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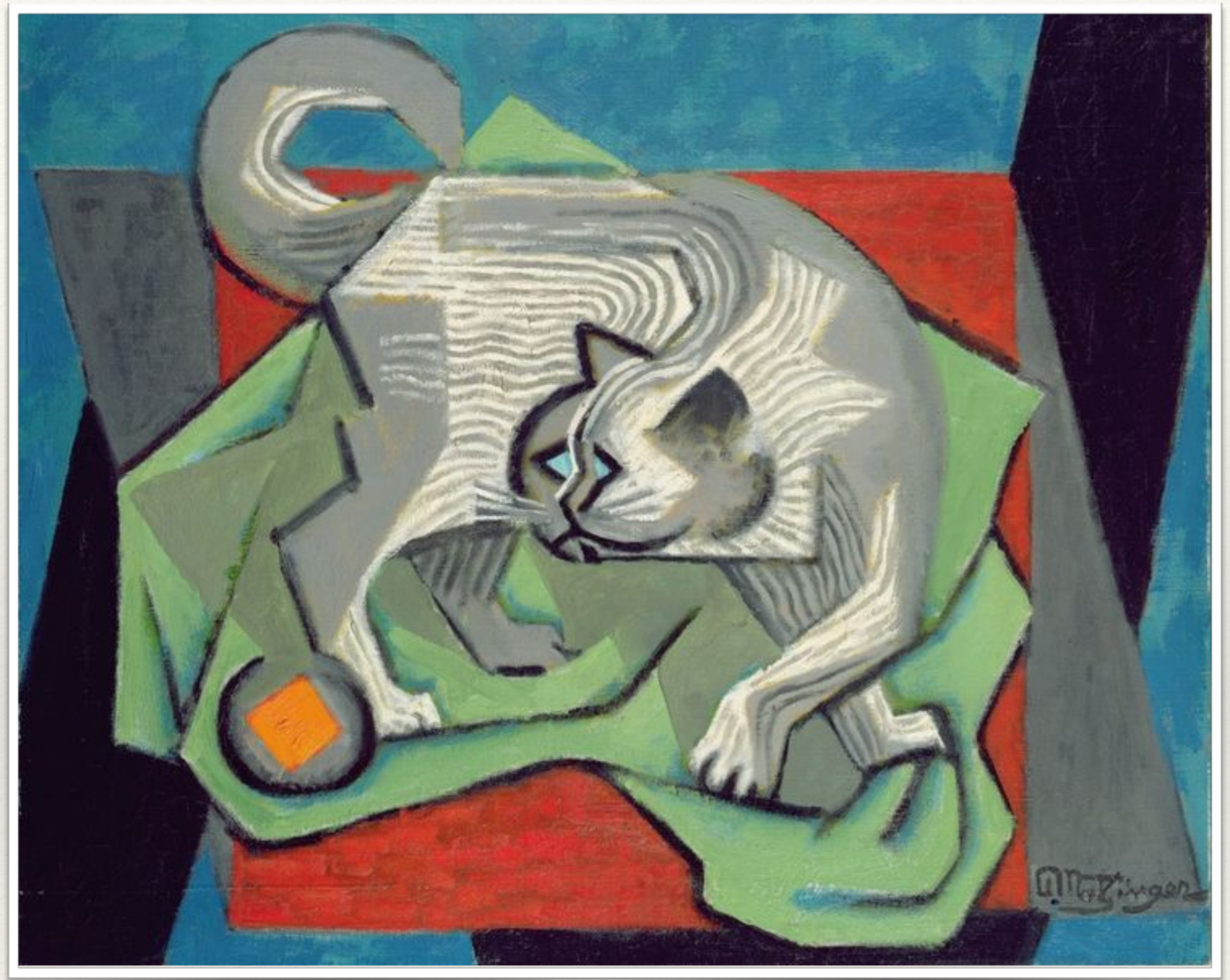
Quantum Mechanics

Lecture 12

Hyperfine structure;
Singlet and triplet states;
Addition of angular momentum;
Clebsch-Gordan coefficients.



