Appendix A

Units and physical quantities

A.1 Physical and plasma constants

The values of physical and plasma constants are given in Table A.1 in SI units and Gaussian units. For the plasma constants, the values in SI units are for n_e is per cubic meter, and the values in Gaussian units are for n_e is per cubic centimeter, and for temperature in kelvin.

Table A.1. Physical and plasma constants

physical quantity		SI units	Gaussian units
speed of light	c	$3.0 \times 10^{8}\,\mathrm{m\,s^{-1}}$	$3.0 \times 10^{10}\,\mathrm{cm\,s^{-1}}$
fundamental charge	e	$1.6 \times 10^{-19}\,\mathrm{C}$	$4.8 \times 10^{-10}\,\mathrm{esu}$
electron mass	m_e	$9.1 \times 10^{-31}\,\mathrm{kg}$	$9.1 \times 10^{-28}\,\mathrm{g}$
proton mass	m_p	$1.67 \times 10^{-27}\,\mathrm{kg}$	$1.67 \times 10^{-24}\,\mathrm{g}$
electron volt	eV	$1.6 \times 10^{-19}\,\mathrm{J}$	$1.6 \times 10^{-12}\,\mathrm{erg}$
(Planck's constant)/2π	$\hbar$	$1.05 \times 10^{-34}\,\mathrm{J\,s}$	$1.05 \times 10^{-27}\,\mathrm{erg\,s}$
classical e^- radius	r_0	$2.8 \times 10^{-15}\,\mathrm{m}$	$2.8 \times 10^{-13}\,\mathrm{cm}$
Thomson cross section	σ_T	$6.65 \times 10^{-29}\,\mathrm{m^2}$	$6.65 \times 10^{-25}\,\mathrm{cm^2}$
critical B field	B_c	$1.44 \times 10^{9}\,\mathrm{T}$	$1.44 \times 10^{13}\,\mathrm{G}$
ε_0		$8.85 \times 10^{-12}\,\mathrm{F\,m^{-1}}$	
μ_0		$1.23 \times 10^{-6}\,\mathrm{H\,m^{-1}}$	
plasma frequency	ω_p	$56.4 n_\mathrm{e}^{1/2}\,\mathrm{s^{-1}}$	$5.64 \times 10^{4} n_\mathrm{e}^{1/2}\,\mathrm{s^{-1}}$
electron gyrofrequency	Ω_e	$1.76 \times 10^{11} B\,\mathrm{s^{-1}}$	$1.76 \times 10^{7} B\,\mathrm{s^{-1}}$
Debye length	λ_D	$69 T_\mathrm{e}^{1/2} n_\mathrm{e}^{-1/2}\,\mathrm{m}$	$6.9 T_\mathrm{e}^{1/2} n_\mathrm{e}^{-1/2}\,\mathrm{cm}$
ion sound speed	v_s	$91 T_e^{1/2}\,\mathrm{m\,s^{-1}}$	$9.1 \times 10^{3}\,\mathrm{cm\,s^{-1}}$

Conversions factors

Conversion factors between quantities in SI and Gaussian units are given in Table A.2.

Table A.2. Conversion factors between SI and Gaussian units

quantity	Gaussian/SI
length	10^2 cm/m
mass	10^3 g/kg
energy	10^7 erg/J
power	10^7 erg s^{-1}/W
force	10^5 dyne/N
charge	3×10^9 statcoul/C
electric field	$\frac{1}{3} \times 10^{-4}$ statvolt cm^{-1}/V m^{-1}
current	3×10^9 statamp/A
current density	3×10^5 statamp cm^{-2}/A m^{-2}
magnetic induction	10^4 G/T

Boltzmann's constant is not used in this book. One should regard the kelvin as a unit of energy, and then Boltzmann's constant is a conversion factor from kelvin to other energy units. This and other conversion factors are given in Table A.3.

Table A.3. Other conversion factors

quantity	factor	inverse
temperature	1.38×10^{-23} J/K	7.24×10^{22} K/J
temperature	8.62×10^{-5} eV/K	1.16×10^4 K/eV
X-ray energy	4.1×10^{-15} eV/Hz	2.4×10^{14} Hz/eV

A.2 Units and dimensional analysis

Natural, SI and Gaussian units

In most of the formal development in this book, the formulae are written in natural units, in which one has$\hbar = c = 1$. The use of natural units is widespread in relativistic quantum mechanics, but it is unusual in classical electrodynamics and plasma physics. To minimize confusion, selected formulae have $\hbar$ and c restored. Such formulae are preceded by a remark to this effect or are said to be in 'ordinary units' or 'SI units'.

The choice of natural units in a covariant classical overcomes some annoying problems with the appearance or non-appearance of c. One example is in Fourier transforming in both space and time. By convention one integrates over $\mathrm{d}t$ and $\mathrm{d}^3\boldsymbol{x}$, and the inverse transform involves integrals over $\mathrm{d}\omega$ and $\mathrm{d}^3\boldsymbol{k}$. On writing these in covariant forms these are replaced by d^4x and d^4k, which involve integrals over $\mathrm{d}x^0 = c\mathrm{d}t$ and $\mathrm{d}k^0 = \mathrm{d}\omega/c$.

These two different conventions lead to Fourier transformed functions with dimensions that differ by a power of c. Another choice that can lead to confusion is whether c is included in the definition of the 4-velocity. A widely used convention is such that a 4-velocity is dimensionless, so that $u^2 = u^\mu u_\mu = 1$. This corresponds to $u^\mu = [\gamma, \gamma\boldsymbol{\beta}]$ with $\gamma = (1 - \beta^2)^{-1/2}$, and with $\boldsymbol{\beta} = \boldsymbol{v}/c$ in ordinary units.

A more serious source of confusion arises from the choice of electromagnetic units. The units on which the formulae in this book are based are SI units. An alternative choice that is used widely is Gaussian units. The different powers of c that appear in electromagnetic formulae with these different choices of units are simply avoided by the use of natural units. In SI units the quantities μ_0 and ε_0 are related by $\mu_0\varepsilon_0 = 1/c^2$, which becomes $\mu_0\varepsilon_0 = 1$ in natural units. Formulae in SI units with $c = 1$ are rewritten in Gaussian units with $c = 1$, by making the replacements $\mu_0 = 4\pi$, $\varepsilon_0 = 1/4\pi$.

Dimensional analysis

To restore $\hbar$ and c in a formula written in natural units one needs to use dimensional analysis. The dimensions of a quantity are written as powers of mass, M, length, L, and time, T. Let the symbol $\ni$ denote 'has the dimension'. One has $\hbar \ni \mathrm{ML^2T^{-1}}$ and $c \ni \mathrm{LT^{-1}}$. Setting $c = 1$ implies that length and time are measured in the same units. A simple physical interpretation is that if time is measured in seconds, then lengths must be measured in light-seconds. In natural units, mass has the same dimensions as inverse time. A physical interpretation is that a mass, m, corresponds to a rest energy mc^2 and to a frequency $mc^2/\hbar$, so that with $\hbar = c = 1$ the mass is denoted by this frequency.

In practice, to use a formula written in natural units, one needs to use dimensional analysis to rewrite it in terms of ordinary units. This involves multiplying a formula a power of $\hbar$ and a power of c, and choosing these powers such that the result has the desired dimensions. Consider, for example, the formula $W = \int[\mathrm{d}^3\boldsymbol{p}/(2\pi)^3](m^2 + |\boldsymbol{p}|^2)^{1/2} f(\boldsymbol{p})$, given the additional information that W is an energy density, that $\boldsymbol{p}$ is a momentum, and that the distribution function, $f(\boldsymbol{p})$, is dimensionless. By assumption the dimensions of the left hand side are $W \ni (\mathrm{ML^2T^{-2}})\mathrm{L^{-3}}$. By assumption, the term $|\boldsymbol{p}|^2 \ni (\mathrm{MLT^{-1}})^2$has the dimensions of of a momentum squared, and the term m^2 must be multiplied by c^2 so that it has the same dimensions. Then the right hand side has the dimensions of momentum to the fourth power, that is $(\mathrm{MLT^{-1}})^4$. We must multiply by powers of $\hbar$ and c so that these dimensions are the same as those of the left hand side. On multiplying by $\hbar^a c^b$, one requires $(\mathrm{ML^2T^{-2}})\mathrm{L^{-3}} = (\mathrm{MLT^{-1}})^4(\mathrm{ML^2T^{-1}})^a(\mathrm{LT^{-1}})^b$, implying $a = -3$, $b = 1$. Thus dimensional analysis implies that when this formula rewritten in ordinary units it becomes $W = \int [\mathrm{d}^3\boldsymbol{p}/(2\pi\hbar)^3](m^2c^4 + |\boldsymbol{p}|^2c^2)^{1/2} f(\boldsymbol{p})$.

Electromagnetic units

On including electromagnetic effects, one needs to add a further dimension, and this is chosen to be the charge, Q. SI units and Gaussian units lead to different dimensions. Charge times electric field has the same dimensions in both sets of units, $q\boldsymbol{E} \ni \mathrm{MLT}^{-2}$. However, charge times magnetic field has different dimensions, being $q\boldsymbol{B} \ni \mathrm{MT}^{-1}$ in SI units and $q\boldsymbol{B} \ni \mathrm{MLT}^{-2}$ in Gaussian units. Moreover, in SI units the square of the charge and the squares of the electric and magnetic field require an additional quantity with dimensions, either ε_0 or $1/\mu_0 = \varepsilon_0/c^2$, to convert them into quantities that involve only M, L, T, whereas in Gaussian units the squares of either electromagnetic field can be expressed in terms of M, L, T directly.

To avoid possible confusion with units, appropriate formulae are written in terms of quantities whose dimensions are clear. In particular, the square of the charge q can be combined with the number density, n, and mass, m, in the plasma frequency, ω_p, with

$$\omega_\mathrm{p}^2 = \frac{q^2 n}{\varepsilon_0 m} = \frac{\mu_0 q^2 n}{mc^2}, \tag{A.2.1}$$

in SI units, and with $\varepsilon_0 = 1/\mu_0 c^2 = 1/4\pi$ in Gaussian units, or in terms of the classical radius, r_0, of the particle:

$$r_0 = \frac{q^2}{4\pi\varepsilon_0 mc^2} = \frac{\mu_0 q^2}{4\pi m}. \tag{A.2.2}$$

The fine structure constant, α_c, is the ratio of the classical radius of the electron to the Compton wavelength $\hbar/mc$:

$$\alpha_c = \frac{e^2}{4\pi\varepsilon_0 \hbar c} = \frac{\mu_0 e^2 c}{4\pi\hbar} \approx \frac{1}{137}. \tag{A.2.3}$$