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Chapter 1

Some elementary functions

This chapter contains some material which you may already know but also some material which you probably do not. Please read all the material in this chapter and answer the questions at the end of the chapter.

1.1 Functions

A function is a black box which takes several inputs and gives one output. Here only functions of a single variable (that is with only one input) are considered. For an input $x$ the output of the function is denoted $f(x)$, $g(x)$, function($x$) or similar. A function is one-to-one if every input gives a different output. The set of inputs of a function is called the domain of the function and the set of outputs of a function is called the range of the function.

The inverse function of $y = f(x)$ is a function $f^{-1}(y)$ such that $f^{-1}(f(x)) = x$ for every $x$. The original function is the inverse of its own inverse function. Inverse functions are very useful for solving equations where $f(x) = c$ and $c$ is known. If $f(x)$ is not one-to-one then the inverse of $f$ cannot be defined.

The rest of this chapter is about real-valued functions on the real line, that is functions whose inputs and outputs are real numbers.

1.2 Polynomial and rational functions

A monomial in $x$ is a term $x^n$ where $n$ is zero or a positive integer. By writing $f(x) = x^2$ we define $f$ to be the function which, when evaluated at $a$, takes the value $a^2$. A polynomial function is a summation of terms consisting of a constant times a monomial. $g(x) = 2.7x^4 + x - \sqrt{15}$ is a polynomial function.

Polynomials of two variables, say $x$ and $y$ are similar except that they have products of powers of $x$ and $y$ where functions of one variable only have powers of $x$. $h(x, y) = 5x^2y^3 + \pi y - x^{15} + \sqrt{2}$ is an example of a polynomial function of two variables. Polynomial functions of more variables are defined similarly.

If $f(x)$ and $g(x)$ are two polynomial functions then

$$h(x) = \frac{f(x)}{g(x)}$$

is known as a rational function.
1.3 Exponential functions

The function $f(x) = 3^{x+2}$ is an exponential function, in fact for any base $A$ and a constant $k$, $g(x) = A^{x+k}$ is an exponential function. $x + k$ is the exponent of the base $A$. If $A > 0$ then the domain of $f(x) = A^x$ is all real numbers and the range is all positive real numbers, as long as $A \neq 1$, in which case $f(x) = 1$. If $A$ is not positive then $A^x$ cannot be defined as a real number for every real number so we will not consider this case for now.

For any $A > 0$

$$A^1 = A \quad \text{and} \quad A^0 = 1 .$$

The rules for manipulating exponentials are

$$A^x A^y = A^{x+y} ,$$

$$A^{-x} = \frac{1}{A^x} ,$$

$$A^x B^x = (AB)^x ,$$

$$A^x = \sqrt[n]{A} \quad \text{and} \quad (A^x)^y = A^{xy} = (A^y)^x .$$

1.4 The logarithmic function

The inverse of the exponential function is a logarithmic function. The logarithmic function is defined for $A > 0$ and $A \neq 1$. If $y = A^x$ then $x = \log_A y$, which is read ‘$x$ equals log base $A$ of $y$’. For example the log base 2 of 32

$$\log_2(32) = \log_2(2^5) = 5 .$$

Since the logarithmic function is the inverse of the exponential function, $B = A^{\log_A B}$ for any positive $A$ and $B$. Note that the domain of the logarithmic function must be the range of the exponential function, as for any function and its inverse.

Taking logarithms of each side of $A = A^1$ and $1 = A^0$ gives

$$\log_A(A) = 1$$

$$\log_A(1) = 0 .$$

By taking the logarithm of both sides of some of the rules for manipulating exponentials we can deduce a similar set of rules for logarithms:

$$\log_A(ab) = \log_A(a) + \log_A(b) ,$$

$$\log_A\left(\frac{1}{a}\right) = -\log_A(a) ,$$

$$\log_A(a^x) = x \log_A(a) .$$
Specifically, the first two rules come from the first two rules from manipulating exponentials, substituting \( x = \log_A(a) \) and \( y = \log_A(b) \) and the third comes from the last rule for exponentials, substituting for only one of \( x \) or \( y \).

Our counting system is base 10, as are the SI prefixes, so it is often easier to estimate logarithms base ten than any other base. For example to estimate \( \log_{10}(3 \times 10^8) = \log_{10}(3) + 8 \) we need only estimate \( \log_{10}(3) \).

Your calculator probably has a log button, which takes logarithms base ten. Note that some people use \( \log(x) \) to mean \( \log_{10}(x) \) but this is not always true and the meaning of \( \log(x) \) depends on the favourite base of the person you are talking to.

1.4.1 Natural base

There is a special number in mathematics, with similar standing to \( \pi \), sometimes called Euler’s constant or Napier’s constant and denoted by \( e \). This constant has the (approximate) value of

\[
e = 2.718128182845904524\ldots
\]

\( e \) is one of a class of numbers known as irrational numbers and like all irrational numbers has neither a terminating nor recurring decimal expansion. This number occurs naturally in calculus via the exponential function \( e^x \), which is its own derivative (this may not make sense until you read the later chapters on the calculus). This property is very useful and makes it the natural base for exponentials and logarithms in physics. The logarithm base \( e \) is called the natural logarithm and gets its own special mathematical symbol

\[
\ln(x) = \log_e(x) .
\]

This equation is read ‘the natural log of \( x \) equals the log base \( e \) of \( x \)’. There is probably a ln button on your calculator. Some people use \( \log(x) \) to mean \( \log_e(x) \) but as this may also mean \( \log_{10}(x) \) you should not write simply \( \log(x) \).

Note also that \( f(x) = e^x \) is called the exponential function, indicating that it is more special than any other exponential function. The notation \( \exp(x) = e^x \) is sometimes used.

1.4.2 Solving exponential equations

If an exponential equation such as \( 3^x = 5 \times 2^x \) is to be solved for \( x \), a base must first be chosen. Here base \( e \) is used and it is recommended you do the same, again because of the special role base \( e \) plays in the calculus. Next take the logarithm of both sides and rearrange using the rules for manipulating logarithms,

\[
\begin{align*}
\ln(3^x) & = \ln(5 \times 2^x) \\
x \ln(3) & = \ln(5) + \ln(2^x) \\
\ln(3)x & = \ln(5) + \ln(2)x \\
x (\ln(3) - \ln(2)) & = \ln(5) \\
x & = \frac{\ln(5)}{\ln(3) - \ln(2)} .
\end{align*}
\]
1.4.3 Changing base

Sometime it is desirable to change the base of a logarithm. Below, the relationship between logarithms of different bases is derived. Using the fact that if \( y = \log_A x \) then \( x = A^y \),

\[
x = A^{\log_A(x)} \\
= A^{\log_A(B) \log_A(x)} \\
= (A^{\log_A B})^{\log_A(x)} \\
= B^{\log_A(x)},
\]

so

\[
\log_B(x) = \frac{\log_A(x)}{\log_A(B)}. \]

Note that this is how you evaluate logarithms of bases other than 10 or \( e \) on your calculator.

1.5 Trigonometric functions

Another class of elementary functions is the trigonometric functions. The basic trigonometric functions are:

\[
sin(\theta) = x = \frac{1}{\csc(\theta)} \\
cos(\theta) = y = \frac{1}{\sec(\theta)} \\
tan(\theta) = \frac{x}{y} = \frac{\sin(\theta)}{\cos(\theta)} = \frac{1}{\cot(\theta)}
\]

Where \( x \) and \( y \) are the projections onto the \( x \) and \( y \) axes respectively of an arrow centred at the origin and of unit length, when the arrow has been rotated through \( \theta \) radians anti-clockwise from the positive \( x \) axis, see figure 1.1. Remember that \( \pi \) radians is 180°. If \( \theta > 2\pi \) then this means that the arrow has undergone more than one full rotation and if \( \theta < 0 \) then the arrow is rotated clockwise.

The unit circle, traced out by the head of the arrow as it rotates, is divided into four quadrants by the axes. These are called the first quadrant, where \( 0 < \theta < \pi/2 \), through to the fourth quadrant where \( 3\pi/2 < \theta < 2\pi \). Each trigonometric function is one-to-one within the first quadrant, but the absolute magnitudes repeat four times around the unit circle at \( \theta, \pi-\theta, \pi+\theta \) and \( 2\pi-\theta(= -\theta) \) radians. Thus, for instance, \( |\sin(\pi+\theta)| = |\sin(\theta)| \).
Some people remember the signs of the functions in each quadrant via the mnemonic All Stations To Central, however it is also easy to remember the signs simply through the unit circle construction. The mnemonic refers to the fact that in the first quadrant all of sin, cos and tan are positive, in the second of those functions only sin is positive, in the third only tan and in the fourth only cos. Thus \( \sin(\pi + \theta) = -\sin(\theta) \) (third quadrant).

There are various useful trigonometric identities, such as \( \cos(\theta) = \sin(\pi/2 - \theta) \); similar identities hold for the pairs cot and tan and cosec and sec. Perhaps the most widely used identity is

\[
\sin^2(\theta) + \cos^2(\theta) = 1.
\]

Figure 1.1 shows that the arrow of unit length is the hypotenuse of a right angled triangle with side of lengths \( x \) and \( y \), applying Pythagoras’ theorem gives the identity.

From this we obtain

\[
\tan^2(\theta) + 1 = \sec^2(\theta) \quad \text{dividing by } \cos^2(\theta),
\]

\[
1 + \cot^2(\theta) = \cosec^2(\theta) \quad \text{dividing by } \sin^2(\theta).
\]

There are relationships for sines and cosines of sums and differences of angles. Given one it is relatively easy to derive the others (you may like to try it).

\[
\begin{align*}
\sin(A + B) &= \sin A \cos B + \cos A \sin B \\
\sin(A - B) &= \sin A \cos B - \cos A \sin B \\
\cos(A + B) &= \cos A \cos B - \sin A \sin B \\
\cos(A - B) &= \cos A \cos B + \sin A \sin B
\end{align*}
\]

In the case \( A = B \) these formulae for the sums reduce to

\[
\begin{align*}
\sin(2A) &= 2\sin(A)\cos(A) \\
\cos(2A) &= \cos^2(A) - \sin^2(A) = 2\cos^2(A) - 1 = 1 - 2\sin^2(A)
\end{align*}
\]

The range of sin is \(-1 \leq \sin(\theta) \leq 1\) and the range of cos is \(-1 \leq \cos(\theta) \leq 1\) but the range of tan \(\theta\) is all real numbers. The domains of all three functions are all real numbers and none of the three are one-to-one. All of them, however, may be made one-to-one by restricting their domains. The convention is to restrict the domains of sin and tan to \(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\) and the domain of cos to \(0 \leq \theta \leq \pi\). Thus the inverse functions \(\arcsin(x) = \sin^{-1}(x)\) and \(\arctan(x) = \tan^{-1}(x)\) both have a range of \(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\) and
arccos(x) = cos^{-1}(x) has range 0 ≤ θ ≤ π. In contrast the domain of arcsin(x) and of arccos(x) is -1 ≤ x ≤ 1 while the domain of arctan(x) is all real numbers.

It is important to remember that arcsin(x) is not the only solution to x = sin(θ), there are infinitely many solutions. This is true for all trigonometric functions. In the above example all the solutions are of the form 2kπ + arcsin(x) or (2k + 1)π - arcsin(x), where k is an integer.

1.6 Problem Set: Functions

1. Show that:
   (a) log_A(ab) = log_A(a) + log_A(b)
   (b) log_A(\frac{1}{a}) = -log_A(a)
   (c) log_A(a^x) = x log_A(a)
   (d) log_A(\frac{a}{b}) = log_A(a) - log_A(b)
   (e) cos(\theta) = sin\left(\frac{\pi}{2} - \theta\right)
   (f) log_A x = 1/ log_x A

2. Estimate the following without using a calculator.
   (a) log_{10}(3)
   (b) log_{10}(30)
   (c) ln(10)
   (d) log_{10}(e)
   (e) log_{0.1}(24.5)
   (f) log_{2}.024

3. Solve the following equations for x:
   (a) 13^x = 6 \times 4^{-2x}
   (b) sin(x + 3) = \frac{1}{2}
Chapter 2

Vectors

In this section we shall look at vector analysis using the traditional definition of a vector as a quantity, such as force, velocity, or acceleration, which possesses both magnitude and direction. A scalar is a quantity, such as volume, mass or energy, which is fully described by a single real number. To distinguish vectors from scalars we shall set all vectors in boldface type, for example \( \mathbf{v} \). In hand-written equations vectors are often denoted \( v \) or \( \vec{v} \).

To represent a vector graphically we may draw a line segment in the direction of the vector with a length proportional to the magnitude of the vector. For convenience, we shall often refer to the representative line segment as though it were the vector itself, and draw it with an arrow showing direction as in figure 2.1.

![Figure 2.1: A single vector](image)

The absolute value \(|\mathbf{v}|\) of a vector \( \mathbf{v} \) is another name for its magnitude. The magnitude is often represented in both print and handwritten work by writing the symbol for the vector as if it were a scalar, rather than a vector. In print this appears as \( v = |\mathbf{v}| \), and in handwritten work as however you draw it! A unit vector is any vector which has an absolute value of one. \( \hat{v} \) is the unit vector in the direction of \( \mathbf{v} \). A vector is said to be zero if and only if its absolute value is zero. The direction of a zero vector is undefined.

Two vectors whose absolute values are equal and whose directions are the same are said to be equal. It is important to realise that the point in space from which a directed line segment is drawn does not change either the absolute value or direction of the vector represented. If two vectors have the same length but opposite directions, each is said to be the negative of the other, as shown in figure 2.2.

![Figure 2.2: A vector and its negative](image)

A vector in two dimensions, that is in a plane, cannot be described by a single real number. However it may be described by a pair of real numbers, and there are many possibilities for such a description. One way is to describe each vector \( \mathbf{r} \) by its magnitude \( r \) and an angle \( \theta \). This is achieved by choosing both a point in the plane, as the origin, and a direction. The vector, represented graphically, is drawn starting from the origin and theta is defined to be the angle between the chosen direction and the vector in the anti-clockwise direction. See figure 2.3 for a diagram of the above construction. The numbers \( r \) and \( \theta \) are called the components or co-ordinates of \( \mathbf{r} \).

Another way to represent a vector \( \mathbf{r} \) by two real numbers is to choose a point as the origin
Figure 2.3: Two methods for describing a vector in a plane

and perpendicular $x$ and $y$ axes. Again a directed line segment representing the vector is drawn starting from the origin. The vector is projected onto the $x$ and $y$ axes by drawing a line perpendicular to each axis from the tip of the vector to the axis. The points at which these lines and the axes intersect are $r_x$ and $r_y$, as shown in figure 2.3. These are called the $x$ and $y$ co-ordinates respectively.

In this section two different coordinate systems have been described. In any coordinate system, one vector is equal to another if and only if each co-ordinate of one is equal to the corresponding co-ordinate of the other. The first example was a polar coordinate system, and the second a Cartesian coordinate system. For now we will work only with Cartesian co-ordinate systems. Other co-ordinate systems are discussed in an appendix.

Vectors in three dimensions can also be represented by directed line segments but require three real numbers to describe each vector. The idea of a Cartesian co-ordinate system can be extended from two to three dimensions by introducing a third axis perpendicular to the two axes used in the two dimensional case. Again other co-ordinate systems are discussed in an appendix.

When using the Cartesian co-ordinate system in three dimensions it is useful to define unit vectors along each of the axes. If we use the standard notation of $x$, $y$ and $z$ for the three axes then it is usual to call the unit vectors directed in the positive direction along these axes $\hat{i}$, $\hat{j}$ and $\hat{k}$ respectively.

2.1 Basic vector operations

2.1.1 Sum and difference of vectors

Graphically two vectors $\mathbf{A}$ and $\mathbf{B}$ are summed using the parallelogram law which is applied by choosing a point as the origin and drawing both $\mathbf{A}$ and $\mathbf{B}$ from this point. Then the parallelogram is completed, using $\mathbf{A}$ and $\mathbf{B}$ for the two remaining sides, and the sum $\mathbf{A} + \mathbf{B}$ is the diagonal of the parallelogram which passes through the origin, as depicted in figure 2.4.

From this definition it is evident that vector addition is commutative and associative
which means

\[ A + B = B + A \]

\[ A + (B + C) = (A + B) + C \]

The difference between two vectors, shown in figure 2.5, is defined to be

\[ A - B = A + (-B) \]

where \(-B\) is the negative of \(B\).

### 2.1.2 Multiplication by a scalar

The vector resulting from the product of a scalar \(c\) and a vector \(B\) is a vector which has an absolute value of \(|c||B|\) and direction the same as \(B\) if \(c > 0\) and opposite to \(B\) if \(c < 0\). Note that the negative of \(B\) is \((-1)B\).

Multiplication by a scalar is distributive,

\[ c(A + B) = cA + cB \]

### 2.2 Components of Vectors

If \(v_x\), \(v_y\) and \(v_z\) are the three Cartesian co-ordinates of a vector \(v\), then

\[ v = v_x\hat{i} + v_y\hat{j} + v_z\hat{k} \]
Given a second vector $\mathbf{u} = u_x\hat{i} + u_y\hat{j} + u_z\hat{k}$ and a scalar $c,$

$$\mathbf{v} + \mathbf{u} = (v_x + u_x)\hat{i} + (v_y + u_y)\hat{j} + (v_z + u_z)\hat{k},$$

$$c\mathbf{v} = cv_x\hat{i} + cv_y\hat{j} + cv_z\hat{k}.$$  

### 2.3 Vector products

#### 2.3.1 Dot product $\mathbf{v} \cdot \mathbf{u}$

In addition to the product of a scalar and a vector, two other types of product are defined in three dimensional vector algebra. The first of these is the dot product which is indicated by placing a dot between the two vectors and is sometimes also called the scalar or inner product. By definition, this is a scalar equal to the product of the absolute values of the two vectors and the cosine of the angle between their positive directions, i.e.

$$\mathbf{v} \cdot \mathbf{u} = ||\mathbf{v}|| ||\mathbf{u}|| \cos(\theta) = uv \cos(\theta).$$

Since $||\mathbf{v}|| \cos(\theta)$ is just the projection of the vector $\mathbf{v}$ in the direction of $\mathbf{u},$ and $||\mathbf{u}|| \cos(\theta)$ is the projection of the vector $\mathbf{u}$ in the direction of $\mathbf{v},$ it follows that the dot product of two vectors is equal to the length of either of them multiplied by the length of the projection of the other upon it.
It is useful to note that the volume of any parallel sided object can be expressed as a dot product, see figure 2.9;

\[ V = \text{Area of base} \times \text{perpendicular height} \]

\[ = |A||L| \cos(\theta), \text{ where } A \text{ is the area of the base and } L \text{ a side} \]

\[ = A \cdot L \]

The area vector \( A \) is defined such that it has magnitude equal to the scalar area, and is perpendicular to the surface.

Figure 2.9: A parallel sided object

The dot product is used widely in physics, for an example of this see the section on energy from forces at the end of the Forces chapter.

The dot product is commutative,

\[ \mathbf{v} \cdot \mathbf{u} = \mathbf{u} \cdot \mathbf{v} , \]

and also distributive over addition,

\[ \mathbf{v} \cdot (\mathbf{u} + \mathbf{w}) = \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{w} . \]

If the dot product of two vectors \( \mathbf{v} \) and \( \mathbf{u} \) is zero then either \( \mathbf{v} = 0 \), \( \mathbf{u} = 0 \) or \( \mathbf{v} \) and \( \mathbf{u} \) are perpendicular (meaning \( \cos(\theta) = 0 \)).

Note that

\[ \mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0 , \]

\[ \mathbf{i} \cdot \mathbf{i} = \mathbf{\hat{j}} \cdot \mathbf{\hat{j}} = \mathbf{\hat{k}} \cdot \mathbf{\hat{k}} = 1 . \]

Hence

\[ \mathbf{v} \cdot \mathbf{u} = (v_x \mathbf{\hat{i}} + v_y \mathbf{\hat{j}} + v_z \mathbf{\hat{k}}) \cdot (u_x \mathbf{\hat{i}} + u_y \mathbf{\hat{j}} + u_z \mathbf{\hat{k}}) = v_x u_x + v_y u_y + v_z u_z . \] (2.1)

In particular, taking \( \mathbf{u} = \mathbf{v} \), we have

\[ \mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2 = v_x^2 + v_y^2 + v_z^2 \]

\[ = \sqrt{v_x^2 + v_y^2 + v_z^2} . \] (2.2)

If we write \( \mathbf{v} \cdot \mathbf{u} = |\mathbf{u}| |\mathbf{v}| \cos(\theta) \), substitute Eq. (2.1) and Eq. (2.2), then solve for \( \cos(\theta) \), we obtain the formula

\[ \cos(\theta) = \frac{v_x u_x + v_y u_y + v_z u_z}{\sqrt{v_x^2 + v_y^2 + v_z^2} \sqrt{u_x^2 + u_y^2 + u_z^2}} . \]
2.3.2 Cross or vector product $v \times u$

The other type of product which we shall consider is the cross product which is indicated by placing a cross between the vectors and is also sometimes called the vector product. By definition, $v \times u$ is a vector of magnitude $|v||u|\sin(\phi)$, where $\phi$ is the smallest angle between $v$ and $u$; the direction of $v \times u$ is perpendicular to the plane containing both $v$ and $u$ and is given by the right hand rule. There are many ways of describing the right hand rule but the best way to get a proper understanding is to look at the diagrams and practice. One way to think about it is to align your palm with $v$ and curl your fingers around, through the smallest angle, towards $u$, then the direction of your thumb gives the direction of $v \times u$. Make sure you use your right hand when you are applying the right hand rule!

\[ a \times b = c \]

\[ b \times a = -c' \]

Figure 2.10: A diagram depicting the right hand rule

The cross product is distributive over addition, so

\[ u \times (v + w) = u \times v + u \times w \].

If you interchange $u$ and $v$ in $u \times v$ the direction of the resultant vector is reversed so the cross product is not commutative. In fact

\[ u \times v = -v \times u \].

Operations which obey this rule are sometimes said to be anticommutative. It is extremely important to remember to preserve the order of vectors in a cross product, otherwise you will find that the direction of your resultant vector is wrong.
If \( \mathbf{u} \times \mathbf{v} = 0 \) then either \( \mathbf{u} = 0 \), \( \mathbf{v} = 0 \) or \( \mathbf{u} \) and \( \mathbf{v} \) are in the same or opposite directions (meaning \( \sin(\theta) = 0 \)).

Any vector is parallel to itself so
\[
\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0.
\]

A co-ordinate system is \textit{right handed} if
\[
\mathbf{i} \times \mathbf{j} = \mathbf{k}.
\]

From this it follows that
\[
\mathbf{j} \times \mathbf{k} = \mathbf{i},
\]
\[
\mathbf{k} \times \mathbf{i} = \mathbf{j}.
\]

Hence
\[
(u_x \mathbf{i} + u_y \mathbf{j} + u_z \mathbf{k}) \times (v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}) = (u_y v_z - u_z v_y) \mathbf{i} + (u_z v_x - u_x v_z) \mathbf{j} + (u_x v_y - u_y v_x) \mathbf{k}.
\]

There are many applications of the cross product in physics. Perhaps the simplest of these is the concept of torque.

### 2.4 Problem Set: Vectors

1. Which of the following are legitimate mathematical operations if \( \mathbf{A}, \mathbf{B} \) and \( \mathbf{C} \) are vectors?
   
   (a) \( \mathbf{A} \cdot (\mathbf{B} - \mathbf{C}) \)
   
   (b) \( (\mathbf{A} - \mathbf{B}) \times \mathbf{C} \)
   
   (c) \( \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) \)
   
   (d) \( \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \)
   
   (e) \( \mathbf{A} \times (\mathbf{B} \cdot \mathbf{C}) \)

2. When two vectors \( \mathbf{A} \) and \( \mathbf{B} \) are drawn from a common point, the angle between them is \( \theta \).
   
   (a) Show that the magnitude of their vector sum is given by \( (A^2 + B^2 + 2AB \cos(\theta))^{1/2} \)
   
   (b) If \( \mathbf{A} \) and \( \mathbf{B} \) have the same magnitude, under what circumstances will their vector sum have the same magnitude as \( \mathbf{A} \) or \( \mathbf{B} \)?
   
   (c) Derive a result analogous to that in (a) for the magnitude of the vector difference \( \mathbf{A} - \mathbf{B} \).
   
   (d) If \( \mathbf{A} \) and \( \mathbf{B} \) have the same magnitude, under what circumstance will \( \mathbf{A} - \mathbf{B} \) have this same magnitude?
   
   (e) Show that \( \mathbf{A} \cdot (\mathbf{B} \times \mathbf{A}) \) is zero for all vectors \( \mathbf{A} \) and \( \mathbf{B} \).
   
   (f) What is the value of \( \mathbf{A} \times (\mathbf{B} \times \mathbf{A}) \)?

3. Find the angle between the two vectors \( \mathbf{A} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k} \) and \( \mathbf{B} = 3\mathbf{i} - 4\mathbf{j} - 5\mathbf{k} \).
4. Prove that two vectors must have equal magnitudes if their sum is perpendicular to their difference.

5. Obtain a unit vector perpendicular to the two vectors: \( \mathbf{A} = -\hat{i} + 2\hat{j} - 5\hat{k} \) and \( \mathbf{B} = 2\hat{i} + 3\hat{j} - 2\hat{k} \)

6. Consider any three vectors \( \mathbf{A}, \mathbf{B} \) and \( \mathbf{C} \):
   
   (a) Prove that: \( \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) \)
   
   (b) Are the two products \( \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \) and \( (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \) equal in either magnitude or direction? Prove your answer.

7. Consider a force \( \mathbf{F} = 3\hat{i} + \hat{j} + 5\hat{k} \) (newtons) acting at the point \( \mathbf{r} = 7\hat{i} + 3\hat{j} + \hat{k} \) (m).

   The torque about a point is \( \mathbf{r} \times \mathbf{F} \).
   
   (a) What is the torque in Nm about the origin? (since torque is a vector you may give your result as a linear combination of \( \hat{i}, \hat{j} \) and \( \hat{k} \)).
   
   (b) What is the torque about the point \((0, 10, 0)\)?
Chapter 3

Complex Numbers

In most cases in physics we work with the real numbers, \( \mathbb{R} \). The set of all real numbers, together with the rules for addition and multiplication, form a mathematical entity known as a field. All fields behave in very similar ways mathematically.

There are times, however, when we may wish to do something for which the real numbers are either cumbersome or inadequate. The classic example is solving the equation \( x^2 + 1 = 0 \). There is no real number which satisfies this equation, so the solution to \( x^2 + 1 = 0 \) is defined to be a number \( i \). This \( i \) is not a real number but does exist. Note that by definition \( i^2 + 1 = 0 \) and so \( i^3 = -i \) and \( i^4 = 1 \). Thus all powers of \( i \) can be expressed as either real numbers or real numbers multiplied by \( i \).

We now define a new field \( \mathbb{C} \), the complex numbers, as the set of all expressions of the form \( z = a + ib \) where \( a \) and \( b \) are real numbers. An unknown complex number is often denoted \( z \) much as our default real variable is often \( x \). We call \( a \) the real component of \( z \), and \( b \) the imaginary component of \( z \), and sometimes write \( a = \Re(z) \) and \( b = \Im(z) \). In handwritten work, these are usually written \( \text{Re}(z) \) and \( \text{Im}(z) \). This new field inherits the operations of addition and multiplication as much as it can from the real numbers, as follows. Let \( z = a + ib \) and \( w = c + id \). Then

\[
\begin{align*}
-z &= -a - ib, \\
z + w &= (a + c) + i(b + d), \\
zw &= ac + i(ad + bc) + i^2bd \\
&= ac - bd + i(ad + bc).
\end{align*}
\]

A very important property of the complex numbers is that every \( n^{\text{th}} \) degree polynomial has exactly \( n \) roots (not necessarily distinct) in \( \mathbb{C} \). So if we work over \( \mathbb{C} \) we shall never run into an equation such as \( x^2 + 1 = 0 \) which has no solution.

It is desirable to obtain an expression for the quotient of two numbers in the form \( x + iy \) where \( x \) and \( y \) are real. To do this, we first define the complex conjugate. If \( w = c + id \) then its complex conjugate, \( w^* \), is given by \( w^* = c - id \); in other words to take the complex conjugate we negate the imaginary component.

Multiplying \( w \) by its complex conjugate yields,

\[
w^*w = (c - id)(c + id) = c^2 - (i^2d^2) = c^2 + d^2,
\]

which is a real number. Strictly speaking, it is the complex number \( c^2 + d^2 + i0 \), but we abbreviate it and treat it as a real number.
So the quotient of two complex numbers is

\[ \frac{z}{w} = \frac{a + ib}{c + id} = \frac{(a + ib)(c - id)}{(c + id)(c - id)} = \frac{(ac + bd) + i(bc - ad)}{c^2 + d^2} = \frac{ac + bd}{c^2 + d^2} + i\frac{bc - ad}{c^2 + d^2}. \]

These definitions of multiplication and division are sometimes cumbersome. It is useful to be able to represent complex numbers in another way which made this sort of manipulation easier.

Complex numbers may be plotted on an Argand diagram, which is a two dimensional plot of imaginary part against real part. Figure 3.1 shows such a diagram, on which complex numbers are drawn like and behave like 2-D vectors. As for vectors, complex numbers can be expressed in other co-ordinate representations.

In polar form, a complex number is specified by a modulus and an argument, and written as \( z = re^{i\theta} \). \( r \) is the modulus, which is the distance of the complex number from \((0,0)\), defined by \( r = \sqrt{z\overline{z}} \). \( \theta \) is the argument, which is the angle of the number from the positive real axis, calculated using \( \tan \theta = \Im(z) / \Re(z) \) and always taking values in the interval \((-\pi, \pi]\).

In this form,

\[ \frac{r_1 e^{i\theta_1} r_2 e^{i\theta_2}}{(r_1 e^{i\theta_1})^n} = \frac{r_1 r_2 e^{i(\theta_1 + \theta_2)}}{r_1^n e^{in\theta_1}} \]

\[ \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)} \]

and

\[ (re^{i\theta})^* = re^{-i\theta} \].
The ease of use of this form makes it well suited to applications in which quantities are multiplied, such as in AC circuit theory. You will see these ideas used in January.

It is a theorem that

\[ e^{i\theta} = \cos \theta + i \sin \theta. \]

This relationship is well worth knowing and allows one to express complex numbers using trigonometric functions, which is very useful when the real or imaginary part of a number in polar form is to be found. The abbreviation cis\( \theta \) is sometimes used to mean \( \cos \theta + i \sin \theta \). If \( z = re^{i\theta} = r \cos \theta + ir \sin \theta \), then \( \Re(z) = r \cos \theta \) and \( \Im(z) = r \sin \theta \). This allows one to convert a number in polar form back to the Cartesian form \( z = x + iy \). In physics it is often useful to use complex numbers to simplify a calculation and then take the real part, which corresponds to the physical effect, to get a final result.

### 3.1 Problem Set: Complex Numbers

1. There are two complex numbers \( a = 3 + i \) and \( b = -2 + 2i \). Find and draw the following on an Argand diagram:
   
   (a) \( a, b, a+b \) and \( a-b \)
   
   (b) \( a^*, b^*, (ab)^* \) (the complex conjugates)

2. Let \( z = 1 + i\sqrt{3} \). Find \( z^{10} \).
Chapter 4

Calculus

Physical problems often involve rates of change or continuous sums. The calculus, a systematic method of solving these problems, is therefore one of the most important mathematical tools available to a physicist.

4.1 Limits

A function, \( f(x) \) is called continuous if you can draw the graph of it without taking your pen off the paper. This is the same as saying that if you consider a number \( a \) then \( f(x) \) must get very close to \( f(a) \) as \( x \) gets very close to \( a \); in mathematics this is written

\[
\lim_{x \to a} f(x) = f(a) ,
\]

and we say the limit of \( f(x) \) as \( x \) approaches \( a \) is \( f(a) \). Unfortunately the functions we work with are not always continuous, in fact \( f(a) \) may not even be defined. However \( \lim_{x \to a} (f(x)) \) may still exist, the classic example is \( \sin(x)/x \) which is not defined at \( x = 0 \) but if you draw a graph of \( \sin(x)/x \) you will see that

\[
\lim_{x \to 0} \left( \frac{\sin(x)}{x} \right) = 1 .
\]

Consider the function \( \lim_{x \to 0} \left( \frac{1}{x^2} \right) \). As you get closer and closer to \( x = 0 \) the function simply keeps increasing. In this case we say the limit is equal to infinity and write \( \lim_{x \to 0} \left( \frac{1}{x^2} \right) = \infty \). Similarly, a limit can be minus infinity.

We may also be interested in what happens to a function as \( x \) gets either very large or small. For example, \( \lim_{x \to -\infty} (e^x) = 0 \).

4.2 Differentiation

Consider lifting a 1 kg mass. If \( x \) is its height, then close to the Earth’s surface we can say that its potential energy \( U = mgx \). A plot of \( U \) vs. \( x \) is shown in Figure 4.1.

The increase in its potential energy \( U \) (denoted by \( \Delta U = U_{\text{final}} - U_{\text{initial}} \), where \( \Delta \) is the Greek symbol delta) is approximately 10 J for every 1 m increase in its height \( (\Delta x) \).
The rate of change of potential energy is then

\[
\frac{\Delta U}{\Delta x} = 10 \text{ J m}^{-1}.
\]

This is also the gradient of the graph, remembering that gradient is rise over run. Now consider a more general function \( y = f(x) \).

The rate of change of \( y \) with respect to \( x \) (the gradient) is itself changing. We need a way to find the instantaneous rate of change of the function at a particular point, say \( x_0 \). Graphically this can be done by finding the gradient of a tangent, as shown, but sometimes an algebraic solution is preferable. It can be seen that gradients of the dotted secants (lines intersecting a curve at two points) approach the gradient of the tangent as the second point approaches the first. If \( h \) denotes the horizontal separation of the secant points, then the gradient of the tangent must be

\[
\text{gradient} = \lim_{h \to 0} \frac{\text{rise}}{\text{run}} = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.
\]

This quantity is called the derivative of \( y = f(x) \) with respect to \( x \), and is denoted by \( \frac{dy}{dx} \).
\[ \frac{d}{dx} f(x) \text{ or } f'(x). \text{ Substituting } f(x) = x^2, \]

\[ \frac{dy}{dx} = \lim_{h \to 0} \frac{(x + h)^2 - x^2}{h} \]
\[ = \lim_{h \to 0} \frac{2xh + h^2}{h} \]
\[ = \lim_{h \to 0} (2x + h) \]
\[ = 2x \text{ (the } h \text{ term vanishes)} . \]

Thus the derivative at \( x = 1 \) is 2, and the derivative at \( x = -2 \) is -4. The above process is called differentiation from first principles and is most often used in physics when approximations to functions are being made. In practice, the following basic identities are used to evaluate derivatives (there are some other trigonometric identities which can be useful; they can be looked up in tables when needed):

\[ \frac{d}{dx} a = 0 \text{ if } a \text{ is a constant} \]
\[ \frac{d}{dx} x^n = nx^{n-1} \]
\[ \frac{d}{dx} e^x = e^x \]
\[ \frac{d}{dx} \ln(x) = \frac{1}{x} \]
\[ \frac{d}{dx} \sin(x) = \cos(x) \]
\[ \frac{d}{dx} \cos(x) = -\sin(x) \]

Sums, products and compositions of the functions shown above are differentiated using
the following results:

\[ \frac{d}{dx} ky(x) = k \frac{d}{dx} y(x) \]

\[ \frac{d}{dx} (u(x) + v(x)) = \frac{d}{dx} u(x) + \frac{d}{dx} v(x) \]

\[ \frac{d}{dx} (u(x)v(x)) = v(x) \frac{d}{dx} u(x) + u(x) \frac{d}{dx} v(x) \quad \text{(Product rule)} \]

\[ \frac{d}{dx} \frac{u(x)}{v(x)} = \frac{v(x) \frac{d}{dx} u(x) - u(x) \frac{d}{dx} v(x)}{v(x)^2} \quad \text{(Quotient rule)} \]

\[ \frac{d}{dx} y(x) = \frac{d}{du} y(u) \frac{d}{dx} u(x) \quad \text{(Chain rule)} \]

For example let \( f(x) = \sin(ax) \) and \( u = ax \) so \( f(x) = \sin(u) \). We know the derivative of \( \sin(u) \) from above, so we can now use the chain rule,

\[ \frac{d}{dx} \sin(ax) = \frac{d}{du} \sin(u) \frac{d}{dx} ax = \cos(u) \cdot a = a \cos(ax) . \]

Also

\[ \frac{d}{dx} \cos(ax) = -a \sin(ax) \]

\[ \frac{d}{dx} e^{f(x)} = f'(x)e^{f(x)} . \]

As an example of an application of the quotient rule we shall differentiate \( \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \).

\[ \frac{d}{d\theta} \frac{\sin(\theta)}{\cos(\theta)} = \frac{\cos(\theta) \frac{d}{d\theta} \sin(\theta) - \sin(\theta) \frac{d}{d\theta} \cos(\theta)}{\cos^2(\theta)} \]

\[ = \frac{\cos^2(\theta) + \sin^2(\theta)}{\cos^2(\theta)} \]

\[ = \frac{1}{\cos^2(\theta)} . \]

Please attempt the problems on differentiation in section 4.5 as differentiation is an essential tool in physics and only practice can make you proficient. Note that although in such examples it is common to differentiate with respect to \( x \) or \( \theta \), in practice you should be able to differentiate with respect to any variable with equal ease.
Other notation that will be encountered includes the following:

- second derivative $\frac{d^2y}{dx^2} = \frac{d}{dx} \frac{dy}{dx}$. For example take $y = x^3$, then $\frac{d^2y}{dx^2} = 6x$.

- $n^{th}$ derivative $\frac{d^n y}{dx^n}$, $n \geq 1$: $y$ differentiated with respect to $x$, $n$ times.

- $y'$ ($y$ prime), $y''$: derivatives of $y$ with respect to $x$.

- $\dot{x}$ ($x$ dot), $\ddot{x}$: derivatives of $x$ with respect to $t$ (time).

### 4.3 Partial Differentiation

If you have a function of two variables, $U(x, y)$, you cannot differentiate with respect to $x$ as described above but you can find the rate of change of $U(x, y)$ with respect to $x$ when $y$ is held constant. This rate of change is called the partial derivative of $U(x, y)$ with respect to $x$ and is denoted

$$\frac{\partial}{\partial x} U(x, y).$$

Partial derivatives are calculated simply by treating all other variables as constants and differentiating. For example, let $g(x, y) = x^2 + xy^2$ then $\frac{\partial g}{\partial y} = 2xy$ and $\frac{\partial g}{\partial x} = 2x + y^2$.

Higher order partial derivatives are denoted

$$\frac{\partial^2 g}{\partial x \partial x} = \frac{\partial^2 g}{\partial x^2},$$

$$\frac{\partial g}{\partial x} \frac{\partial g}{\partial x} = \frac{\partial^2 g}{\partial x \partial y} = \frac{\partial^2 g}{\partial y \partial x} = \frac{\partial^2 g}{\partial x \partial y}.$$

### 4.4 Algebraic Manipulation of Differentials

If we write the chain rule as $\frac{df}{dx} = \frac{df}{du} \frac{du}{dx}$, then it appears that the $du$ terms can just be cancelled although this is not mathematically rigorous. As physicists we sometimes consider $dx$ to be an infinitesimally small portion of $x$ (where $x$ represents any scalar or vector quantity), and treat it as an algebraic quantity, writing

$$F = -\frac{dU}{dx},$$

$$dU = -F \, dx.$$
We can take differentials of both sides of an equation.

\[ pV = nRT \]

\[ p \, dV = nR \, dT \quad (p, \, n, \, R, \text{ constant}) \]

or \[ V \, dp = nR \, dT \quad (V, \, n, \, R \text{ constant}) \]

or \[ p \, dV + V \, dp = nR \, dT \quad \text{(both} \, p \text{ and} \, V \text{ variables)} \]

We can solve simultaneous equations in differentials, e.g.

\[ \frac{dy}{dx} = 2 \]

\[ dx = 3 \, dr \]

\[ dy = 6 \, dr \]

Please complete these problems on differentiation before proceeding.

### 4.5 Problem Set: Differentiation

Throughout this problem set when you are asked to differentiate you should either find the derivative or partial derivative as appropriate.

1. Differentiate with respect to \( y \):
   (a) \( y^2 + C \)
   (b) \( 15y^3 - 3Dy^2 + 8y - 1 \)
   (c) \( y^2 + 2y + 1 \)
   (d) \( Ay + \frac{B}{y} \)

2. Use the chain rule to differentiate with respect to \( x \):
   (a) \( (x + 1)^2 \)
   (b) \( (x + y)^3 \)
   (c) \( 2(3x + 2)^5 \)
   (d) \( (4x - 3)^4 \)
   (e) \( (x^2 + 3h)^4 \)
   (f) \( \frac{1}{3}(2x^3 - 2)^9 \)
   (g) \( (\frac{1}{x} + 1)^{-4} \)
   (h) \( \left(\frac{1}{x^2 + 2}\right)^5 \)

3. Use the chain rule to find the following:
   (a) \( \frac{d}{d\theta} \sin(\theta + \pi) \)
   (b) \( \frac{d}{dz} \cos(2z - 3) \)
(c) \( \frac{d}{dp}e^{p^2} \)

(d) \( \frac{d}{dx}\sin(x^3 + 3x) \)

(e) \( \frac{d}{dx}F\tan(L^2) \)

(f) \( \frac{d}{dy}\tan\left(\frac{1}{y}\right) \)

(g) \( \frac{d}{dx}\ln(x^2) \)

(h) \( \frac{d}{dx}\ln(Ax^3 + 1) \)

4. Use the product rule to differentiate with respect to \( z \):

(a) \( z \sin(z) \)

(b) \( z^2e^z \)

(c) \( \sin(z) \cos(z) \)

(d) \( \sin(z) \ln(z) \)

(e) \( (3z^3 + 2z) \ln(z) \)

(f) \( \frac{\sin(z)}{z} \)

5. Find the following by first applying the product rule, then using the chain rule:

(a) \( \frac{d}{dx}\sin(x)\cos(x^2) \)

(b) \( \frac{d}{dy}3\ln(y^3)\cos(y) \)

(c) \( \frac{d}{dx}(z^3 + \mu)^4e^z \)

(d) \( \frac{d}{dx}\cos^6(\lambda)\sin^3(\lambda) \)

6. Differentiate the following by using the chain rule multiple times if necessary:

(a) \( \frac{d}{da}\sin(\sin(a^2)) \)

(b) \( \frac{d}{d\theta}\ln(\cos(\theta)) \)

(c) \( \frac{d}{dx}\sin(e^{x^3}) \)

(d) \( \frac{d}{dx}(\tan(\sin(\ln(x))))^6 \)

4.6 Integration

Consider a general curve \( y = f(x) \). Integration is concerned with finding the area \( PQRS \) under the curve, as shown in figure 4.3.
The area can be divided into strips and the area of a strip from \( x_1 \) to \( x_2 \) can be approximated by \( f(x_1)(x_2 - x_1) \). Thus if the area is divided into \( n \) strips by the points \( x_0 (= P), x_1, x_2, ..., x_n (= Q) \) the total area

\[
A \simeq \sum_{i=0}^{n-1} f(x_i)(x_{i+1} - x_i) = f(x_0)(x_1 - x_0) + f(x_1)(x_2 - x_1) + \cdots + f(x_{n-1})(x_n - x_{n-1}) ,
\]

where \( \sum \) (the Greek uppercase letter sigma) denotes a sum. The limit of the sum as \( n \) tends to infinity (so the strips become very narrow) gives the area exactly. We call this limit the *integral from \( P \) to \( Q \) of \( f(x) \) with respect to \( x \) and denote it by

\[
\int_{P}^{Q} f(x) \, dx = \lim_{n \to \infty} \left( \sum_{i=0}^{n-1} f(x_i)(x_{i+1} - x_i) \right) .
\]

In practice integrals are not found using this type of sum except to find an approximate numeric value. Instead we use the Fundamental Theorem of Calculus,

\[
\int_{a}^{b} \frac{d}{dx} F(x) \, dx = F(b) - F(a) .
\]

If we have a function \( f(x) \) then any function \( F(x) \) such that \( F'(x) = f(x) \) is called an antiderivative of \( f(x) \). If \( G(x) = F(x) + C \) where \( C \) is constant then \( G'(x) = F'(x) = f(x) \) so \( G(x) \) is also an antiderivative of \( f(x) \). Note that \( [F(x)]_a^b = F(b) - F(a) \) by definition; this notation is widely used when evaluating integrals.

We also define the *indefinite integral* which, in contrast to the *definite integral* discussed above, has no limits. This is defined by

\[
\int f(x) \, dx = F(x) + C ,
\]

where \( F(x) \) is any antiderivative of \( f(x) \) and \( C \) is an arbitrary constant. Indefinite integrals describe the anti-derivatives of a function and are useful for tabulating integrals and practicing integration; in solving physical problems definite integrals are almost always required.
Table of standard integrals

\[ \int a f(x) \, dx = a \int f(x) \, dx \]

\[ \int (f(x) + g(x)) \, dx = \int f(x) \, dx + \int g(x) \, dx \]

\[ \int x^n \, dx = \frac{x^{n+1}}{n+1} + C \quad (\text{for } n \neq -1) \]

\[ \int \frac{1}{x} \, dx = \ln(x) + C \]

\[ \int e^x \, dx = e^x + C \]

\[ \int \cos(x) \, dx = \sin(x) + C \]

\[ \int \sin(x) \, dx = -\cos(x) + C \]

For example, the indefinite integral

\[ \int x^2 \, dx = \frac{1}{3} x^3 + C \]

and the definite integral

\[ \int_1^2 x^2 \, dx = \left. \frac{1}{3} x^3 \right|_1^2 \]

\[ = \frac{1}{3} (2^3 - 1^3) \]

\[ = \frac{7}{3} . \]

4.7 More difficult integrals

Integration is generally more difficult than differentiation. Two integration techniques are discussed below; substitution, which is a consequence of the chain rule for differentiation, and integration by parts which is a consequence of the product rule. Some integrals cannot be expressed in terms of elementary functions.
4.7.1 Substitution

Sometimes an expression can be substituted for the integration variable, simplifying the integrand. This is effectively the reverse of the chain rule for differentiation. Take for example

$$\int x \sin(x^2 + 1) \, dx .$$

First one needs to find a simpler expression for part of the integral, such as setting $x^2 + 1 = u$. But once this has been substituted into the integrand, $\int x \sin(u) \, dx$ cannot be evaluated until the relationship between $du$ and $dx$ is known. This is found by differentiating $u$ with respect to $x$. Here, $\frac{du}{dx} = 2x$, and thus one has $dx = \frac{du}{2x}$. Once this has been substituted into the integral,

$$\int \frac{1}{2} \sin(u) \, du = -\frac{1}{2} \cos(u) + C = -\frac{1}{2} \cos(x^2 + 1) + C .$$

It is necessary to substitute back for the original variable in the last step. If it were a definite integral then it would be evaluated after we have substituted the original variable, or the limits of integration would have been recalculated in terms of $u$.

Using this technique we can evaluate a more general class of functions than given earlier. For instance, take

$$\int e^{ax} \, dx .$$

In this case take $u = ax$, then $\frac{du}{dx} = a$, and we can say $dx = \frac{1}{a} du$, then perform the integral,

$$\int \frac{1}{a} e^{u} \, du = \frac{e^{u}}{a} + C = \frac{e^{ax}}{a} + C .$$

Another, more general, example is

$$\int \frac{f'(x)}{f(x)} \, dx .$$

Set $u = f(x)$, then $dx = \frac{du}{f(x)}$, and substitute to get

$$\int \frac{du}{u} = \ln(u) + C = \ln(f(x)) + C .$$
Also:
\[
\int \cos(ax) \, dx = \frac{1}{a} \sin(ax) + C ,
\]
\[
\int \sin(ax) \, dx = -\frac{1}{a} \cos(ax) + C .
\]

Sometimes a useful substitution may appear to make the integral more complex. For example we can use a trigonometric substitution to perform the integral
\[
\int \frac{1}{\sqrt{4 - x^2}} \, dx .
\]
Let \( x = 2 \sin(\theta) \) then \( dx = 2 \cos(\theta) \, d\theta \) with \( \theta = \sin^{-1}\left(\frac{1}{2}x\right) \). The integral becomes
\[
\int \frac{1}{\sqrt{4 - x^2}} \, dx = \int \frac{1}{\sqrt{4 - 4 \sin^2(\theta)}} \, 2 \cos(\theta) \, d\theta
\]
\[
= \int \frac{1}{2 \cos(\theta)} \, 2 \cos(\theta) \, d\theta
\]
\[
= \theta + C
\]
\[
= \sin^{-1}\left(\frac{1}{2}x\right) + C .
\]

Being able to choose an appropriate substitution is a skill which is only acquired with practice. Note that with definite integrals the limits could be transformed into the new variable (using the appropriate equation), or the original variable substituted prior to evaluation, as stated earlier.

### 4.7.2 Integration by Parts

This is useful when the integrand can be expressed as the product of one function for which an antiderivative is known and another function which has a simple derivative. Derived from the product rule for differentiation, the integration by parts formula is
\[
\int v(x) \ u'(x) \, dx = u(x)v(x) - \int u(x) \ v'(x) \, dx .
\]
Take for example
\[
\int x \ e^x \, dx .
\]
We know already that the antiderivative of \( e^x \) is \( e^x \), which is no more complicated; this leaves us with \( x \) which has the derivative 1. So by applying the integration by parts result the problem will be simplified. Thus we set

\[
\begin{align*}
v(x) &= x & u'(x) &= e^x \\
v'(x) &= 1 & u(x) &= e^x ,
\end{align*}
\]

which yields

\[
\int x e^x \, dx = xe^x - \int 1 \cdot e^x = xe^x - e^x + C .
\]

For a definite integral the limits are evaluated for each term once no further integration is to be performed upon it.

Sometimes integration by parts may be applied even when it does not appear possible at first. A well known example is \( \int \ln(x) \, dx \). Let \( u(x) = \ln(x) \) and so \( u'(x) = \frac{1}{x} \), and let \( v'(x) = 1 \) with \( v(x) = x \). Integrating,

\[
\int \ln(x) \, dx = \int 1 \ln(x) \, dx = x \ln(x) - \int x \frac{1}{x} \, dx = x \ln(x) - x + C .
\]

### 4.7.3 Recurring integrals

Consider the integral \( \int \sin(x)e^x \, dx \), and integrate by parts with \( v(x) = \sin(x) \) and \( u'(x) = e^x \) (so \( v'(x) = \cos(x) \) and \( u(x) = e^x \)), giving

\[
\int \sin(x) \, e^x \, dx = \sin(x)e^x - \int \cos(x) \, e^x \, dx .
\]

Integrating by parts again, using \( v(x) = \cos(x) \) and \( u'(x) = e^x \) so \( v'(x) = -\sin(x) \) and \( u(x) = e^x \) gives

\[
\int \sin(x) \, e^x \, dx = \sin(x)e^x - \left( \cos(x)e^x - \int -\sin(x)e^x \, dx \right) .
\]

If we let \( I = \int \sin(x)e^x \, dx \), then we have

\[
I = \sin(x)e^x - \cos(x)e^x - I ,
\]

so

\[
\int \sin(x)e^x \, dx = I = \frac{1}{2}e^x \left( \sin(x) - \cos(x) \right) .
\]
It is not unusual to use a process of integration which leads to a recurring integral.

4.8 Taylor Series

Consider a function \( y = f(x) \). Close to any point, the tangent at that point approximates the function, so

\[
y \approx f(x_0) + f'(x_0)(x - x_0)
\]

By considering higher derivatives, better approximations to the function can be found; as the number of derivatives considered tends to infinity the approximate function tends to \( f(x) \). We call this limit the Taylor series of \( f(x) \) about \( x_0 \). The Taylor series expansion for a function is given by

\[
f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \cdots
\]

\( f^{(n)}(x_0) \) is \( \frac{d^n}{dx^n} f(x) \) evaluated at \( x_0 \) and \( n \) factorial is \( n! = n \cdot (n-1) \cdots 2 \cdot 1 \).

A useful example of a Taylor series is that for the function \( f(x) = e^x \) expanded about \( x = 0 \). For all \( n = 0, 1, \ldots \) \( f^{(n)}(x) = e^x \) so \( f^{(n)}(0) = 1 \), giving

\[
e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots
\]

Using the result from complex numbers that \( e^{ix} = \cos x + i \sin x \) we can find the Taylor series for \( \cos x \) and \( \sin x \):

\[
\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots
\]

\[
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots
\]

4.9 Miscellania

- Whatever mathematical tricks you may be able to perform, there is no way of evaluating \( \int y \, dx \) unless you can find the relationship between \( y \) and \( x \). Look for this relationship in the physical problem.

- The following result is useful in kinematics.

\[
\frac{d}{dx} \left( \frac{1}{2} v^2 \right) = \frac{d}{dv} \left( \frac{1}{2} v^2 \right) \frac{dv}{dx} = v \frac{dv}{dx} = \frac{dx}{dt} \frac{dv}{dx} = \frac{dv}{dt}
\]

- \( \int ds \) gives displacement, while \( \int |ds| \) is distance. Be careful not to use a scalar when you want a vector or vice versa.
• If $ds$ represents the length of a curve segment, then by Pythagoras’ Theorem $ds^2 = dx^2 + dy^2$, hence the length of a curve is

$$s = \int_C \sqrt{dx^2 + dy^2}$$

$$= \int_C \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx .$$

• It is important to recognize that when a function gives a negative value $dA$ becomes negative, and thus the contribution to the integral becomes negative. Thus, to find the total absolute area between a curve and the $x$-axis, one must integrate the absolute value of the integrand. This often means that the integral must be split into parts (between zeroes of the function) and the absolute value of each integral added.

4.10 Problem Set: Integral Calculus

1. Perform the following integrals.

   (a) $\int (x^3 + 2) \, dx$

   (b) $\int (3y^2 + 2y) \, dy$

   (c) $\int_0^3 (x^2 + 2x + 1) \, dx$

   (d) $\int_{-1}^1 (x + x^2) \, dx$

   (e) $\int_0^{\pi/2} \sin(\theta) \, d\theta$

   (f) $\int_0^{2\pi} \cos(\phi) \, d\phi$

   (g) $\int_2^3 (e^x + 1) \, dx$

   (h) $\int (Az^2 + \frac{B}{z}) \, dz$

   (i) $\int \sin(\theta) \cos(\theta) \, d\theta$

2. Use substitution to integrate:

   (a) $\int (x + 1)^6 \, dx$

   (b) $\int (2x + 3)^4 \, dx$

   (c) $\int \sin(\theta + \phi) \, d\theta$

   (d) $\int 2x(x^2 + 1)^3 \, dx$

   (e) $\int xe^{x^2} \, dx$

   (f) $\int (3x^2 + 2) \cos(x^3 + 2x) \, dx$

   (g) $\int \frac{y}{y^2 + 1} \, dy$
(h) \( \int \cos(\phi)e^{\sin(\phi)} \, d\phi \) \hspace{1cm} (j) \( \int \frac{\sin(\theta)}{\cos^4(\theta)} \, d\theta \)

(i) \( \int \frac{e^x}{1 + e^x} \, dx \) \hspace{1cm} (k) \( \int \tan(x) \, dx \)

(hint: write \( \tan(x) \) as a fraction)

3. Use integration by parts to perform the following integrals.

(a) \( \int x \cos(x) \, dx \) \hspace{1cm} (c) \( \int x \ln(x) \, dx \)

(b) \( \int y^2 e^y \, dy \)

4. Perform the integral

\[ \int \frac{1}{\sqrt{1 - x^2}} \, dx \]

Hint: use a trigonometric substitution.
Chapter 5

Simple differential equations

5.1 A brief overview

A differential equation is an equation involving derivatives. We have already solved many differential equations using integration. For example if

$$\frac{dA}{dx} = f(x) ,$$

then integrating both sides with respect to \( x \) gives

$$A = \int f(x) \, dx + C .$$

There are many types of differential equations. A first order differential equation is one involving first derivatives, a second order differential equation has second and first derivatives, etc. Some first order differential equations can be solved using simple methods while others can be more complicated to solve.

*Separable* differential equations can be solved by considering the differentials as algebraic quantities and rearranging them so that each side of the equation is written in terms of only one variable, with the differentials in the numerators. The solution is then found by integrating both sides.

Consider a situation in which the acceleration of a body is proportional to its velocity,

$$\frac{dv}{dt} = kv . \quad (5.1)$$

We rearrange such that all terms in \( v \) are on one side and all terms in \( t \) on the other,

$$\frac{dv}{v} = k \, dt .$$
Now integrating both sides yields

\[ \int \frac{dv}{v} = \int k \, dt \]

\[ \ln(v) = kt + C \]

\[ v = e^{kt+C} \]

\[ v = e^C e^{kt} \]

When \( t = 0, v = e^C = v_0 \) so

\[ v = v_0 e^{kt} \quad (5.2) \]

To check a solution to a differential equation substitute it back into the original differential equation and make sure that both sides are equal. In the above case, differentiating 5.2 and substituting it into 5.1 gives

\[ \frac{dv}{dt} = v_0 k e^{kt} \]

\[ kv = kv_0 e^{kt} \]

Thus, the differential equation has been satisfied.

Second order differential equations (involving second derivatives) are generally solved by substituting ‘trial solutions’ which are known to satisfy the differential equation. This will be learned with practice; however note that the following important equation (of Simple Harmonic Motion, or SHM)

\[ m \frac{d^2x}{dt^2} + kx = 0 \]

has the solution \( x = A \sin(\omega t + \phi) \), where \( \omega = \sqrt{k/m} \), \( A \) and \( \phi \) are constants relating to the initial state of a system (an \( n^{\text{th}} \) order differential equation has a solution with \( n \) arbitrary constants). You should substitute to verify that this solution satisfies the equation.
Chapter 6

Calculus in Higher Dimensions

What we have done in calculus so far has been in one dimension or variable, for instance integrating along a line. It is often necessary to integrate in more than one dimension in order to find a physical quantity.

6.1 Multivariate Integration

In order to integrate a function in multiple dimensions one variable is required for each dimension. In two dimensions we are interested in integrating a function over a certain region of a surface. We have a function which takes different values on different infinitesimally small area elements, and we wish to add up the total value of the function over some region $A$. This is denoted

$$\iint_A f \, dA .$$

In order to evaluate this, it is desirable to choose variables to describe position on the surface, and use the theory developed for one dimension. One such choice for a plane is the Cartesian co-ordinates $x$ and $y$, in which case $dA = dx\,dy$ (we decompose the region into tiny squares) and the integral becomes

$$\iint_A f(x, y) \, dx\,dy .$$

Next the boundary of the region $A$ needs to be represented as a function of $x$ and $y$. The region of integration defines limits on the values of $x$ and $y$, which are ordinary limits of one-dimensional integrals. This is not always an easy task. As a first example, let $A$ be the rectangle with one corner at the origin, and the opposite corner at $(a, b)$. Then integrating over this region means that we want to integrate over all points at which $0 \leq x \leq a$ and $0 \leq y \leq b$. So the double integral becomes two single integrals, one over $x$ and one over $y$, and we know the limits. So either

$$\iint_A f(x, y) \, dx\,dy = \int_0^a \left( \int_0^b f(x, y) \, dy \right) \, dx ,$$

or

$$\iint_A f(x, y) \, dx\,dy = \int_0^b \left( \int_0^a f(x, y) \, dx \right) \, dy .$$
Which should we choose? If this made a difference to the result, then this method would not work. Fortunately, in very many cases (and for our purposes in all cases) a theorem called *Fubini’s Theorem* applies and tells us that it doesn’t matter; the two give the same result. Choose the order that is most convenient. When integrating we treat all variables other than the one over which we are integrating as constants, and integrate as in one dimension. We repeat the process for the second integral.

As a second, more complicated example, consider instead the region $A$ to be a right-angled triangle with its corners at $(0, 0)$, $(a, 0)$ and $a, b$, as in figure 6.3. First we integrate over $y$, so fix $x$ and look at the allowed range of $y$-values. $y$ may range from 0 to $bx/a$ in this case, and so the limits for the inner integral over $y$ are 0 to $bx/a$, which depends on $x$. This is not a problem because performing this integral converts all expressions in $x$ and $y$ to expressions in $x$ alone. In general each integral may depend on variables over which integrals have not yet been performed, but not the current variable of integration. The outer integral is performed over $x$, from 0 to $a$ as that corresponds to the geometry.

We could have chosen to take the integral over $x$ first. If we do so, we fix $y$, and then $x$ may range from $ay/b$ to $a$. So in this case Fubini’s Theorem tells us that

$$
\int_0^a \left( \int_0^{bx/a} f(x, y) \, dy \right) \, dx = \int_0^b \left( \int_{ay/b}^a f(x, y) \, dx \right) \, dy , \quad (6.1)
$$

and either of these expressions may be chosen to evaluate the double integral. Sometimes one will be easier to perform than the other; note though that you must be careful to define the region over which you are integrating and change the limits accordingly if you wish to interchange the order of integration.

In some cases the integrand can be expressed as a product of functions of a single variable. For example taking $x$ and $y$ as the co-ordinates, let $f(x, y) = g(x)h(y)$. Then

$$
\int \int_A f(x, y) \, dxdy = \int_a^b \left( \int_c^d f(x, y) \, dy \right) \, dx \\
= \int_a^b \left( \int_c^d g(x)h(y) \, dy \right) \, dx \\
= \int_a^b g(x) \left( \int_c^d h(y) \, dy \right) \, dx \\
= \int_a^b g(x) \left( H(d) - H(c) \right) \, dx \\
= \left( H(d) - H(c) \right) \int_a^b g(x) \, dx \\
= \left( H(d) - H(c) \right) \left( G(b) - G(a) \right) .
$$

The integrals have been separated and evaluated exactly as they would be in one dimension, using the Fundamental Theorem of Calculus.
6.2 Line Integrals

Sometimes in physics we need to integrate a function along a line. This just means that the variable of integration is constrained to a known path. The most important case of this which you will encounter, other than arc length which is mentioned in the calculus chapter, is the case of a vector field integrated along a line.

