Prof. Steven Flammia

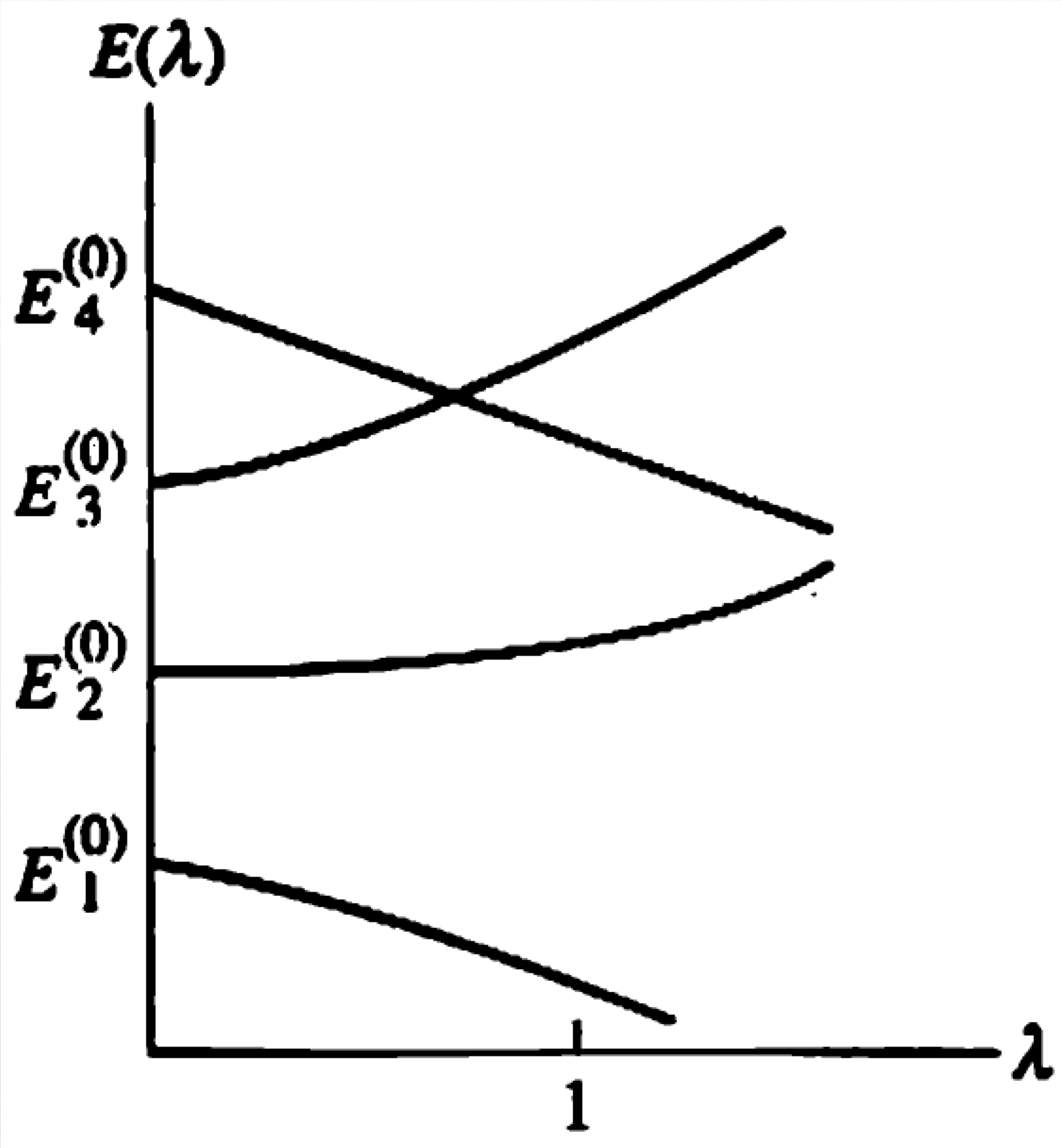
Quantum Mechanics

Lecture 9

Non-degenerate perturbation theory;
Example: quantum harmonic oscillator.

Weakly interacting systems

Consider a non-degenerate interacting system:



We want to find all of the eigenstates:

Idea: suppose λ is small and expand as a power series:

Matching term by term

Plug in the series expansion ansatz and define separate equations term by term:

First-order energy shift

Take the inner product with the zeroth-order eigenstates to derive:

$$H_0 | \phi_n^{(1)} \rangle + H_1 | \phi_n^{(0)} \rangle = E_n^{(0)} | \phi_n^{(1)} \rangle + E_n^{(1)} | \phi_n^{(0)} \rangle$$

Use nth eigenstate

First-order correction to the eigenstates

Take the inner product with the zeroth-order eigenstates to derive:

$$H_0 | \phi_n^{(1)} \rangle + H_1 | \phi_n^{(0)} \rangle = E_n^{(0)} | \phi_n^{(1)} \rangle + E_n^{(1)} | \phi_n^{(0)} \rangle$$

Use kth eigenstate, $k \neq n$.

First-order correction to the eigenstates

Now expand in the zeroth-order basis:

$$H_0 | \phi_n^{(1)} \rangle + H_1 | \phi_n^{(0)} \rangle = E_n^{(0)} | \phi_n^{(1)} \rangle + E_n^{(1)} | \phi_n^{(0)} \rangle$$

Use kth eigenstate, $k \neq n$.

First-order correction to the eigenstates

What about $\langle \phi_k^{(0)} | \phi_n^{(1)} \rangle$? Use normalization condition:

Second-order energies

The second order energies follow from similar considerations:

$$H_0 | \phi_n^{(2)} \rangle + H_1 | \phi_n^{(1)} \rangle = E_n^{(0)} | \phi_n^{(2)} \rangle + E_n^{(1)} | \phi_n^{(1)} \rangle + E_n^{(2)} | \phi_n^{(0)} \rangle$$

In general, computing the *n*th order energy shift requires knowing the (*n*-1)th order corrections to the energy.

Example: perturbed harmonic oscillator

Consider a charged particle in a 1D harmonic potential, and an applied electric field that leads to a linear potential term:

$$H = \frac{\hat{p}_x^2}{2m} + \frac{1}{2}m^2\omega^2\hat{x}^2 - q|\mathbf{E}|\hat{x} \qquad H_0 = \frac{\hat{p}_x^2}{2m} + \frac{1}{2}m^2\omega^2\hat{x}^2 \qquad H_1 = -q|\mathbf{E}|\hat{x}$$

The unperturbed energies and states are simply:

$$E_n^{(0)} = \left(n + \frac{1}{2}\right)\hbar\omega \qquad |\phi_n^{(0)}\rangle = |n\rangle$$

Now recall the harmonic oscillator raising and lowering operators:

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a} + \hat{a}^\dagger\right) \qquad \hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$$

Example: perturbed harmonic oscillator

The first order energy corrections are: $E_n^{(1)} = \langle \phi_n^{(0)} | H_1 | \phi_n^{(0)} \rangle$

We have to go to second order to see an effect: $E_n^{(2)} = \sum_{k \neq n} \frac{|\langle \phi_k^{(0)} | H_1 | \phi_n^{(0)} \rangle|^2}{E_n^{(0)} - E_k^{(0)}}$

Example: perturbed harmonic oscillator

Thus the total corrected energies to second order are:

We can check explicitly the accuracy in this case by completing the square:

The eigenstates are just translated copies of the original number states.