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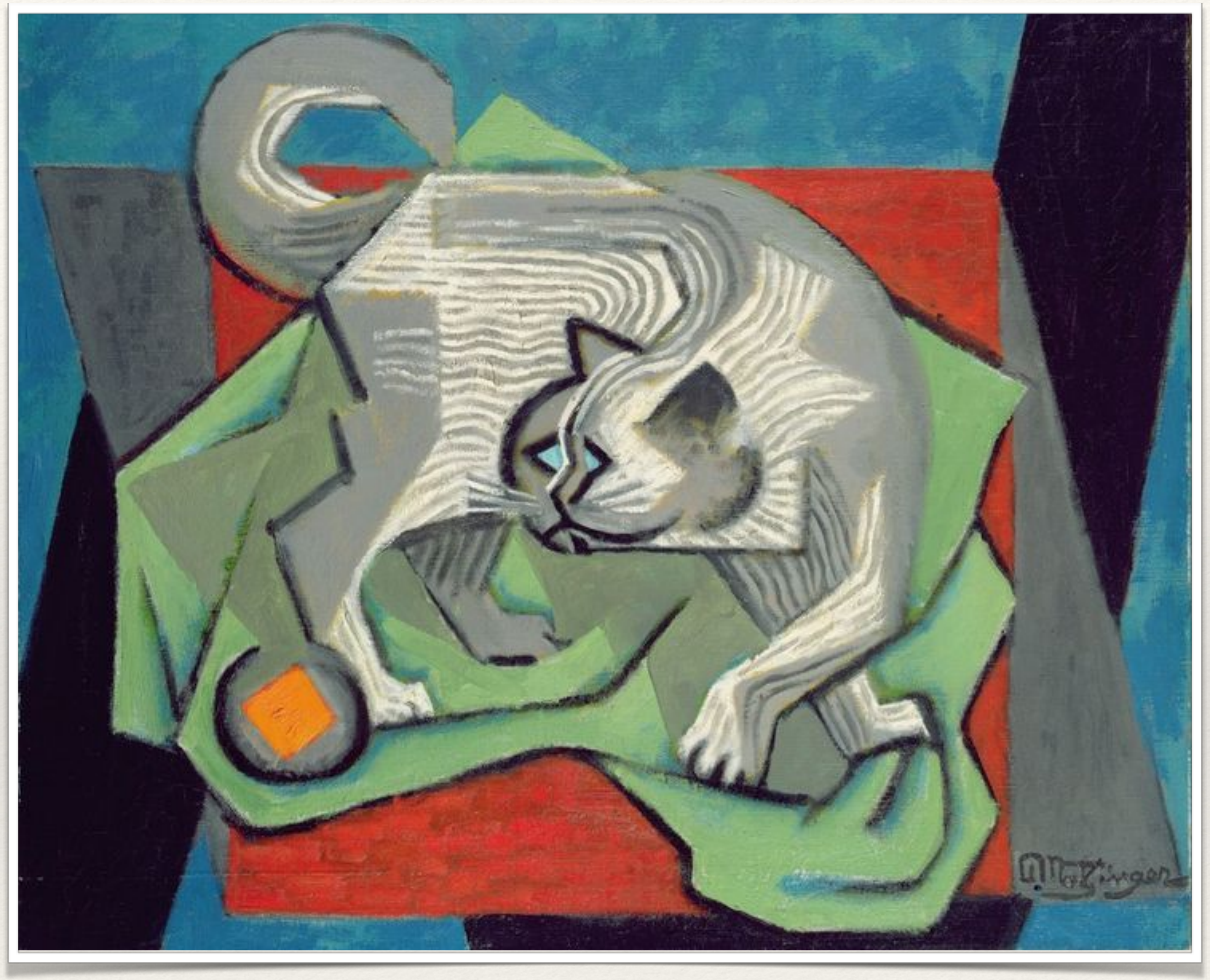
Quantum Mechanics

Lecture 3

Translation and momentum operators;
Time-independent Schrödinger equation;
Uncertainty relations;
Canonical commutation relations.



A quick recap

Time dynamics is unitary and generated by H, the Hamiltonian:

$$U(\mathrm{d}t) = 1 - \frac{i}{\hbar} H \mathrm{d}t \qquad U(t)^\dagger U(t) = 1 \qquad H = H^\dagger \qquad U(t)\,|\,\psi(0)\rangle = |\,\psi(t)\rangle$$

When H is independent of time, Schrödinger equation is solved by:

$$U(t) = \mathrm{e}^{-iHt/\hbar} \qquad U(t)U(s) = U(t+s) \qquad [H, U(t)] = 0 \qquad \langle H \rangle = \langle E \rangle = \text{const.}$$

Wave functions in the position basis and relation between position eigenstates:

$$|\,\psi\rangle = \int_{-\infty}^{\infty} \langle x\,|\,\psi\rangle\,|\,x\rangle \mathrm{d}x \;\Rightarrow\; \psi(x) := \langle x\,|\,\psi\rangle \qquad \Pr(a < x < b) = \int_a^b |\,\langle x\,|\,\psi\rangle\,|^2 \mathrm{d}x$$

$$\langle x\,|\,x'\rangle = \delta(x - x')$$ Dirac delta function

Real-space wave functions

This suggests defining the *wave function* $\psi(x)$:

What is the Born rule probability for finding the particle at x?

