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Quantum Mechanics

Lecture 5

Quiz 1;
Eigenvalue spectrum;
Angular momentum quantum numbers;
Matrix elements.

A quick recap

Rotation operators are generated by angular momentum (AM):

$$R(\theta\mathbf{n}) = e^{i\theta\mathbf{n}\cdot\mathbf{J}/\hbar} \qquad \mathbf{n}\cdot\mathbf{J} = n_x J_x + n_y J_y + n_z J_z$$

AM operators obey the relations:

$$[J_i, J_j] = i\hbar \sum_k \epsilon_{ijk} J_k \qquad J^2 = J_x^2 + J_y^2 + J_z^2 \qquad [J_z, J^2] = 0$$

Define raising and lowering operators that obey:

$$J_\pm = J_x \pm iJ_y \qquad J_\pm^\dagger = J_\mp \qquad [J_z, J_\pm] = \pm\hbar J_\pm \qquad [J^2, J_\pm] = 0$$

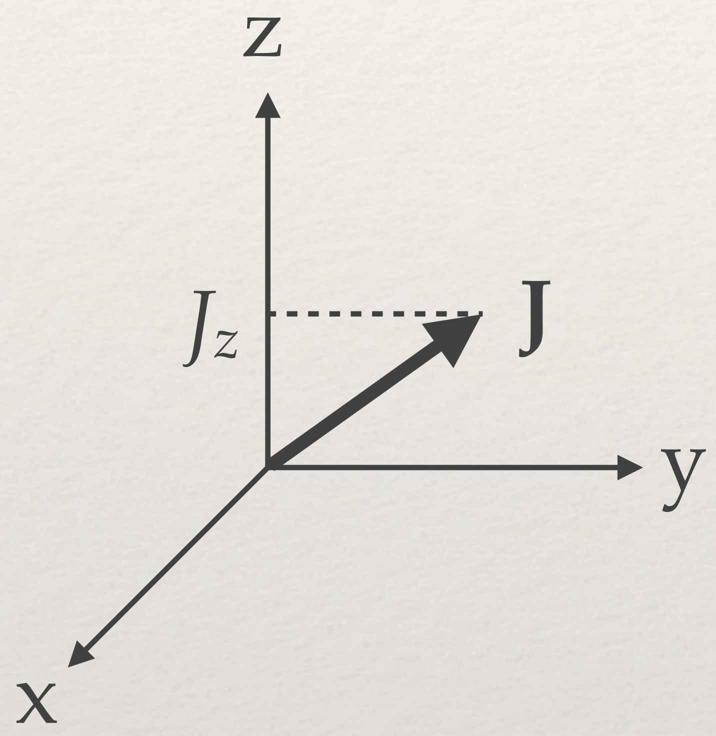
Define simultaneous eigenstates of J^2 and J_z:

$$J^2 |\lambda, m\rangle = \lambda\hbar^2 |\lambda, m\rangle \qquad J_z |\lambda, m\rangle = m\hbar |\lambda, m\rangle \qquad J_\pm |\lambda, m\rangle \propto |\lambda, m \pm 1\rangle$$

Eigenvalue spectrum

We can guess the physical meaning of m: it is essentially the z-component of AM.

What is the physical meaning of λ? Is it just total AM squared? We have:



"z-component squared cannot be more than total AM squared"

Eigenvalue spectrum

Therefore there must exist a *maximum* value of m: call it j. We must have:

Now calculate:

Eigenvalue spectrum

Similarly, there must exist a *minimum* value of m: call it j'. We must have:

Now calculate:

Eigenvalue spectrum

These two values of λ must be self-consistent:

This implies:

We can therefore exactly compute λ:

The quantity j is more fundamental than λ, so we will re-label our states as:

Quantized values

What are the allowed values of j and m? Start with $m = j$ and work down:

Allowed values for j are:

Matrix elements

What do the raising and lowering operators look like in this basis?

We know:

We also have:

We can always make a choice of phase such that:

And similarly for the lowering operator we find:

Matrix elements

Putting these expressions together we find:

And similarly for the lowering operator we find:

From these expressions, we can write explicit matrices that act on the space of spin-j states.