By this we mean that the component of the vector in the direction of the path at each point is to be summed; this type of construction should immediately suggest a dot product. Denoting the path by \( C \), the vector function by \( \mathbf{F} \) and the differential displacement along the path by \( ds \), this type of line integral is written

\[
\int_C \mathbf{F} \cdot ds .
\]

6.3 Area and volume integrals

The length \( x \) of a line between \( S \) and \( T \) is given by

\[
x = \int_T^S dx = S - T ,
\]

and we can extend this to find the area \( A \) of a region, writing

\[
A = \iint_{\text{region}} dA .
\]

The double integral signs denote that integration is to be performed over two variables. We need to specify the region and parametrize \( dA \), the infinitesimal area element. We want to choose the shape of the area element to best fit the boundary. One choice is to split the area into little squares of side lengths \( dx \) and \( dy \), so \( dA = dx \cdot dy \). Consider a rectangle with side lengths \( a \) and \( b \) in the \( x \)- and \( y \)-directions respectively, as in figure 6.1.

![Figure 6.1: A rectangle split into infinitesimal pieces](image)

Then we can specify the region as the limits of two integrals,
\[
\int\!\int_{\text{rectangle}} dA = \int\!\int_{\text{rectangle}} \, dx \, dy = \int_{0}^{b} \left( \int_{0}^{a} \, dx \right) \, dy .
\]

We can separate these integrals as discussed in the previous section. Evaluating the \( x \) integral gives \( a \) and the \( y \) integral \( b \), and

\[ A = ab \]

as expected. We could also choose the area element such that \( dA = b \, dx \), as in figure 6.2.

![Figure 6.2: A rectangle split into infinitesimal strips](image)

We have performed one integral without writing it down. We can then evaluate the other,

\[ A = \int_{0}^{a} b \, dx = ab . \]

In general to find areas we need to divide the area under consideration into strips of known length. Take for instance a right-angled triangle, split up into infinitesimal strips as in Figure 6.3.

![Figure 6.3: A triangle split into infinitesimal strips](image)

The height of each strip is \( y \), and the width \( dx \). Thus \( dA = y \, dx = (bx/a) \, dx \). Then

\[ A = \int\!\int_{\text{triangle}} dA = \frac{b}{a} \int_{0}^{a} x \, dx = \frac{1}{2} ab . \]
Compare this with the result from before, equation 6.1, with $f(x, y) = 1$.

Volumes are calculated in a similar way. One decomposition of the volume $V$ is into infinitesimal cubes so that $dV = dx\, dy\, dz$. Consider a cube of side length $s$. Its volume is

$$V = \iiint_{\text{cube}} dV = \int_0^s \left( \int_0^s \left( \int_0^s \, dx \right) \, dy \right) \, dz = s^3.$$  

As discussed with respect to areas, sometimes one or more integrals can be evaluated before anything is written. Consider a cylinder along the $z$ axis with radius $r$ and height $h$. The volume can be split into infinitesimally thin disks of radius $r$ and thickness $dz$. Then $dV = \pi r^2 dz$, and

$$V = \iiint_{\text{cylinder}} dV = \int_0^h \pi r^2 \, dz = \pi r^2 h.$$  

Next, consider a right-angled cone.

![Figure 6.4: Dividing up a right-angled cone](image)

Divide the cone into vertical slices as shown in Fig. 6.4. The cross-section of each slice is a circle of radius $y$, area $\pi y^2$, and the thickness is $dx$, so the volume $dV$ is given by

$$dV = \pi y^2 \, dx = \pi x^2 \, dx \quad \text{(right-angled: } y = x),$$
and

\[
V = \int_0^L \pi x^2 \, dx
\]

\[
= \pi \left[ \frac{1}{3} x^3 \right]_0^L
\]

\[
= \frac{1}{3} \pi L^3 .
\]

### 6.4 Integration in polar co-ordinates

In problems with rectangular symmetry we use the cartesian co-ordinates \(x, y\) and \(z\). In problems with circular or spherical symmetry it is more convenient to use spherical polar co-ordinates \(r, \theta\) and \(\phi\).

#### 6.4.1 Two dimensional polar co-ordinates

It is necessary to write the area element in polar co-ordinates. In Cartesian co-ordinates, \(dA = dx \, dy\), but care must be taken when relating \(dA\) to the differentials \(dr\) and \(d\theta\). Figure 6.5 depicts the geometry.

![Figure 6.5: A circle split into infinitesimal rectangles](image)

Consider the region swept out by changing \(r\) by \(dr\), and \(\theta\) by \(d\theta\). The arc length of the curved sides is \(r \, d\theta\) (\(\theta\) in radians, as always), and is approximately the same on both sides as \(dr\) is infinitesimal. Also, since \(d\theta\) is infinitesimal the arc approximates a straight line. So the area is that of a rectangle with sides \(r \, d\theta\) and \(dr\),

\[
dA = r \, dr \, d\theta .
\]

This is the area element in plane polar co-ordinates. This can be used to derive the area
of a circle, radius $R$. So $r$ can range from 0 to $R$, and $\theta$ can take the values from 0 to $2\pi$.

$$
A = \int \int_{\text{circle}} dA = \int_0^{2\pi} \int_0^R r \, dr \, d\theta
$$

$$
= \left( \int_0^{2\pi} d\theta \right) \times \left( \int_0^R r \, dr \right)
$$

$$
= 2\pi \times \frac{1}{2} R^2
$$

$$
= \pi R^2.
$$

Note that the constant $\pi$ in this equation comes from the definition of the radian.

### 6.4.2 Cylindrical polar co-ordinates

Cylindrical polar co-ordinates are polar co-ordinates in two dimensions and Cartesian in the third. The co-ordinate variables are often $(r, \theta, z)$ and the volume element is

$$dV = r \, dr \, d\theta \, dz.$$

You might like to try to derive the volume of a cylinder or a cone from this. (Hint: most of the triple integral was done in the last example.)

### 6.4.3 Spherical polar co-ordinates

Spherical co-ordinates are more complicated again than two dimensional polar co-ordinates. The volume element is derived in Appendix A. If a system is described by $(r, \theta, \phi)$ being the distance from the centre of a sphere, the angle from the north pole and the angle around the equator from some reference direction, then

$$dV = r^2 \sin(\theta) \, dr \, d\theta \, d\phi.$$

The $r^2$ seems necessary on dimensional grounds, but what about the $\sin(\theta)$? Imagine the Earth described by polar co-ordinates. Then $\theta$ near 0 means that the position is close to the north pole, $\theta = \frac{\pi}{2}$ is the equator, and $\theta = \pi$ the south pole. $\sin(\theta)$ is small near the poles and large near the equator. Moving around the Earth, i.e. changing $\phi$, the longitude, causes a greater motion along the surface near the equator than near the poles, so it encompasses more volume for the same change in $\phi$. The $\sin(\theta)$ makes the volume elements small near the poles and larger at the equator.

These co-ordinates are often convenient. For instance, in deriving the volume of a sphere
we integrate over the three polar co-ordinates \( r, \theta \) and \( \phi \): 

\[
dV = (dr)(r d\theta)(r \sin(\theta) d\phi)
\]

\[
V = \int_0^R \int_0^\pi \int_0^{2\pi} r^2 \sin(\theta) d\phi d\theta dr
\]

\[
= \left[ \int_0^R r^2 dr \right] \left[ \int_0^\pi \sin(\theta) d\theta \right] \left[ \int_0^{2\pi} d\phi \right]
\]

\[
= \left[ \frac{1}{3} R \right] \left[ -(-1) - (-1) \right] [2\pi - (0)]
\]

\[
= \frac{4}{3} \pi R^3 .
\]

### 6.5 Centre of Mass

Firstly, we need to define what we mean by “centre of mass”. The geometric centre of an object might be an easier starting point. The geometric centre is the ‘average position’ of a shape. We know what it is intuitively, for instance the geometric centre of a sphere is just its centre, but it is desirable to define it formally.

The average value of a set of quantities is the sum of all elements divided by the number of elements. For example, the average value of the set of \( p_i \) \([p_1, p_2, \ldots, p_n]\) is

\[
\frac{\sum_{i=1}^n p_i}{n}.
\]

Note that \( n = \sum_{i=1}^n (1) \).

One can also define a weighted average, in which some of the \( p_i \)'s are weighted more heavily than others. They are weighted by some set of \( w_i \), and the weighted average of the \( p_i \) weighted by the \( w_i \) is

\[
\frac{\sum_{i=1}^n p_i w_i}{\sum_{i=1}^n w_i}.
\]

Note that the denominator is the sum over the weighting elements, so the standard average has \( w_i = 1 \).

Using these ideas, consider a collection of \( n \) points on a line, and label their positions by \( x_i \). Then the geometric centre of those points is

\[
x_{g.c.} = \frac{\sum_{i=1}^n x_i}{n}.
\]

The centre of mass is the average position weighted by mass. If there were a mass \( m_i \) at
each point \( x_i \), then the centre of mass would be

\[
x_{cm} = \frac{\sum_{i=1}^{n} x_i m_i}{\sum_{i=1}^{n} m_i}.
\]

Note that the denominator is the total mass. For an example of where the geometric centre and centre of mass lie, look at figure 6.6.

![Figure 6.6: The positions of the geometric centre and the centre of mass for \( m_2 \) several times larger than \( m_1 \).](image)

In the case of a continuous line of points, or a continuum of mass (e.g. anything solid) there are infinitely many infinitesimal points of mass. The sum is replaced by an integral, and the average is taken in the same way,

\[
x_{cm} = \frac{\int x \, dm}{\int dm}.
\]

In one dimension, the mass is characterized by a ‘linear mass density’, mass per unit length, sometimes denoted \( \lambda \). Then \( dm = \lambda \, dx \), and the integral can be evaluated if \( \lambda(x) \) is known.

If point masses as well as a continuum are present, then

\[
x_{cm} = \sum_{i=1}^{n} x_i m_i + \int x \, dm
\]

where \( m \) is the total mass.

To find the centre of mass of a body in three dimensions we find each component of the centre of mass separately; \( \mathbf{x}_{cm} = (x_{cm}, y_{cm}, z_{cm}) \) and we determine \( x_{cm}, y_{cm} \) and \( z_{cm} \) separately.

\( x_{cm} \) is found by adding up all the \( x \, dm \) terms over the entire mass and dividing the result by the total mass,

\[
x_{cm} = \frac{\int x \, dm}{\int dm}.
\]

The denominator is the total mass.
When finding the centre of mass, it is convenient to write $dm = \rho dV$, where $\rho$ is the density; $\rho$ may be a function of position or it may be constant. The volume integral is converted into a triple integral and performed as described earlier.

Symmetry is an invaluable tool in physics. If an object is symmetric about some point in the $x$ direction, then that point must be the $x$ component of the centre of mass.

6.5.1 Example: Centre of mass of a rod

Consider a rod of length $L$.

If the rod has uniform density, then

$$m = \int_{\text{rod}} dm = \int_0^L \lambda dx = L\lambda$$

and

$$\int_{\text{rod}} x dm = \int_0^L \lambda x dx = \frac{1}{2}L^2\lambda,$$

so

$$x_{cm} = \frac{1}{2}L.$$

If the density is proportional to the distance from the end of the rod, $\lambda = \kappa x$, then

$$m = \int_{\text{rod}} dm = \int_0^L \kappa x dx = \frac{1}{2}L^2\kappa$$

and

$$\int_{\text{rod}} x dm = \int_0^L \kappa x^2 dx = \frac{1}{3}L^3\kappa$$

and so

$$x_{cm} = \frac{2}{3}L.$$

The centre of mass to the right (large $x$), because the density is high on that side.

6.5.2 Example: Centre of mass of a cone

What is the centre of mass of an isotropic cone of height $h$ and maximum radius $R$?
The centre of mass must lie on the axis of the cone, as it is symmetric about this axis. Let the distance from the top of the cone be $dz$, and the density of the cone be $\rho$. Then, cutting the cone into discs, we have

$$dm = \pi r(z)^2 \rho dz .$$

In a cone, $r$ is proportional to $z$. But we know that $r(h) = R$, so $r(z) = Rz/h$. Putting it all together yields

$$m = \int_{cone} dm = \int_0^h \frac{\pi R^2 \rho}{h^2} z^2 dz = \frac{\pi \rho R^2 h^3}{3h^2} = \frac{\pi \rho R^2 h}{3} ,$$

$$\int_{cone} z dm = \int_0^h \frac{\pi R^2 \rho}{h^2} z^3 dz = \frac{\pi \rho R^2 h^2}{4}$$

and finally

$$z_{c.o.m} = \frac{3}{4} h .$$

This is in the fat end of the cone, which makes sense.

You can also do this problem defining $z = 0$ at the fat end of the cone. Try it and check that it is consistent with the example above.

### 6.6 Moment of Inertia

The moment of inertia of an object about an axis is a measure of its resistance to being made to spin or stop spinning about that axis. For example, try spinning around in a swivel chair with your arms and legs held out - it is harder to rotate than if your limbs are close to your body. The moment of inertia depends on the spatial distribution of the mass in the system. When you hold your legs and arms out you have a larger moment of inertia than when you sit normally.

Bodies have different moments of inertia about different axes. The moment of inertia about a given axis is found by calculating the integral

$$I = \int r^2 dm .$$

The $r$ in this integral is not the $r$ used in spherical polar co-ordinates. It is not the distance from a single point on the axis to $dm$, it is the shortest distance from the axis to $dm$, as in cylindrical polar co-ordinates. The same method for dealing with $dm$ is used as was presented in the section on centre of mass, i.e. $dm = \rho dV$ and $dV$ is expressed in terms of co-ordinate variables. Often the total mass will be substituted into the final expression to eliminate $\rho$.

The simplest example of a moment of inertia of an extended body is that of a ring about its central axis. In this case $r = R$ is constant, and so $I = \int R^2 dm = MR^2$. 

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6.6.1 Example: Moment of inertia of a rod

Consider a rod of length $L$ and mass $M$. We want to find the moments of inertia a) about an axis perpendicular to the rod through its centre and b) about an axis perpendicular to the rod through its end, as in figure 6.7.

In case a) we set $r = 0$ at the centre of the rod, so $r = \pm L/2$ at the ends. If the rod has uniform density, then $dm = \lambda dr$, and $M = \lambda L$. Then

$$I = \int_{\text{rod}} r^2 dm$$

$$= \int_{-L/2}^{L/2} r^2 \lambda dr$$

$$= \frac{1}{3} \lambda \left( \left( \frac{L}{2} \right)^3 - \left( -\frac{L}{2} \right)^3 \right)$$

$$= \frac{1}{12} ML^2.$$

In case b), $r = 0$ at one end and $r = L$ at the other, so

$$I = \int r^2 dm$$

$$= \int_{0}^{L} r^2 \rho dr$$

$$= \frac{1}{3} \rho L^3$$

$$= \frac{1}{3} ML^2.$$

It is customary to express the answer in terms of the total mass, as this is a more readily measurable quantity than the density.
6.6.2 Example: Moment of inertia of a rectangular plate

Consider a rectangular plate of uniform density with side lengths $a$ and $b$, rotated about an axis perpendicular to the plate and passing through one corner as in figure 6.8.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{rectangle_split_up.png}
\caption{A rectangle split up with co-ordinates around the axis}
\end{figure}

By Pythagoras’ theorem $r^2 = x^2 + y^2$. Also, $dm = \sigma dA$ (where $\sigma$ is the mass per unit area) and $dA = dx \, dy$, using Cartesian co-ordinates because they are most convenient for a rectangular plate. Using the definition of moment of inertia,

\[ I = \int_{\text{plate}} r^2 \, dm \]

\[ = \int_0^b \int_0^a \sigma (x^2 + y^2) \, dx \, dy \]

\[ = \int_0^b \int_0^a \sigma x^2 \, dx \, dy + \int_0^b \int_0^a \sigma y^2 \, dx \, dy \]

\[ = \sigma \frac{1}{3} (a^3 b + ab^3) \]

\[ = \frac{1}{3} M (a^2 + b^2) \]

You will find tables of the moments of inertia of common, isotropic bodies such as cylinders, spheres, etc in textbooks, but it is important to be able to derive them if needed. Practice is important, as usual.

6.7 Problem Set: Calculus in Higher Dimensions

1. Find the centre of mass of an isotropic hemisphere.

2. Find the mass of water that can fit into a vessel described by $z = x^2 + y^2$, of height $z_{\text{max}} = 9 \, \text{cm}$. (Hint: what shape is a cross-section perpendicular to the $z$-axis?)

3. Given the moment of inertia $I = \int r^2 \, dm$, where $r$ is the distance of each element (of mass $dm$) from an axis of rotation, find the moment of inertia of

(a) a cylinder about the axis through its center and parallel to the length.
(b) a wedge about a vertical axis at a corner at the thick edge, as in figure 6.9.

(Hint: substitute for $dm$ in terms of $dr$.)

Figure 6.9: The wedge and the axis in question
Appendix A

Coordinate Systems

This appendix is a modified extract from a set of mathematics notes prepared as a reference at about the level of the Easter training school, and is reproduced here with permission. It explains with diagrams the three main coordinate systems used in physics, and goes through the derivation of the gradient from first principles for each. It is not expected that you will understand all of this material before January.

A.1 Cartesian Coordinates

The gradient of a function of a single variable is its derivative, i.e. the rate at which $f(x)$ varies as $x$ is varied. It is desirable that a gradient be defined for functions of multiple variables. This case is a bit more complicated, however, since not only the magnitude of the change at any given point, but also its direction, must be specified.

![Figure A.1: A volume element in Cartesian Coordinates](image)

The ‘gradient’ of a function is the change in the value of that function in the direction of interest divided by the length over which it changes. An infinitesimal change in the function takes place over the infinitesimal change in length, the line element, in the direction of interest. The line element in a direction along a coordinate axis is the length of the appropriate side of the volume element in the coordinate system. In figure A.1, a Cartesian volume element is shown. So for instance, in Cartesian coordinates, the gradient in the $x$-direction is

$$
\frac{F(x + dx, y, z) - F(x, y, z)}{dx} = \frac{\partial F}{\partial x}.
$$

This shows that the value of the change in the function with a differential change in $x$ is the partial derivative in that direction. The gradient of a function $F$ in any coordinate system is defined as the quantity obtained by differentiating the function in the directions of each coordinate, and adding these vectorially. In other words, in Cartesian coordinates,
the gradient of a function $F$ is

$$\nabla F = \frac{\partial F}{\partial x} \hat{x} + \frac{\partial F}{\partial y} \hat{y} + \frac{\partial F}{\partial z} \hat{z}.$$  

(A.2)

The symbol $\nabla$ is called ‘del’ or ‘nabla’ and is an operator, which means that it has no value until but operates on a function. Its expression in Cartesian coordinates is

$$\nabla = \frac{\partial}{\partial x} \hat{x} + \frac{\partial}{\partial y} \hat{y} + \frac{\partial}{\partial z} \hat{z}.$$  

(A.3)

The gradient is in the direction of most rapid change of a function with position. An example is the force vector acting on a particle in a potential field; the force acts in the direction in which the particle is most likely to ‘roll’ down the hill. Hence the force, as in one dimension, is the negative of the gradient of the potential, so in 3D space $\mathbf{F} = -\nabla U$.

### A.2 Cylindrical Polar Coordinates

![Figure A.2: A volume element in Cylindrical Polar Coordinates](image_url)

This coordinate system specifies the radius from some axis $z$, the angle around it and the distance up it. This is most useful (unsurprisingly) in problems exhibiting some form of cylindrical symmetry or axial symmetry. Here, because we have a changing angle as a parameter instead of the usual lengths, the length over which a function $F$ changes by an amount $dF$ in the $\theta$ direction is not simply $d\theta$ but $r d\theta$. So the gradient in these new coordinates is

$$\nabla F = \frac{\partial F}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial F}{\partial \theta} \hat{\theta} + \frac{\partial F}{\partial z} \hat{z}.$$  

The infinitesimal displacement is $d\mathbf{r} = dr \hat{r} + r d\theta \hat{\theta} + dz \hat{z}$ and the volume element is $dV = r dr d\theta dz$. 


Here we have the distance from the origin, $r$, the angle from the top (altitude), $\theta$, and the angle around the circle (azimuth) $\phi$. Again, because we have angles used here, the relations for the gradient will not be as simple as the Cartesian case. The gradient is

$$\nabla \mathbf{F} \cdot d\mathbf{F} = \frac{\partial F}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial F}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial F}{\partial \phi} \hat{\phi}.$$

The displacement is $d\mathbf{l} = dr \hat{r} + r d\theta \hat{\theta} + r \sin \theta d\phi \hat{\phi}$ and the volume element is given by $dV = r^2 \sin \theta \, dr \, d\theta \, d\phi$. For areas constrained to lie on the surface of a sphere, the area element is $dA = R^2 \sin \theta \, d\theta \, d\phi$. 