A quick recap

The spin and orbital AM of an electron couple to give **fine structure**.

$$H_{SO} \propto \mathbf{S} \cdot \mathbf{L} \qquad [\mathbf{S}, \mathbf{L}] = 0$$

Some of old quantum numbers become "bad" and must be replaced:

$$[H_{SO}, S_z] \neq 0\,,\ [H_{SO}, L_z] \neq 0$$

We defined a new total AM operator: $\mathbf{J} = \mathbf{S} + \mathbf{L}\,, \quad [S^2, J_z] = [L^2, J_z] = 0$

The symmetries of H_{SO} are now:

$$[H_{SO}, S^2] = [H_{SO}, L^2] = [H_{SO}, J^2] = [H_{SO}, J_z] = 0$$

$$(l, m_l, s, m_s) \rightarrow (l, s, j, m_j)$$

old quantum numbers

new quantum numbers

(n is still good, too)

A quick recap

The new quantum numbers j, m_j are expressions of conservation of AM:

$$m_j = m_l + \frac{1}{2}\,,\ j = l \pm \frac{1}{2}$$

The new states that diagonalize H_{SO} are linear combinations of the old ones:

$$|\chi_1\rangle = |\,l, m_j - \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\rangle\,, \quad |\chi_2\rangle = |\,l, m_j + \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\rangle$$

$$|\,l \pm \frac{1}{2}, m_j\rangle = \alpha_{\pm}\,|\chi_1\rangle \pm \beta_{\pm}\,|\chi_2\rangle$$

(Just some coefficients… we can look them up.)

The corrections to the energy are found by taking expected values.

$$E_{SO}^{(1)} = \langle n, j, m_j \,|\, H_{SO} \,|\, n, j, m_j\rangle$$

Hyperfine structure

We can follow this recipe again to understand the **hyperfine structure**, i.e. the splitting in energies due to spin-spin coupling between the electron and proton.

For simplicity, we restrict our discussion to the hydrogen ground state $n = 1$.

Before perturbation, 4 degenerate states:

Introducing a total AM operator is a very good idea:

Total angular momentum is conserved

Total F and F_z also give good quantum numbers for H_{HF}:

Note:

We conclude that f and $m_f = m_i + m_s$ are good quantum numbers for H_{HF}.

Matrix elements for H_{HF}

What are the matrix elements $\langle \chi_j | H_{HF} | \chi_k \rangle$ of H_{HF} in the unperturbed basis?

Rewrite in terms of raising and lowering operators:

Matrix elements are now straightforward to calculate. Example:

Eigenvalues

The matrix in this basis is: (all other terms vanish)

$$H_{HF} = \frac{2A}{\hbar^2}\mathbf{S}\cdot\mathbf{I} \longrightarrow \begin{pmatrix} A/2 & & & \\ & -A/2 & A & \\ & A & -A/2 & \\ & & & A/2 \end{pmatrix}$$

Diagonalizing this gives us the new eigenstates and eigenvalues.

Eigenvalues:

Eigenvectors:

Quantum numbers

The new total AM quantum numbers are constrained

For consistency, use:

Two choices:

Again we see that total angular momentum adds up nicely:

In terms of the new f quantum number, the states have a natural interpretation:

Addition of angular momenta

General question:
Given two spins, j_1 and j_2, what are the allowed values of total AM?

For each value of the total spin, we can have a z-component:

We want to be able to answer questions like (for example):

A spin $j_1 = 3/2$ and spin $j_2 = 1$ particle have total spin $J = 3/2$ and $M = 1/2$.
Now measure the z-component of each individually.

What is the probability of finding $(m_1, m_2) = (+3/2, -1)$?
What about $(m_1, m_2) = (+1/2, 0)$? What about $(m_1, m_2) = (-1/2, 1)$?

Clebsch-Gordan coefficients

A general solution to this problem is given by the Clebsch-Gordan coefficients:

The **Clebsch-Gordan coefficients** are just the basis expansion coefficients.

These are tabulated, and you can **look them up**.

Clebsch-Gordan tables

Returning to our example question:

$$C^{j_1\,j_2\,J}_{m_1\,m_2\,M} = \langle j_1\,m_1\,j_2\,m_2\,|\,J\,M\rangle$$

$j_1 = \frac{3}{2}\,, j_2 = 1$

$m=\frac{1}{2}$

m_1, m_2 \ j	$\frac{5}{2}$	$\frac{3}{2}$	$\frac{1}{2}$
$\frac{3}{2}$, -1	$\sqrt{\frac{1}{10}}$	$\sqrt{\frac{2}{5}}$	$\sqrt{\frac{1}{2}}$
$\frac{1}{2}$, 0	$\sqrt{\frac{3}{5}}$	$\sqrt{\frac{1}{15}}$	$-\sqrt{\frac{1}{3}}$
-$\frac{1}{2}$, 1	$\sqrt{\frac{3}{10}}$	$-\sqrt{\frac{8}{15}}$	$\sqrt{\frac{1}{6}}$

Use the Born rule to calculate the answers: