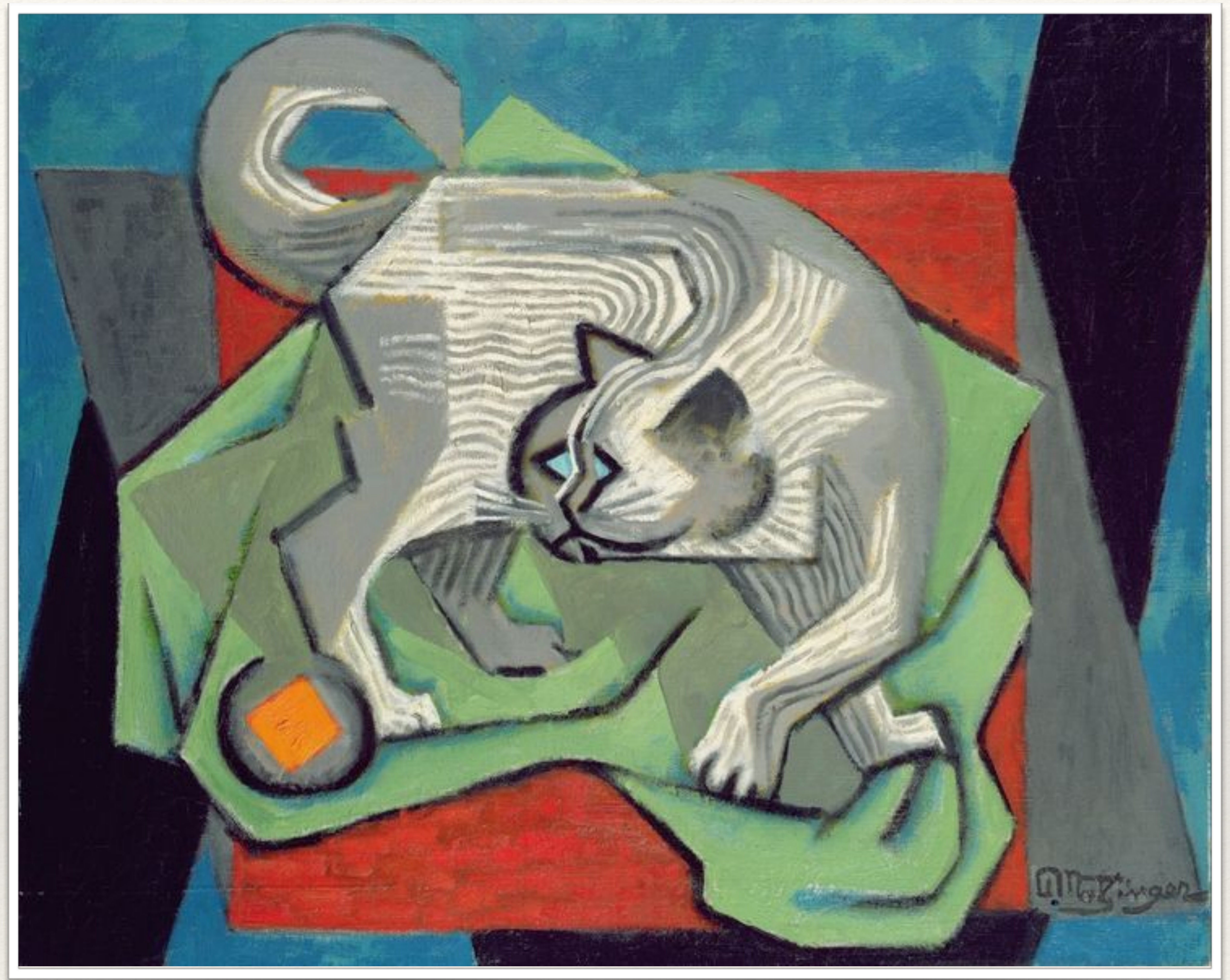
Prof. Steven Flammia

Quantum Mechanics

Lecture 2

Time evolution and the Schrödinger equation;
The Hamiltonian as the generator of time translations;
Wave functions in infinite-dimensional Hilbert spaces;
Position operator and position basis.



Commuting operators

Consider the case of two **nondegenerate** operators A and B
Suppose they are Hermitian and that they commute.

More generally, commuting Hermitian operators share a common eigenbasis.
(The proof can be done by generalizing the above argument.)

To track commutativity (or lack thereof), introduce the **commutator**:

Many nice algebraic identities…

Unitary time evolution

Let's look at the unitary operator that translates a state in time:

Recall, it must be unitary to conserve probability.

Rather than study the most general such operator, Taylor expand for small time:

Unitarity at first order in dt implies:

The Schrödinger equation

What about at large times? We can expand again, but around *t*.

Schrödinger equation,
operator form

When *H* is time-independent, the general solution is:

The Schrödinger equation

Applying both sides to some initial state $|\psi(0)\rangle$, we find

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}U(t) = H\,U(t)$$

Schrödinger equation, state vector form

Schrödinger equation, operator form

Is this still unitary for all t, not just $\mathrm{d}t$? Assuming H is independent of t:

The Hamiltonian operator

Let's continue assuming that H is independent of t.

Recall that H has units of energy.

It commutes with $U(t)$:

It is self-adjoint, so it is an observable with real eigenvalues.

What is the expected value of H?

We therefore define H to be the Hamiltonian or energy operator.

The Hamiltonian operator

What are the eigenstates of H? The *energy eigenstates*:

The energy eigenstates are “stationary” with respect to time:

Superpositions of energy eigenstates have non-trivial dynamics.
Example:

Time dependence of expected values

What about time dependence of expected values more generally?

Operators A that are independent of time are conserved iff they commute with H.

Position basis

Our derivation of the Schrödinger equation was completely general. But let's focus on a special case more challenging than spin degrees of freedom: *position*.

Unlike spin, which takes a finite set of values, position is a *continuous* variable.

In analogy with spin, let's consider a 1D line and define a position operator:

We should be able to expand any state in the *position basis*. Because position is a continuous variable, the resolution of the identity takes an integral form:

Real-space wave functions

This suggests defining the *wave function* $\psi(x)$:

What is the Born rule probability for finding the particle at x?

What is the Born rule probability for finding the particle between x and $x+\mathrm{d}x$?

Position eigenstates and the Dirac delta function

Are the eigenstates of the position operator valid wave functions?

The Dirac delta function:

The Dirac delta function is **not** a normalizable wave function, so position eigenstates are not physically realizable.

Expected values and overlaps

The trick is always to insert a resolution of the identity.