What is the Born rule probability for finding the particle between x and $x+\mathrm{d}x$?

Position eigenstates and the Dirac delta function

Are the eigenstates of the position operator valid wave functions?

The Dirac delta function:

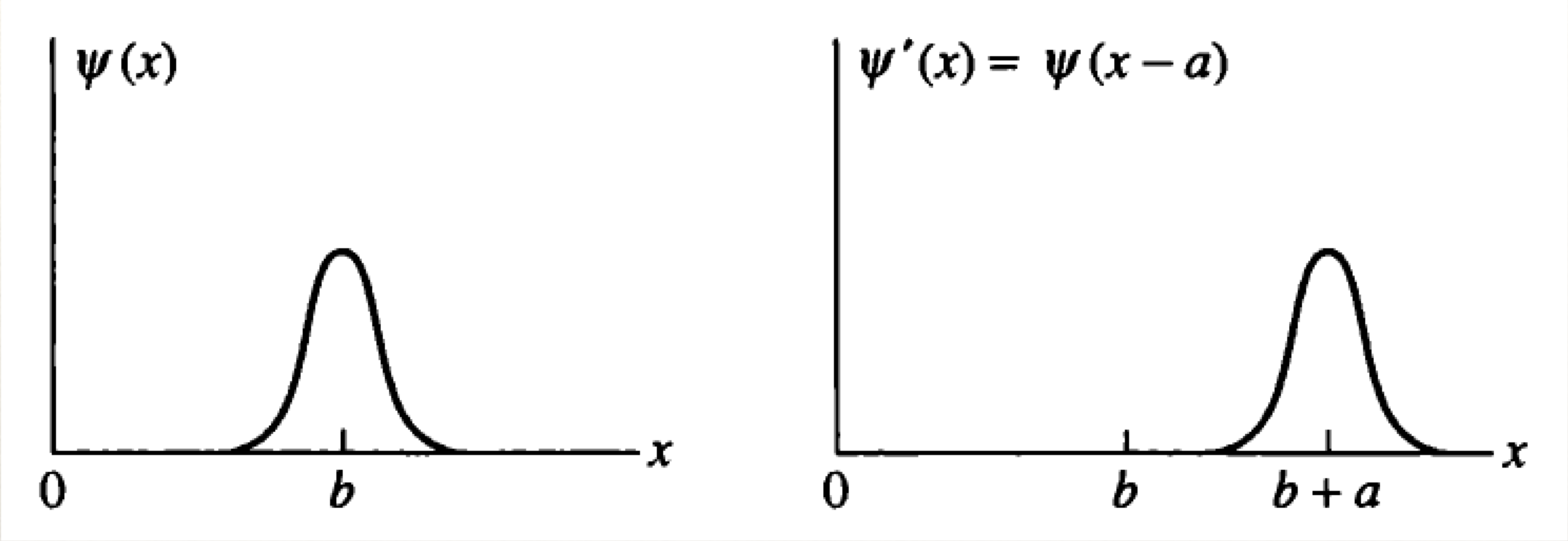
The Dirac delta function is **not** a normalizable wave function, so position eigenstates are not physically realizable.

Translation operator

A natural operation on the line is to translate a wave function.

Define:

This acts by pushing the wave function to the right by a.



This is invertible and unitary:

The wave function in the position basis transforms as:

Generator of translation

As we did with time translation, let us look at the generator of space translation.

And as before, unitarity demands that

Exponentiation gives finite-size translations:

Commutator with position

The generator of translation **does not** commute with the position operator.

This is an operator, so we can act both sides on any state.
For a position eigenstate:

Momentum operator

We can identify the generator of translation with the *linear momentum* operator.

Why? First, the units are correct:

Second, it reproduces classical "p=mv" in expectation:

Momentum operator in the position basis

How is the momentum operator represented in the position basis?

Schrödinger equation in 1D

In the position basis, we can write the Schrödinger equation for a 1D system as:

Therefore:

Time-independent Schrödinger equation in 1D

Starting with an energy eigenstate leads to the time-independent equation:

Build general solutions using superpositions of solutions for each E.

Uncertainty relations and canonical commutation relations

Using the Cauchy-Schwarz inequality, it is easy to show the Heisenberg uncertainty relation:

In more than one dimension, we have independent linear momenta that have pairwise commutation relations: