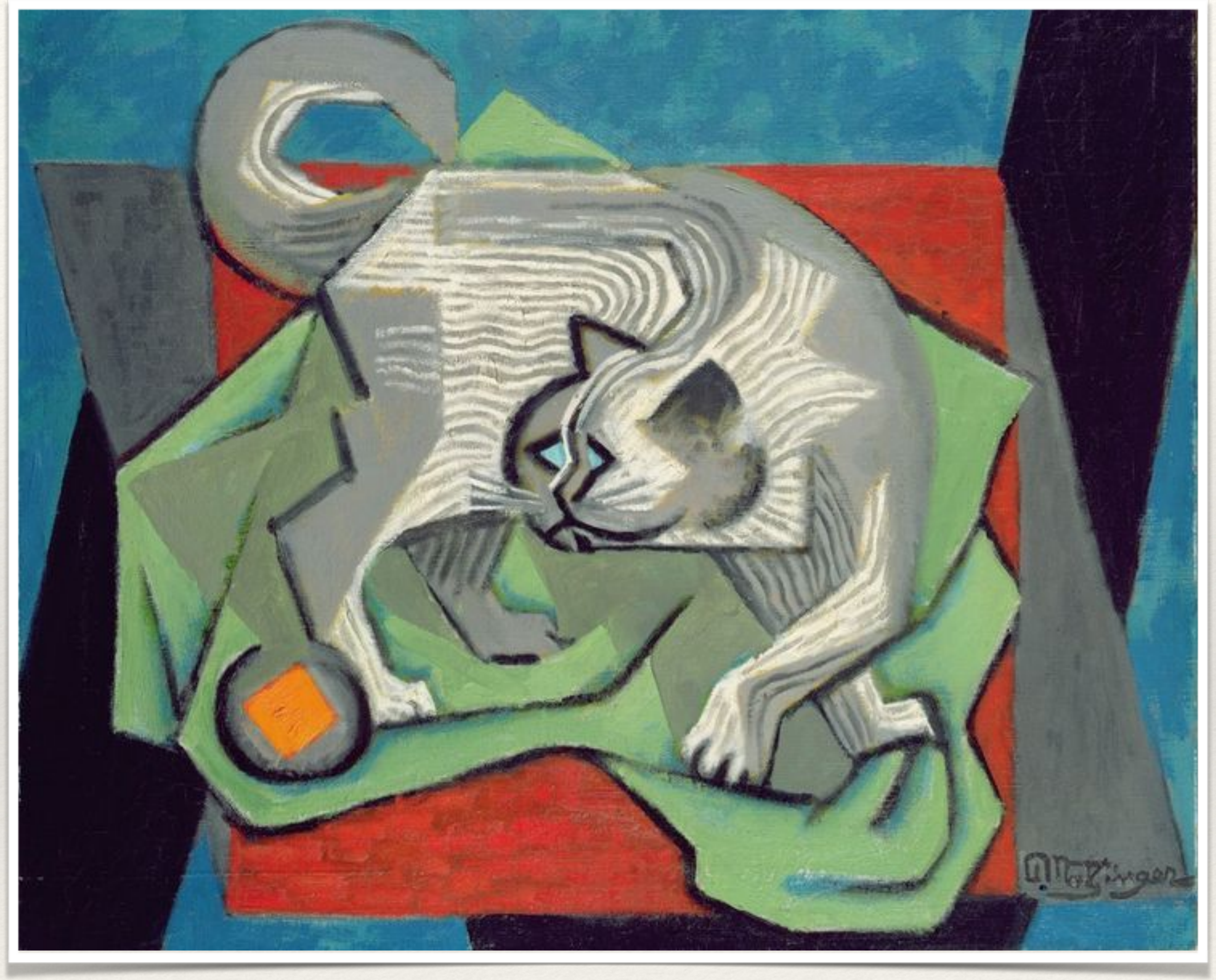
Guest lecture by Dr. Arne Grimsmo

Quantum Mechanics

Lecture 18

Harmonic oscillator redux:
Coherent states;
Quantum phase space.



Simple harmonic oscillator

The simple harmonic oscillator is one of the most important models in all of physics. Let's give a lightning review of the basics.

$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2 \qquad [\hat{x}, \hat{p}] = i\hbar$$

To solve the Hamiltonian, we introduce **creation and annihilation operators**:

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} + \frac{i}{m\omega} \hat{p} \right) \qquad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} - \frac{i}{m\omega} \hat{p} \right) \qquad [\hat{a}, \hat{a}^\dagger] = 1$$

Inverting these equations for position and momentum, we find:

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} \left(\hat{a} + \hat{a}^\dagger \right) \qquad \hat{p} = -i\sqrt{\frac{m\omega\hbar}{2}} \left(\hat{a} - \hat{a}^\dagger \right)$$

Simple harmonic oscillator

In terms of $\hat{a}^\dagger, \hat{a}$ or the **number operator** $\hat{N}$, the Hamiltonian becomes

$$H = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) = \hbar\omega\left(\hat{N} + \frac{1}{2}\right)$$

The eigenstates and energies are given in terms of the **number states**:

$$H|n\rangle = E_n|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle$$

The creation and annihilation operators act on the number states as follows:

$$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle\,, \quad \hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle\,, \quad \hat{a}^\dagger\hat{a}|n\rangle = \hat{N}|n\rangle = n|n\rangle$$

They are sometimes called **raising and lowering operators** because of this.

Simple harmonic oscillator

The number states form a complete orthonormal basis:

$$\langle m | n \rangle = \delta_{mn} \qquad \sum_{n=0}^{\infty} |n\rangle\langle n| = 1$$

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle , \quad c_n = \langle n | \psi \rangle$$

We can create them by applying the raising operator to the vacuum

$$|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}} |0\rangle$$

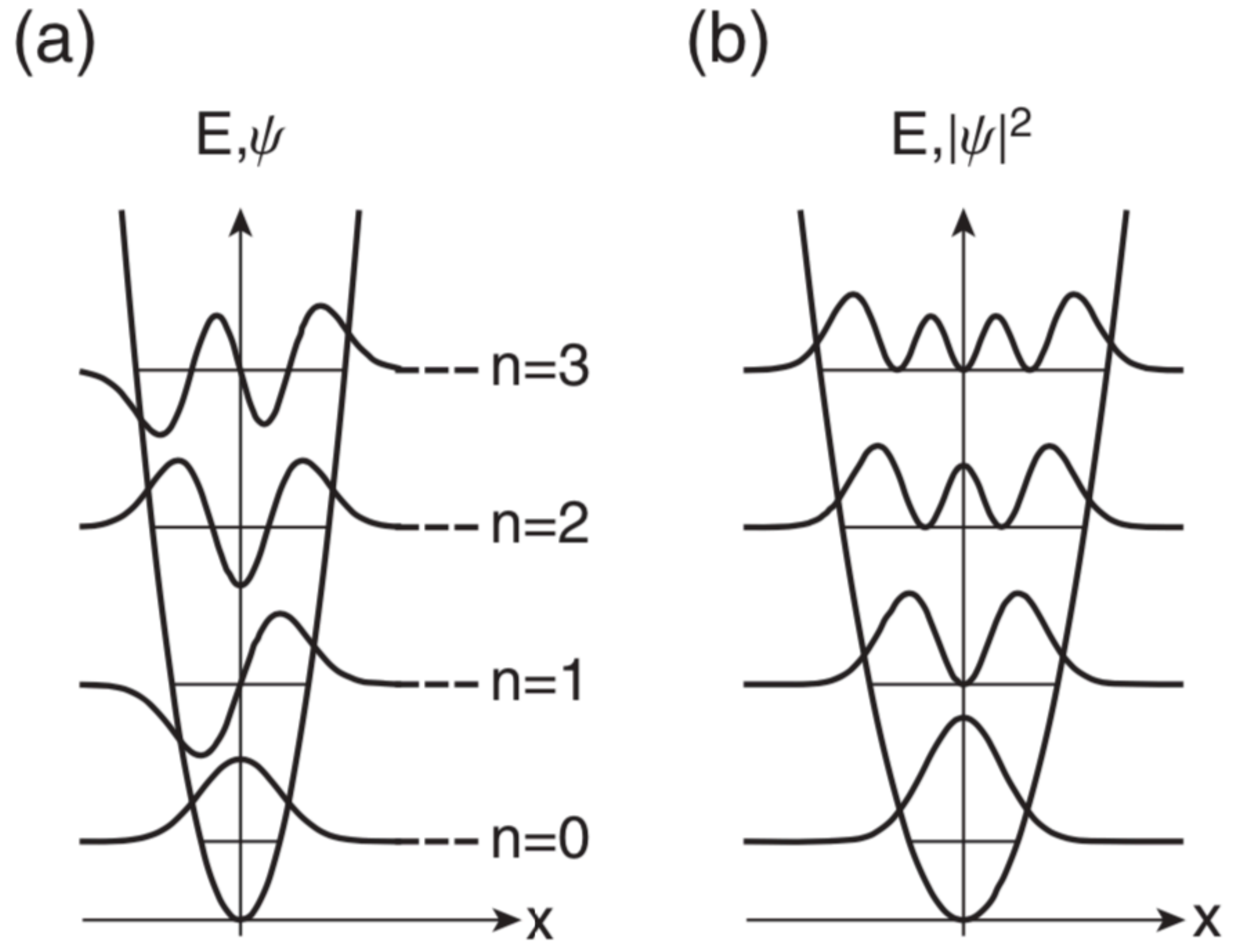


FIGURE 9.6 Energy eigenstate (a) wave functions and (b) probability densities of the harmonic oscillator.

Number state uncertainty

The uncertainty in position or momentum is easy to compute with $\hat{a}^\dagger, \hat{a}$:

$$(\Delta \hat{x})^2 = \langle n | \hat{x}^2 | n \rangle - \langle n | \hat{x} | n \rangle^2$$

Recall: $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} \left(\hat{a} + \hat{a}^\dagger \right)$

We have:

$$\langle n | \hat{x} | n \rangle \propto \langle n | \left(\hat{a} + \hat{a}^\dagger \right) | n \rangle = \left(\sqrt{n} \langle n | n-1 \rangle + \sqrt{n+1} \langle n | n+1 \rangle \right) = 0$$

Similarly, we have:

$$\begin{aligned} \langle n | \hat{x}^2 | n \rangle &= \frac{\hbar}{2m\omega} \langle n | \left(\hat{a} + \hat{a}^\dagger \right)^2 | n \rangle \\ &= \frac{\hbar}{2m\omega} \langle n | \cancel{\hat{a}^2} + \hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger + \cancel{\hat{a}^{\dagger 2}} | n \rangle \\ &= \frac{\hbar}{2m\omega} \langle n | 2\hat{N} + 1 | n \rangle = \frac{\hbar}{m\omega} \left(n + \frac{1}{2} \right) \end{aligned}$$

Recall:
$[\hat{a}, \hat{a}^\dagger] = 1$

Uncertainty and the large-n limit

A similar calculation for p shows that

$$(\Delta\hat{x})^2 = \frac{\hbar}{m\omega}\left(n + \frac{1}{2}\right) \qquad (\Delta\hat{p})^2 = m\omega\hbar\left(n + \frac{1}{2}\right)$$

As we expect from Heisenberg, even in the ground state there is uncertainty:

$$\Delta\hat{x}\Delta\hat{p} = \hbar\left(n + \frac{1}{2}\right) \geq \frac{\hbar}{2}$$

Uncertainty **increases** as n increases!

Thus, the number states cannot directly correspond to a classical limit with a well-defined mass on a spring. Perhaps we should have expected this, since they are eigenstates and have no dynamics. But it begs the question:

What are the "most classical" states of the harmonic oscillator?

Coherent states

The coherent states are defined as eigenstates of the annihilation operator:

$$\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle \qquad \alpha \in \mathbb{C}$$

Since the annihilation operator is not self-adjoint, the eigenvalue can be, and generally is, complex.

In the number basis, they look like:

$$|\alpha\rangle = C_\alpha \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\,|n\rangle$$

$C_\alpha = \mathrm{e}^{-|\alpha|^2/2}$ is a normalization constant.

The eigenvalue property follows easily:

$$\hat{a}\,|\alpha\rangle = C_\alpha \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\hat{a}\,|n\rangle = C_\alpha \sum_{n=1}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\sqrt{n}\,|n-1\rangle$$

$$= C_\alpha \sum_{n=1}^{\infty} \frac{\alpha\,\alpha^{n-1}}{\sqrt{(n-1)!}}\,|n-1\rangle = \alpha\,|\alpha\rangle$$

Coherent state time evolution

The coherent states are normalized, but **not** orthogonal:

$$\langle \alpha | \alpha \rangle = 1 \qquad \langle \beta | \alpha \rangle = C_\alpha C_\beta \mathrm{e}^{\alpha\beta^*}$$

However, they are nearly orthogonal when α or β have large magnitude.

Coherent states evolve in time as follows: $|\alpha(t)\rangle = \mathrm{e}^{-i\omega t/2} |\alpha \mathrm{e}^{-i\omega t}\rangle$

$$|\alpha(t)\rangle = \mathrm{e}^{-iHt/\hbar} |\alpha\rangle = C_\alpha \sum_{n=0}^{\infty} \frac{\alpha^n \mathrm{e}^{-i(n+1/2)\omega t}}{\sqrt{n!}} |n\rangle$$

$$= \mathrm{e}^{-i\omega t/2} C_\alpha \sum_{n=0}^{\infty} \frac{\left(\alpha \mathrm{e}^{-i\omega t}\right)^n}{\sqrt{n!}} |n\rangle = \mathrm{e}^{-i\omega t/2} |\alpha \mathrm{e}^{-i\omega t}\rangle$$

Coherent states remain coherent states under time evolution! Only α changes.

Coherent state expected values

The expected values of position and momentum change with time for a coherent state as they would for a classical mass on a spring.

$$\langle \hat{x}(t) \rangle = \sqrt{\frac{\hbar}{2m\omega}} 2|\alpha|\cos(\omega t - \phi), \quad \langle \hat{p}(t) \rangle = -\sqrt{\frac{\hbar m\omega}{2}} 2|\alpha|\sin(\omega t - \phi), \quad \alpha = |\alpha| e^{i\phi}$$

$$\begin{aligned}
\langle \alpha(t) | \hat{x} | \alpha(t) \rangle &= \sqrt{\frac{\hbar}{2m\omega}} \langle \alpha(t) | \left(\hat{a} + \hat{a}^\dagger \right) | \alpha(t) \rangle \\
&= \sqrt{\frac{\hbar}{2m\omega}} \langle \alpha e^{-i\omega t} | \left(\hat{a} + \hat{a}^\dagger \right) | \alpha e^{-i\omega t} \rangle \\
&= \sqrt{\frac{\hbar}{2m\omega}} \left(\alpha e^{-i\omega t} + \alpha^* e^{i\omega t} \right) = \sqrt{\frac{\hbar}{2m\omega}} 2|\alpha|\cos(\omega t + \phi)
\end{aligned}$$

Coherent states have minimal uncertainty

Coherent states have the minimal uncertainty allowed by quantum mechanics.

$$\langle\alpha|\hat{x}^2|\alpha\rangle = \frac{\hbar}{2m\omega}\langle\alpha|\left(\hat{a}^2 + \hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger + \hat{a}^{\dagger 2}\right)|\alpha\rangle$$

$$= \frac{\hbar}{2m\omega}\langle\alpha|\left(\hat{a}^2 + 2\hat{a}^\dagger\hat{a} + 1 + \hat{a}^{\dagger 2}\right)|\alpha\rangle$$

$$= \frac{\hbar}{2m\omega}\left(\underline{\alpha^2 + 2|\alpha|^2} + 1 + \underline{\alpha^{*2}}\right)$$

$$\langle\alpha|\hat{x}|\alpha\rangle^2 =$$

$$= \frac{\hbar}{2m\omega}(\alpha + \alpha^*)^2$$

$$= \frac{\hbar}{2m\omega}\left(\underline{\alpha^2 + 2|\alpha|^2 + \alpha^{*2}}\right)$$

The α-dependent terms cancel:

$$\Rightarrow \quad \Delta\hat{x}^2 = \langle\hat{x}^2\rangle - \langle\hat{x}\rangle^2 = \frac{\hbar}{2m\omega}$$

Similarly:

$$\Delta\hat{p}^2 = \frac{\hbar m\omega}{2}$$

$$\Rightarrow \quad \Delta\hat{x}\Delta\hat{p} = \frac{\hbar}{2}$$

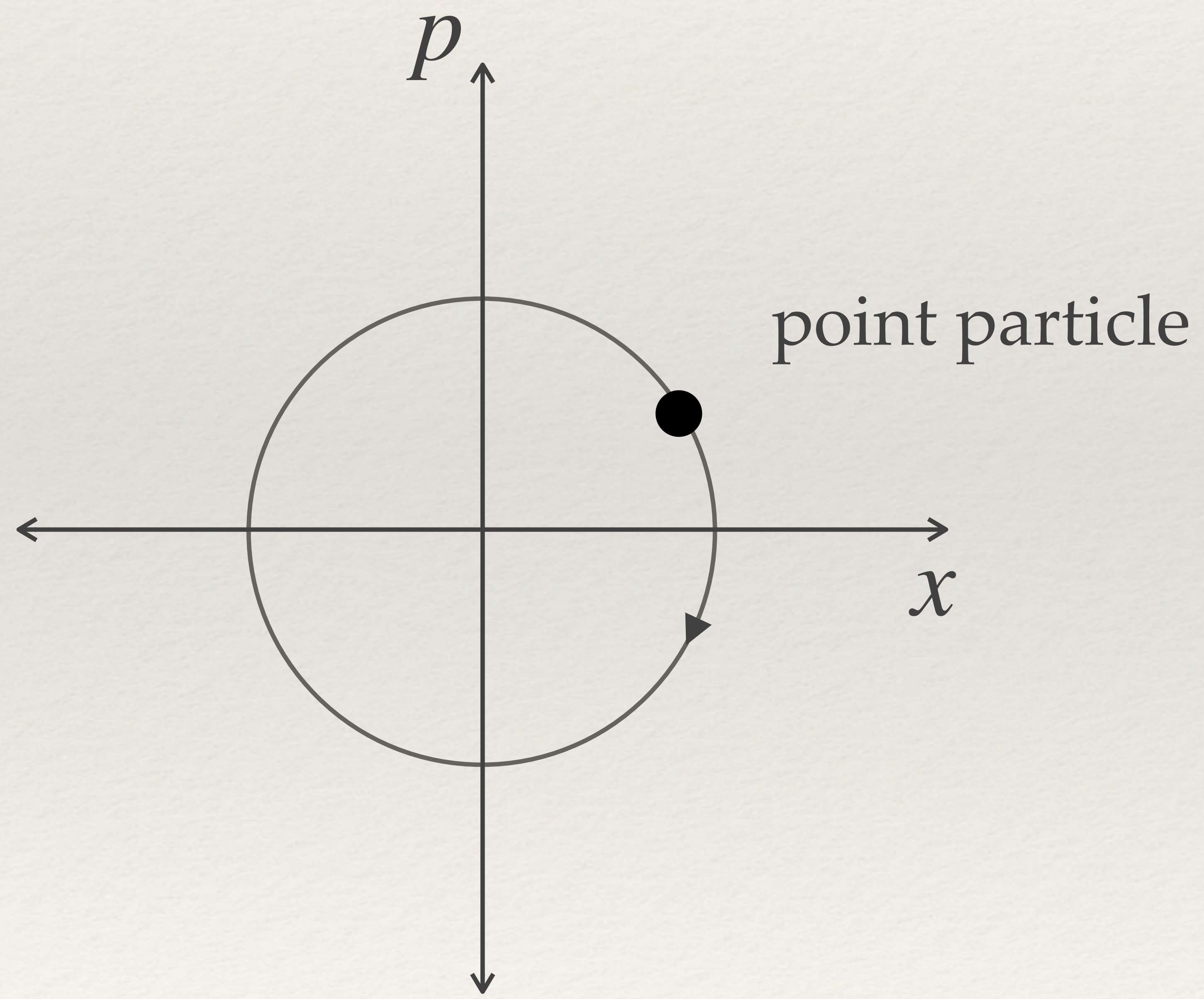
Minimal uncertainty for all α and t!

Classical vs. quantum phase space

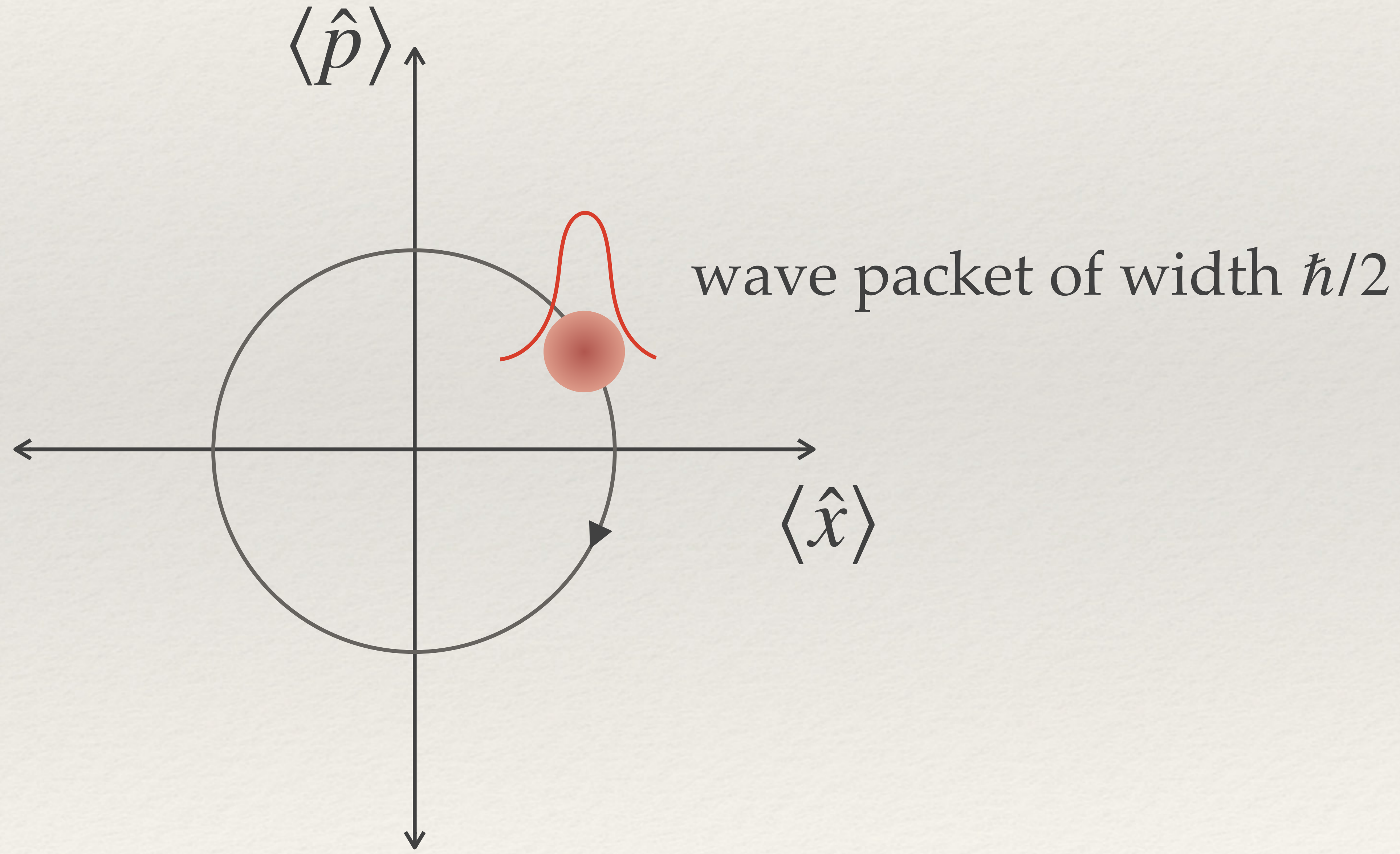
The coherent states suggest the following analogy with classical mechanics.

In classical phase space, point particles evolve along trajectories labeled by coordinates $(x(t), p(t))$.

In quantum phase space, a *distribution* evolves along trajectories labeled by *expected values*, $\left(\langle \hat{x}(t) \rangle, \langle \hat{p}(t) \rangle\right)$.



Simple harmonic oscillator trajectory in classical phase space.



Coherent state distribution evolving as a trajectory in "quantum phase space".

The Wigner function (non-examinable)

One way to make sense of quantum phase space is with the *Wigner function*. Starting from a wave function ψ, we transform it as follows:

$$W(x,p) = \frac{1}{\pi\hbar}\int_{-\infty}^{\infty} \langle x - y\,|\,\psi\rangle\langle\psi\,|\,x + y\rangle\, \mathrm{e}^{2iyp/\hbar}\,\mathrm{d}y$$

This formula can be inverted to yield

$$\psi(x) \simeq \int_{-\infty}^{\infty} W(x/2,p)\,\mathrm{e}^{ipx/\hbar}\,\mathrm{d}p$$

(equals, modulo an overall phase)

Thus, the Wigner function is a **faithful** representation of a quantum state. It allows us to visualize states and dynamics in phase space, as we will see next lecture.