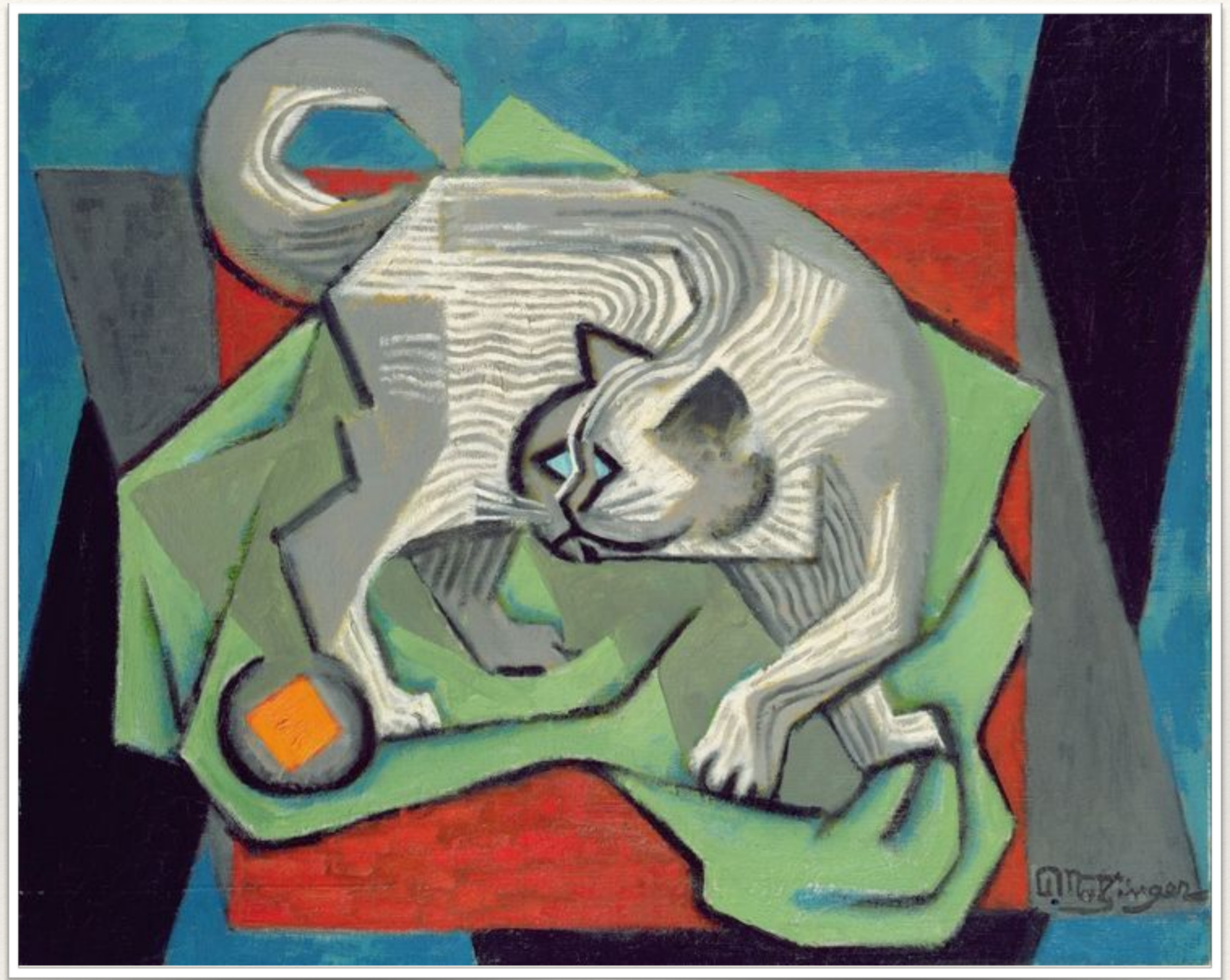
Prof. Steven Flammia

Quantum Mechanics

Lecture 11

Quiz 2;
Spin-orbit coupling;
Fine structure of Hydrogen.



A quick recap

Suppose a complicated Hamiltonian splits into two pieces,

$$H = H_0 + \lambda H_1$$

and that an eigenspace of H_0 with energy E is **degenerate** with N states.

$$|\chi_j\rangle := |\phi_E^{(0)}, j\rangle \qquad j = 1, \ldots, N \qquad H_0 |\chi_j\rangle = E |\chi_j\rangle \qquad E = E^{(0)}$$

Then the first order corrections to the eigenstates and eigenvalues are found by diagonalizing the perturbing Hamiltonian:

$$\sum_{j=1}^{N} |\chi_j\rangle\langle\chi_j| = 1_E \qquad 1_E H_1 1_E |\psi_E\rangle = E^{(1)} |\psi_E\rangle$$

Perturbations will in general break the degeneracy.

Spin-orbit coupling

Moving charges generate currents, and hence magnetic fields:

This B-field will couple to the electron spin:

This argument ignores relativistic effects. A complete treatment includes an effect called Thomas precession and gives the spin-orbit Hamiltonian:

Spin-orbit coupling

We will consider the spin-orbit interaction as a perturbation that will split the degenerate states of a hydrogen atom. To that end, define:

To do degenerate perturbation theory, we must diagonalize:

Total angular momentum is conserved

Total J and J_z also give good quantum numbers for H_{SO}:

Note:

We conclude that s, l, j and $m_j = m_l + m_s$ are **all** good quantum numbers for H_{SO}.

Degenerate perturbation theory for H_{SO}

There are now $2n^2$ degenerate states for each n (due to electron spin). We can again use symmetry to our advantage to diagonalize in each subspace.

Of the original eigenstates, which can possibly contribute as a linear combination of the new eigenstates? At most two states are consistent:

Matrix elements for H_{SO}

What are the matrix elements $\langle \chi_j | H_{SO} | \chi_k \rangle$ of H_{SO} in this basis?

Rewrite in terms of raising and lowering operators:

Matrix elements are now straightforward to calculate. Example:

Eigenvalues

The matrix in this basis (for fixed l, n) is:

Diagonalizing this gives us the new eigenstates and eigenvalues.

Eigenvectors

The eigenvectors are as follows:

Recall: $m_j = m_l + \frac{1}{2}$, $j = l \pm \frac{1}{2}$, $|\chi_1\rangle = |\, l, m_l, \frac{1}{2}, \frac{1}{2}\rangle$, $|\chi_2\rangle = |\, l, m_l + 1, \frac{1}{2}, -\frac{1}{2}\rangle$

As in the Stark effect, the new energy eigenstates will now have distinct energies.

Energy corrections

The final step is to compute the corrections to the energy at 1st order: