V_c^3} \left( \frac{\sinh H_e}{H_e} \right) \exp \left( -\frac{\omega^2 + \Delta_e^2}{2|k|^2V_c^2} \right) \]

with \( H_e = \hbar \omega / 2m_eV_c^2 \). The absorption coefficient for Landau damping is defined by

\[ \gamma_L(k) = -\frac{\text{Im} K^L(\omega, k)}{\text{Re} K^L(\omega, k)} . \]

The Landau damping factor for IAWs with quantum recoil is

\[ \gamma_L(k) = -\sqrt{\frac{\pi}{8}} \frac{\omega_e^4}{|k|^3V_c^3} \left[ \frac{m_e}{m_i} \left( \frac{\sinh H_e}{H_e} \right) + \left( \frac{T_e}{T_i} \right)^{3/2} \left( \frac{\sinh H_i}{H_i} \right) \exp \left( -\frac{\omega_e^2 + \Delta_e^2}{2|k|^2V_c^2} \right) \right] . \]

In the absence of \( \Delta_i \) and \( H_i \) and for \( y_{<,e} \ll 1 \) and \( y_{>,i} \gg 1 \), the absorption coefficient (16) becomes

\[ \gamma_L(k) = -\sqrt{\frac{\pi}{8}} \frac{\omega_e^4}{|k|^3V_c^3} \left[ \frac{m_e}{m_i} \left( \frac{\sinh H_e}{H_e} \right) + \left( \frac{T_e}{T_i} \right)^{3/2} \left( \frac{\sinh H_i}{H_i} \right) \right] \left( 1 + \frac{1}{2} \frac{\omega_e^2}{k^2V_c^2} \right) . \]

In the absence of the quantum recoil of the electrons, \( H_e \), \( \Delta_e \to 0 \), Eq. (17) reproduces the conventional expression for Landau damping for IAWs.

The effect of quantum recoil on the absorption coefficient \( \gamma_L(k)/|\omega_e| \) as a function of \( |k|\lambda_{De} \) is plotted in Fig. 1. The inclusion of the quantum recoil, through the parameter \( H_e \), reduces the Landau damping rate. A physical explanation is given in Sec. V.

\[ \gamma_L(k) = -\frac{\omega_e}{1 + \left| k \right|^2 \lambda_{De}^2 H_e^2} \left[ \frac{m_e}{m_i} \left( \frac{\sinh H_e}{H_e} \right) + \left( \frac{T_e}{T_i} \right)^{3/2} \left( \frac{\sinh H_i}{H_i} \right) \right] \left( 1 + \frac{1}{2} \frac{\omega_e^2}{k^2V_c^2} \right) . \]
the IAW is long compared to the de Broglie length of a thermal electron. While it is plausible that quantum mechanical diffusion applies only over distances long compared to the de Broglie length, this is not included explicitly in the Bohm term.

The inclusion of the quantum recoil in Landau damping is easily understood in the quantum approach. Quantum mechanically, net absorption may be interpreted in terms of the difference between true absorption \(p \rightarrow h \mathbf{k} \rightarrow \mathbf{p}\) and stimulated emission, \(p \rightarrow \mathbf{p} - h \mathbf{k}\). According to Eq. (B1), \(f_\alpha(p)\) may be interpreted in terms of the occupation number, and this difference is proportional to \(f_\alpha(p) - f_\alpha(p - h \mathbf{k})\). For a Maxwellian distribution, this difference can be expressed in terms of a sinh function, same as in Eq. (16). The quantum result differs from its classical counterpart by a factor \(\sinh\left(\frac{\omega}{V_a}\right)\), same as in Eq. (5).

**APPENDIX A: DERIVATION OF EQ. (8)**

Equation (5) is the nonrelativistic form for the longitudinal part of the response tensor including the quantum recoil. For a Maxwellian distribution, the integral may be evaluated in terms of the familiar plasma dispersion function. The basic result we need is

\[
\int d^3p \frac{f_\alpha(p)}{\omega - k \cdot v} = \frac{n_a}{\sqrt{2|k|V_a}} \phi(y_a), \quad y_a = \frac{\omega}{\sqrt{2kV_a}},
\]

(A1)

where the plasma dispersion function is defined by

\[
\phi(y) = -\frac{y}{\sqrt{\pi}} \int_0^\infty dt e^{t^2 - yt^2}.
\]

(A2)

An alternative form for \(\phi(y)\) for real \(y\) is

\[
\phi(y) = 2ye^{-y^2} \int_0^y dt e^{t^2} - iy \sqrt{\pi} e^{-y^2}.
\]

(A3)

Expansions of the real part give

\[
\phi(y) = y^2 - \frac{4}{3} y^4 + \cdots \quad \text{for } y^2 \ll 1 \quad \text{and}
\]

\[
1 + \frac{1}{2y^2} + \frac{3}{4y^4} + \cdots \quad \text{for } y^2 \gg 1.
\]

(A4)

Equation (5) is reduced to two forms of the term (A1) by writing

\[
\frac{1}{(\omega - k \cdot v)^2 - \hbar^2 k^2/4m_a^2} = \frac{m_a}{h^2 k^2/4m_a^2} \sum_{\alpha=\pm} \frac{1}{\frac{\omega - k \cdot v} {\hbar} \mp \frac{\hbar k^2}{2m_a}}.
\]

(A5)

Then, Eq. (A1) implies that

\[
\int d^3p \frac{f_\alpha(p)}{\omega - k \cdot v \pm \hbar \mathbf{k}^2/2m_a} = \frac{n_a}{\sqrt{2|k|V_a}} \phi(y_{\alpha \pm})
\]

(A6)

with

\[
y_{\alpha \pm} = \frac{\omega \mp \hbar \mathbf{k}^2/2m_a}{\sqrt{2|k|V_a}} = \frac{\omega \mp \Delta_a}{\sqrt{2|k|V_a}}.
\]

(A7)

Then, Eq. (5) with Eqs. (A5)–(A7) gives

\[
K_\alpha^i(\omega, k) = 1 - \sum_{\alpha=\pm} \frac{\omega_{\alpha \pm}^2}{2|k|V_a} \frac{1}{2\Delta_a} \left[ \phi(y_{\alpha \pm}) - \phi(y_{\alpha \mp}) \right],
\]

(A8)

which is the desired generalization of the response function with the quantum recoil included.

**APPENDIX B: RELATIVISTIC QUANTUM FORM FOR THE RESPONSE TENSOR**

Let the tensor \(\Pi^{\mu\nu}(k)\) represents the relativistic quantum expression, where quantum recoil is included, here \(k\) denotes the 4-vector \(k=[\omega, \mathbf{k}]\). It is shown in Ref. 7 [Eq. (8.3.14) therein] that the fully quantum relativistic response tensor may be written (in natural units \(\hbar=c=1\)) as

\[
\Pi^{\mu\nu}(k) = -2e^2 \int d^3p \frac{\bar{n}(p)}{V} \frac{\langle kp \rangle^2}{(2\pi)^3} \frac{1}{(k^2/2)^2} \alpha^{\mu\nu}(k, p)
\]

(B1)

with \(kp=\gamma m(\omega - k \cdot v), k^2=(\omega^2-k^2)\), and where \(\bar{n}(p)\) is the occupation number summed over electrons and positrons, related to the classical distribution function by Eq. (B4). The 4-tensor in the numerator of Eq. (B1) is

\[
\alpha^{\mu\nu}(k, p) = g^{\mu\nu} - \frac{k_{\mu}u_{\nu} + k_{\nu}u_{\mu}}{ku} + \frac{k^2 u_{\mu}u_{\nu}}{ku^2}
\]

(B2)

with \(u=p/m\), and where \(g^{\mu\nu}\) is the metric tensor (diagonal \(+1, -1, -1, -1\)).

Detailed discussions of the quantum recoil are given in Chaps. 5 and 8 of Ref. 7. The quantum recoil term is the term \((k^2/2)^2\) in the denominator of Eq. (B1). The quantum recoil for emission and absorption follow from the transitions \(p \rightarrow p' - k, p \rightarrow p' + k\), through the requirement that the particle must be on its mass shell: \(p^2 = p'$^2 = m^2\). These relations imply that \(pk \pm k^2/2 = 0\), which translates into \(\omega - k \cdot v = \hbar(\omega^2/k^2-\omega^2)/2\gamma m = 0\) in ordinary units. The term \(\omega^2/c^2\) is absent in a nonrelativistic treatment. Specifically, \(p \rightarrow p \mp h \mathbf{k}\) and \(p \mp h \mathbf{k}\)\(2m=p^2/2m+h\omega\) implies that \(\omega - k \cdot v = \hbar k^2/2m = 0\).

Rewriting the general form (B1) in terms of the dielectric 3-tensor, \(K_{ij}(\omega, \mathbf{k})\), gives

\[
K_{ij}(\omega, \mathbf{k}) = \delta_{ij} - \sum_{\alpha=\pm} \frac{e^2}{\varepsilon_0 m_a} \int d^3p \frac{\delta_{ij}}{V} \frac{(\omega - k \cdot v)^2}{(\omega - k \cdot v)^2 - \hbar^2 k^2/4m_a^2} \frac{\langle kp \rangle^2}{(2\pi)^3} \frac{1}{(k^2/2)^2} \alpha^{\mu\nu}(k, p)
\]

\[
\times \left[ \delta_{ij} + \frac{k_\mu u_\nu + k_\nu u_\mu}{(\omega - k \cdot v)^2} + \frac{k^2 u_\mu u_\nu}{(\omega - k \cdot v)^2} \right][f_\alpha(p)]
\]

(B3)

with \(p=\gamma m_0\mathbf{v}\) and \(\gamma=(1-v^2/c^2)^{-1/2}\). The form (B3) includes all relativistic quantum effects. The electrons are treated as
unpolarized spin $-1/2$ particles. No assumption is made concerning degeneracy, with $f_e(p)$ related to the quantum mechanical occupation number, $\bar{n}(p)$, averaged over spin states, by

$$\frac{2\bar{n}_s(p)}{(2\pi\hbar)\frac{3}{2}} = f_e(p). \quad (B4)$$

For a completely degenerate distribution, one has $\bar{n}_s(p)=1$ below the Fermi energy, and $\bar{n}_s(p)=0$ above the Fermi energy, and for a nondegenerate distribution, $f_e(p)$ is a Maxwellian.