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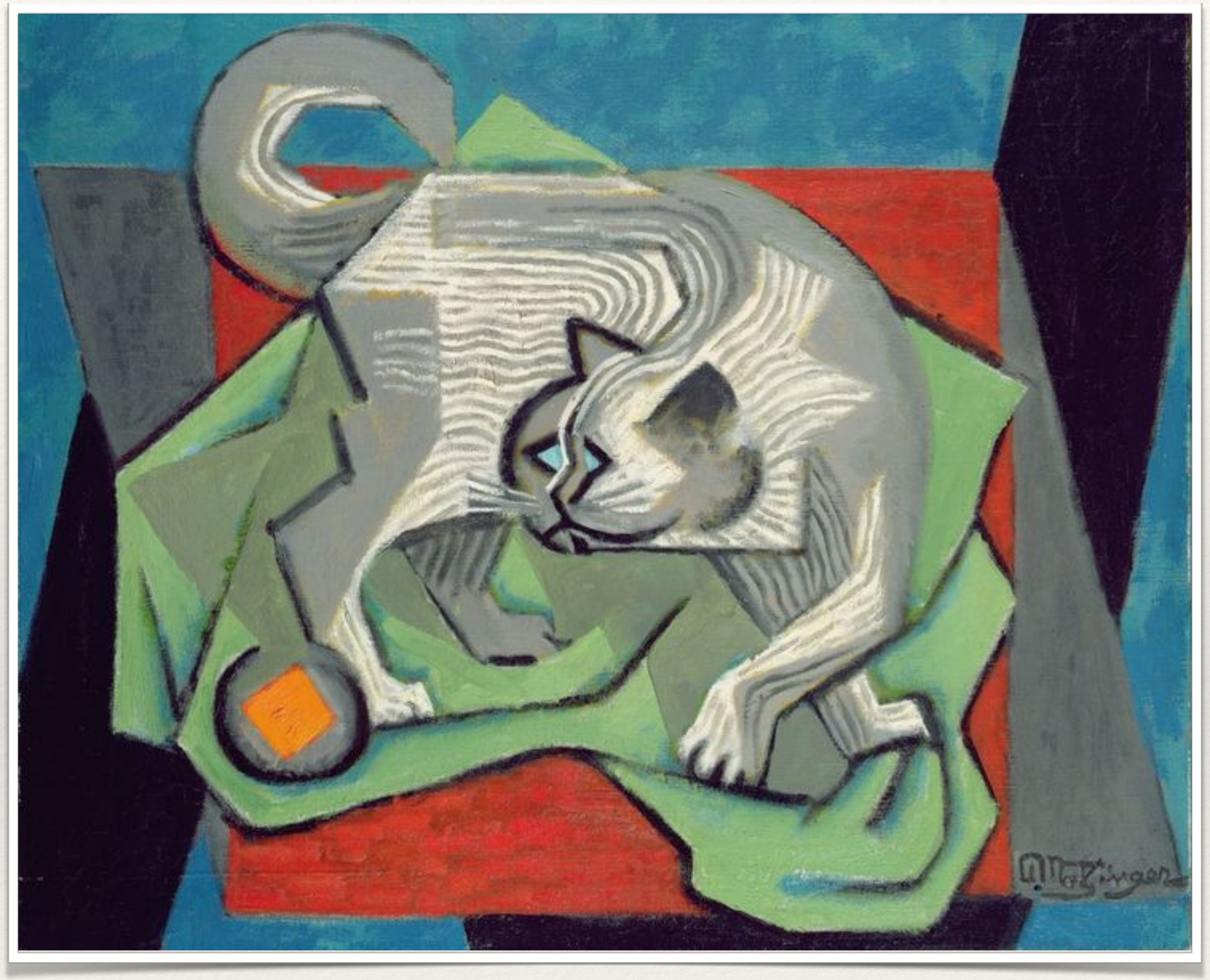
Quantum Mechanics

Lecture 5

Quiz 1;
Eigenvalue spectrum;
Angular momentum quantum numbers;
Matrix elements.



A quick recap

Rotation operators are generated by angular momentum (AM):

$$R(\theta\mathbf{n}) = \mathrm{e}^{i\theta\mathbf{n}\cdot\mathbf{J}/\hbar} \qquad \mathbf{n}\cdot\mathbf{J} = n_xJ_x + n_yJ_y + n_zJ_z$$

AM operators obey the relations:

$$[J_i, J_j] = i\hbar\sum_k \epsilon_{ijk}J_k \qquad J^2 = J_x^2 + J_y^2 + J_z^2 \qquad [J_z, J^2] = 0$$

Define raising and lowering operators that obey:

$$J_\pm = J_x \pm iJ_y \qquad J_\pm^\dagger = J_\mp \qquad [J_z, J_\pm] = \pm\hbar J_\pm \qquad [J^2, J_\pm] = 0$$

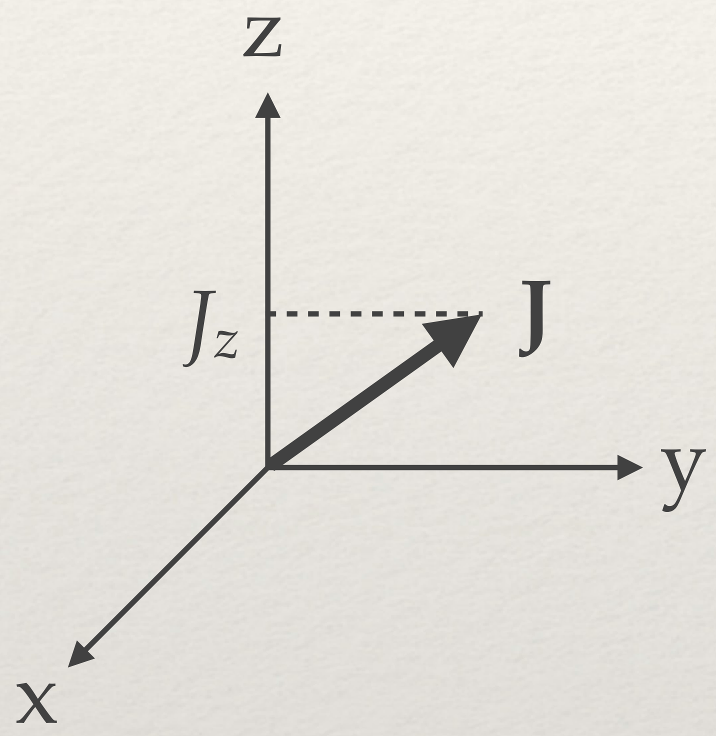
Define simultaneous eigenstates of J^2 and J_z:

$$J^2|\lambda, m\rangle = \lambda\hbar^2|\lambda, m\rangle \qquad J_z|\lambda, m\rangle = m\hbar|\lambda, m\rangle \qquad J_\pm|\lambda, m\rangle \propto |\lambda, m \pm 1\rangle$$

Eigenvalue spectrum

We can guess the physical meaning of m: it is essentially the z-component of AM.

What is the physical meaning of λ? Is it just total AM squared? We have:



"z-component squared cannot be more than total AM squared"

$$m^2 \leq \lambda$$

This follows from:

$$\langle \lambda, m | J_x^2 + J_y^2 | \lambda, m \rangle \geq 0 \iff \langle \lambda, m | J^2 - J_z^2 | \lambda, m \rangle \geq 0$$

$$\begin{aligned} \langle \lambda, m | J^2 - J_z^2 | \lambda, m \rangle &= \langle \lambda, m | \lambda \hbar^2 - m^2 \hbar^2 | \lambda, m \rangle \\ &= (\lambda - m^2) \hbar^2 \langle \lambda, m | \lambda, m \rangle \\ &= (\lambda - m^2) \hbar^2 \geq 0 \end{aligned}$$

Eigenvalue spectrum

Therefore there must exist a *maximum* value of m: call it j. We must have:

$$J_+|\lambda, j\rangle = 0$$

Otherwise we create a state having $m^2 = (j+1)^2 > \lambda$.

Now calculate:

$$J_-J_+ = (J_x - iJ_y)(J_x + iJ_y) = J_x^2 + J_y^2 + i[J_x, J_y] = J^2 - J_z^2 - \hbar J_z$$

$$J_-J_+|\lambda, j\rangle = 0 \;\Rightarrow\; (J^2 - J_z^2 - \hbar J_z)|\lambda, j\rangle = 0$$

$$\Rightarrow\; (\lambda - j^2 - j)\hbar^2|\lambda, j\rangle = 0 \qquad \Rightarrow\; \lambda = j(j+1)$$

Eigenvalue spectrum

Similarly, there must exist a *minimum* value of m: call it j'. We must have:

$$J_- |\lambda, j'\rangle = 0$$

Otherwise we create a state having $m^2 = (j'-1)^2 > \lambda$.

Now calculate:

$$J_+J_- = (J_x + iJ_y)(J_x - iJ_y) = J_x^2 + J_y^2 - i[J_x, J_y] = J^2 - J_z^2 + \hbar J_z$$

$$J_+J_- |\lambda, j'\rangle = 0 \;\Rightarrow\; (J^2 - J_z^2 + \hbar J_z) |\lambda, j'\rangle = 0$$

$$\Rightarrow\; (\lambda - j'^2 + j')\hbar^2 |\lambda, j'\rangle = 0 \qquad \Rightarrow\; \lambda = j'(j' - 1)$$

Eigenvalue spectrum

These two values of λ must be self-consistent:

$$\lambda = j(j+1) = j'(j'-1)$$

This implies:

$$j' = -j \qquad \text{or} \qquad j' = j+1$$

valid

violates our inequality, so discard it.

We can therefore exactly compute λ:

$$\lambda = j(j+1)$$

The quantity j is more fundamental than λ, so we will re-label our states as:

We'll use this notation from now on: $|\lambda, m\rangle \longrightarrow |j, m\rangle$

Note: $J^2 |j, m\rangle = j(j+1)\hbar^2 |j, m\rangle$

Quantized values

What are the allowed values of j and m? Start with $m = j$ and work down:

$$J_-|j,j\rangle = |j,j-1\rangle$$

$\downarrow$

$$J_-|j,j-1\rangle = |j,j-2\rangle$$

$\downarrow$

$$J_-|j,j-2\rangle = |j,j-3\rangle$$

$\downarrow$

…

Must reach $-j$.

$\downarrow$

$$J_-|j,-j+1\rangle = |j,-j\rangle$$

$2j+1$ total states:

$$m = j, j-1, j-2, \ \ldots\ , -j$$

Since we take an integer number of steps, the distance j-($-j$) = $2j$ must be an integer.

Allowed values for j are: $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ \ldots$

Matrix elements

What do the raising and lowering operators look like in this basis?

We know: $J_+|j,m\rangle = c_+\hbar|j,m+1\rangle$ and $J_-|j,m\rangle = c_-\hbar|j,m-1\rangle$

$$\langle j,m|J_-J_+|j,m\rangle = c_+^*c_+\hbar^2\langle j,m+1|j,m+1\rangle = |c_+|^2\hbar^2$$

We also have:

$$\langle j,m|J_-J_+|j,m\rangle = \langle j,m|J^2 - J_z^2 - \hbar J_z|j,m\rangle = \left(j(j+1) - m^2 - m\right)\hbar^2\langle j,m|j,m\rangle$$

We can always make a choice of phase such that:

$$c_+ = \sqrt{j(j+1) - m(m+1)}$$

And similarly for the lowering operator we find:

$$c_- = \sqrt{j(j+1) - m(m-1)}$$

Matrix elements

Putting these expressions together we find:

$$\langle j, m' | J_+ | j, m \rangle = c_+ \hbar \langle j, m | j, m+1 \rangle = \sqrt{j(j+1) - m(m+1)}\,\hbar \delta_{m', m+1}$$

And similarly for the lowering operator we find:

$$\langle j, m' | J_- | j, m \rangle = c_- \hbar \langle j, m | j, m-1 \rangle = \sqrt{j(j+1) - m(m-1)}\,\hbar \delta_{m', m-1}$$

From these expressions, we can write explicit matrices that act on the space of spin-j states.