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Quantum Mechanics

Lecture 6

AM matrices: spin 1/2 example;
Reduction of the two-body problem;
Angular momentum revisited;
Commutation relations;
Simultaneous eigenstates.

A quick recap

Angular momentum eigenstates satisfy:

$$J^2 |j, m\rangle = j(j+1)\hbar^2 |j, m\rangle \qquad J_z |j, m\rangle = m\hbar |j, m\rangle$$

The eigenvalues are constrained:

Allowed values for j are: $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$

$2j+1$ total states: $m = j, j-1, j-2, \ldots, -j$

The matrix elements of the raising and lowering operators are:

$$\langle j, m' | J_\pm | j, m\rangle = \sqrt{j(j+1) - m(m \pm 1)}\,\hbar\,\delta_{m', m\pm 1}$$

Example: spin 1/2

Let's derive the spin operators for a spin-1/2 system using these formulas.

Recall: $\langle j, m' | J_\pm | j, m\rangle = \sqrt{j(j+1) - m(m \pm 1)}\hbar\delta_{m',m\pm 1}$

Set $j = 1/2$.

Now m = +1/2 or -1/2 only.

Example: spin 1/2

Let's derive the spin operators for a spin-1/2 system using these formulas.

Recall: $\left\langle \frac{1}{2}, m' \middle| J_\pm \middle| \frac{1}{2}, m \right\rangle = \sqrt{\frac{3}{4} - m(m \pm 1)}\,\hbar\,\delta_{m',m\pm1}$

Example: spin 1/2

Let's derive the spin operators for a spin-1/2 system using these formulas.

Recall: $J_{\pm} = J_x \pm iJ_y$

These formulas exactly recover the Pauli spin matrices in the z-basis!

Two-body Hamiltonian with interaction

Consider a Hamiltonian with two interacting particles that are otherwise free:

Position ket in 3D:

Total state space:

Total linear momentum:

The potential energy depends only on the **distance** between the particles.

Transform to center-of-mass and relative coordinates:

Total linear momentum.

Total mass.

Center-of-mass position

Relative linear momentum.

Reduced mass.

Relative position.

Reduced Hamiltonian

Rewrite the Hamiltonian in the new coordinates:

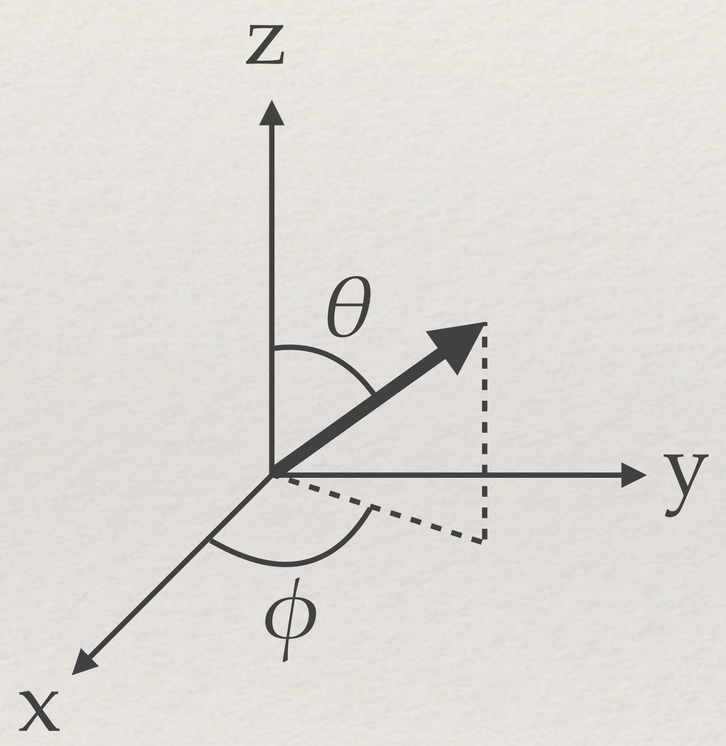
Energy eigenstates can be labeled by total momentum P:

We can always choose a co-moving frame so that:

Angular momentum operator revisited

The new Hamiltonian is radially symmetric, so we expect AM conservation.

To show this, consider an AM operator L_z and it's associated rotation operator:



Spherical coordinates

Commutation relations

Repeating the argument with cyclic symmetry, we conclude that:

This implies commutation relations with position and momentum:

Commutation relations

Repeating the argument with cyclic symmetry, we conclude that:

$$\mathbf{L} = \hat{\mathbf{r}} \times \hat{\mathbf{p}}$$

This implies commutation relations with position and momentum:

The Hamiltonian conserves AM

We have established that the Hamiltonian conserves angular momentum:

$$H = \frac{\hat{\mathbf{p}}^2}{2\mu} + V(|\hat{\mathbf{r}}|)$$

There is nothing special about the z direction… the same is true for x and y!
But L_z does not commute with L_x or L_y, so we can only choose one simultaneous symmetry.

Simultaneous eigenstates

The rotational symmetry establishes the following commutations relations:

$$H = \frac{\hat{\mathbf{p}}^2}{2\mu} + V(|\hat{\mathbf{r}}|) \qquad [L_z, H] = [L^2, H] = [L_z, L^2] = 0$$

Therefore, a simultaneous eigenbasis exists for all three of H, L^2, L_z:

Next lecture, we will see how this allows us to decouple the angular and radial parts of the wave function and solve the Schrödinger equation separately for each part.