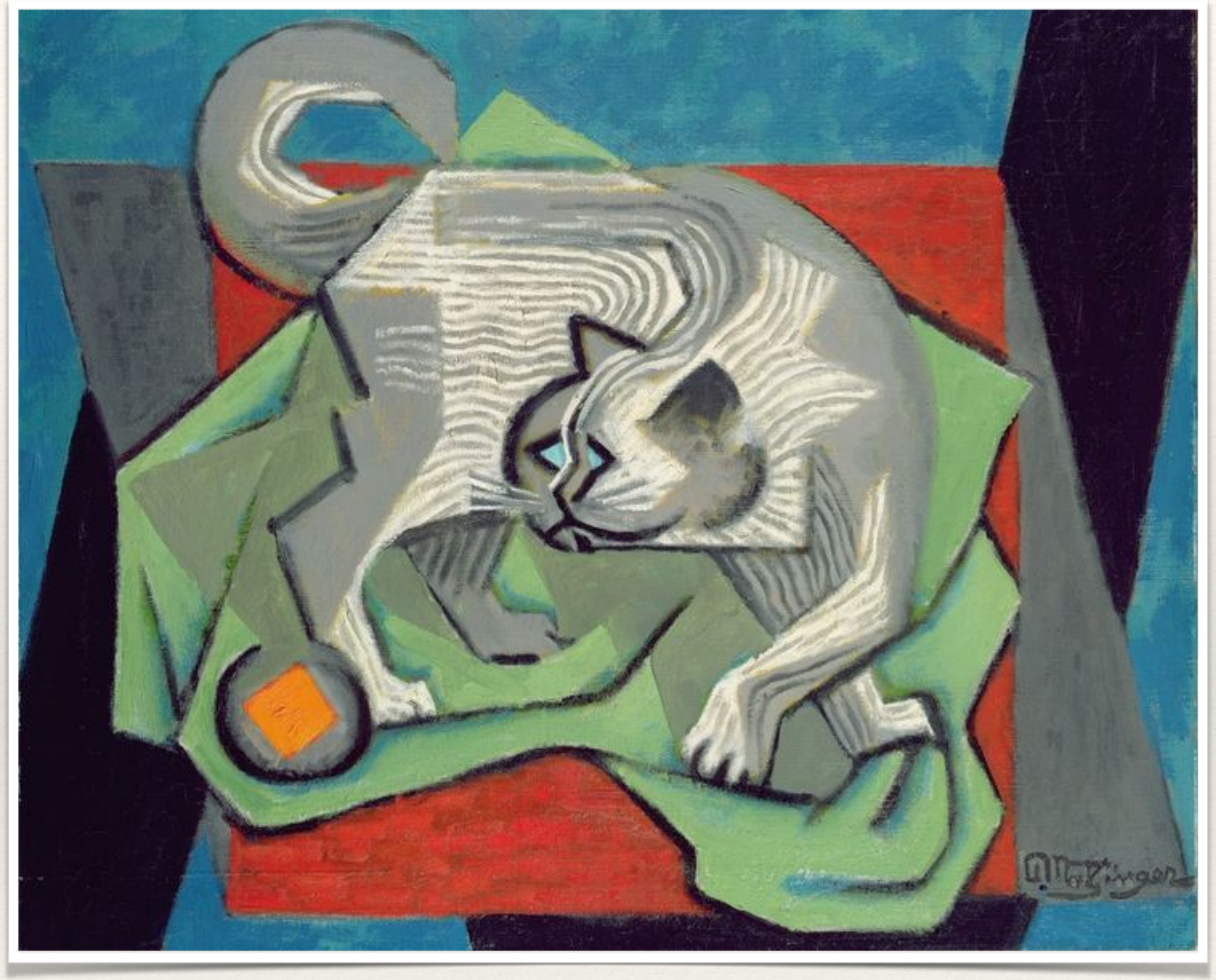
Prof. Steven Flammia

Quantum Mechanics

Lecture 14

Quiz 3;
Quantum Hamiltonian for the EM field;
Zeeman splitting;
Aharonov-Bohm effect.



Classical electrodynamics

The classical Lorentz force law is a velocity-dependent force

This is **not a conservative force**, i.e. it cannot be derived from a potential. Deriving the quantum Hamiltonian for the EM field is therefore delicate.

Recall that the **B** and **E** fields can be expressed in terms of a scalar and vector potential, ϕ and **A**:

The Lagrangian, from which we derive equations of motion, is given by:

Classical equations of motion

The classical equations of motion are given by the Euler-Lagrange equations. The Hamiltonian formulation, with $\dot{x}_i = v_i$, has canonical momentum:

Thus, **canonical momentum is not just** mv! The classical Hamiltonian is

Quantum Hamiltonian

Thus we expect that the correct quantum Hamiltonian is obtained by:

However, in the presence of the **B** field, **velocities don't commute**:

Let's expand the square:

Gauge transformations

We can simplify this Hamiltonian be a careful choice of gauge.

$$H = \frac{-\hbar^2}{2m}\nabla^2 + \frac{i\hbar q}{2mc}\left(\mathbf{A}\cdot\nabla + \nabla\cdot\mathbf{A}\right) + \frac{q^2}{2mc^2}\mathbf{A}^2 + q\phi$$

Any transformation on the gauge fields of the following form leaves **B**, **E** alone:

The **Coulomb gauge** is a choice that sets:

In the Coulomb gauge the Hamiltonian becomes:

Constant B-field (Coulomb gauge)

The result is:

$$H = \frac{-\hbar^2}{2m}\nabla^2 + \frac{i\hbar q}{mc}\mathbf{A}\cdot\nabla + \frac{q^2}{2mc^2}\mathbf{A}^2 + q\phi$$

"paramagnetic term" (the $\frac{i\hbar q}{mc}\mathbf{A}\cdot\nabla$ term)   "diamagnetic term" (the $\frac{q^2}{2mc^2}\mathbf{A}^2$ term)

Let's consider an example with constant **B**-field.

The vector potential can be expressed as: $\mathbf{A} = \frac{1}{2}\mathbf{B}\times\mathbf{r}$ (check: $\nabla\times\mathbf{A} = \mathbf{B}$)

Paramagnetic term:

Diamagnetic term:

Zeeman splitting

The ratio of the dia- and paramagnetic terms is tiny in any case where an electron is bound to an atom and when B is less than a few Tesla.

The paramagnetic term is small compared to the Coulomb energy scale:

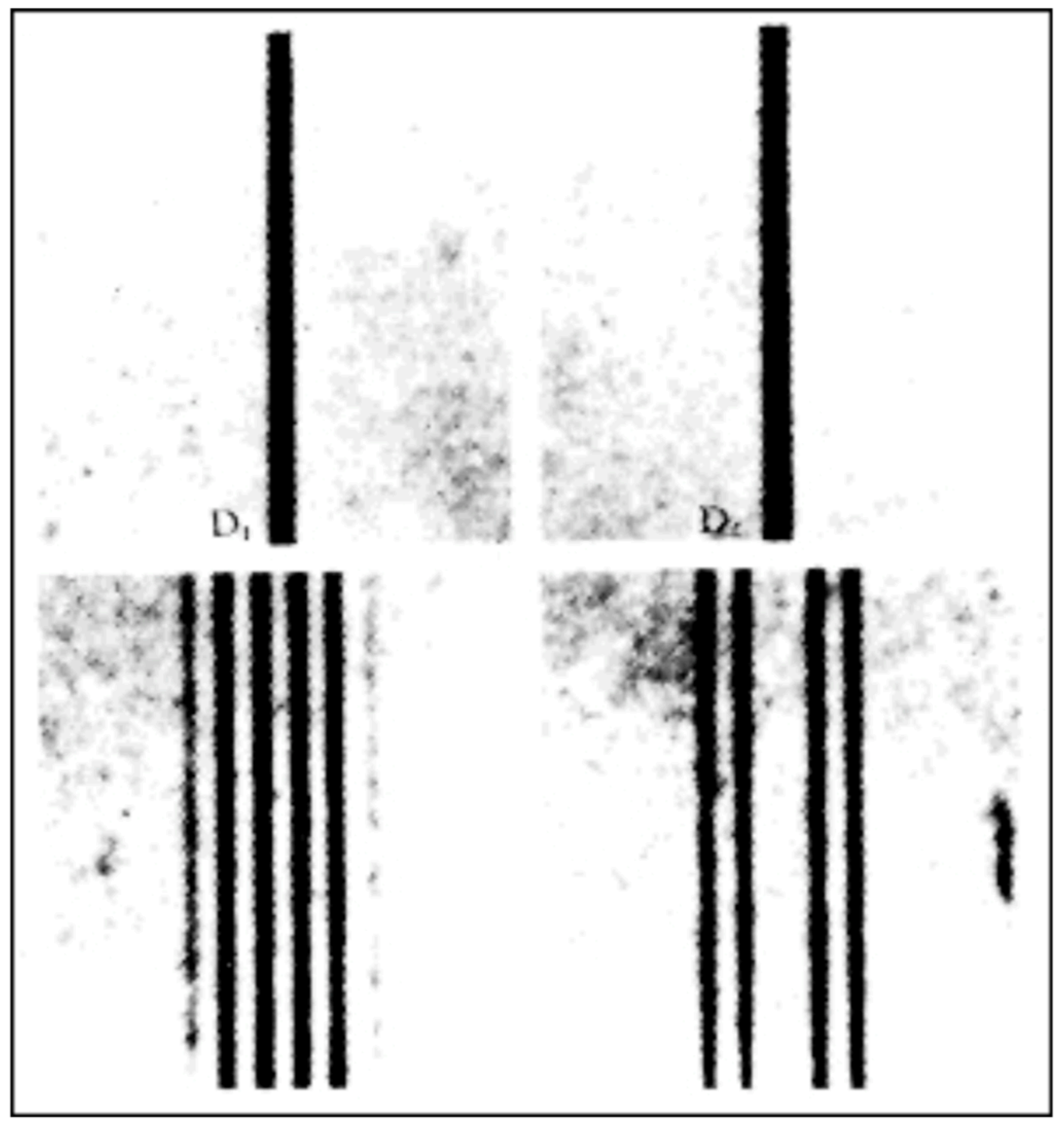
We therefore want to solve:

Zeeman splitting

Notice that L_z still commutes with H, so m_l is still a good quantum number:

$$H = \frac{\hat{\mathbf{p}}^2}{2m} - \frac{e^2}{|\hat{\mathbf{r}}|} + \frac{eB}{2mc} L_z$$

The new eigenstates are unchanged.



Splitting of Sodium D lines, from Zeeman's original paper (1897)

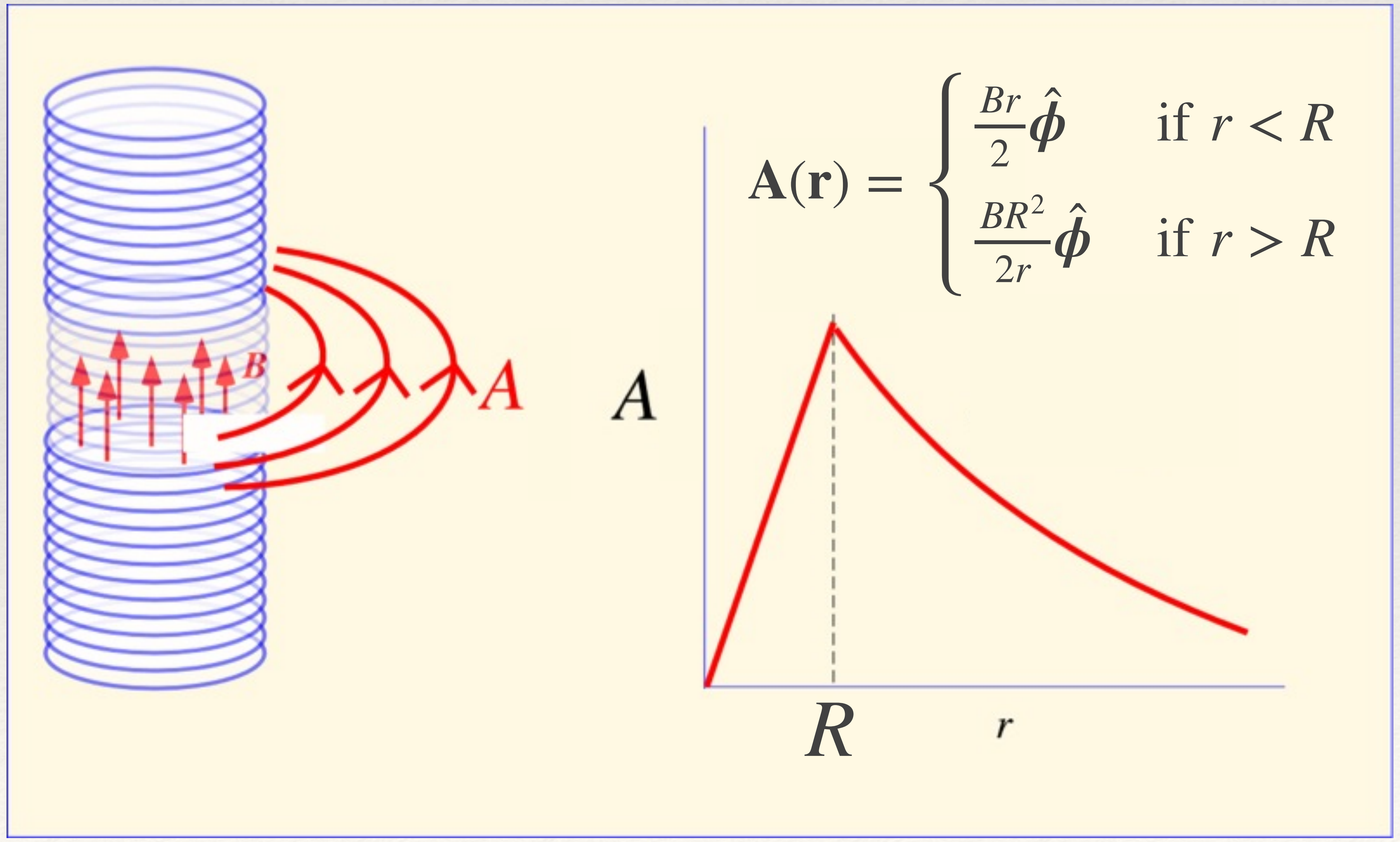
Magnetic field leads to splitting of degeneracy of the $(2l+1)$ states in the l subspace.

Consequences of gauge invariance

Gauge transformations change the wavefunction, but not in an observable way:

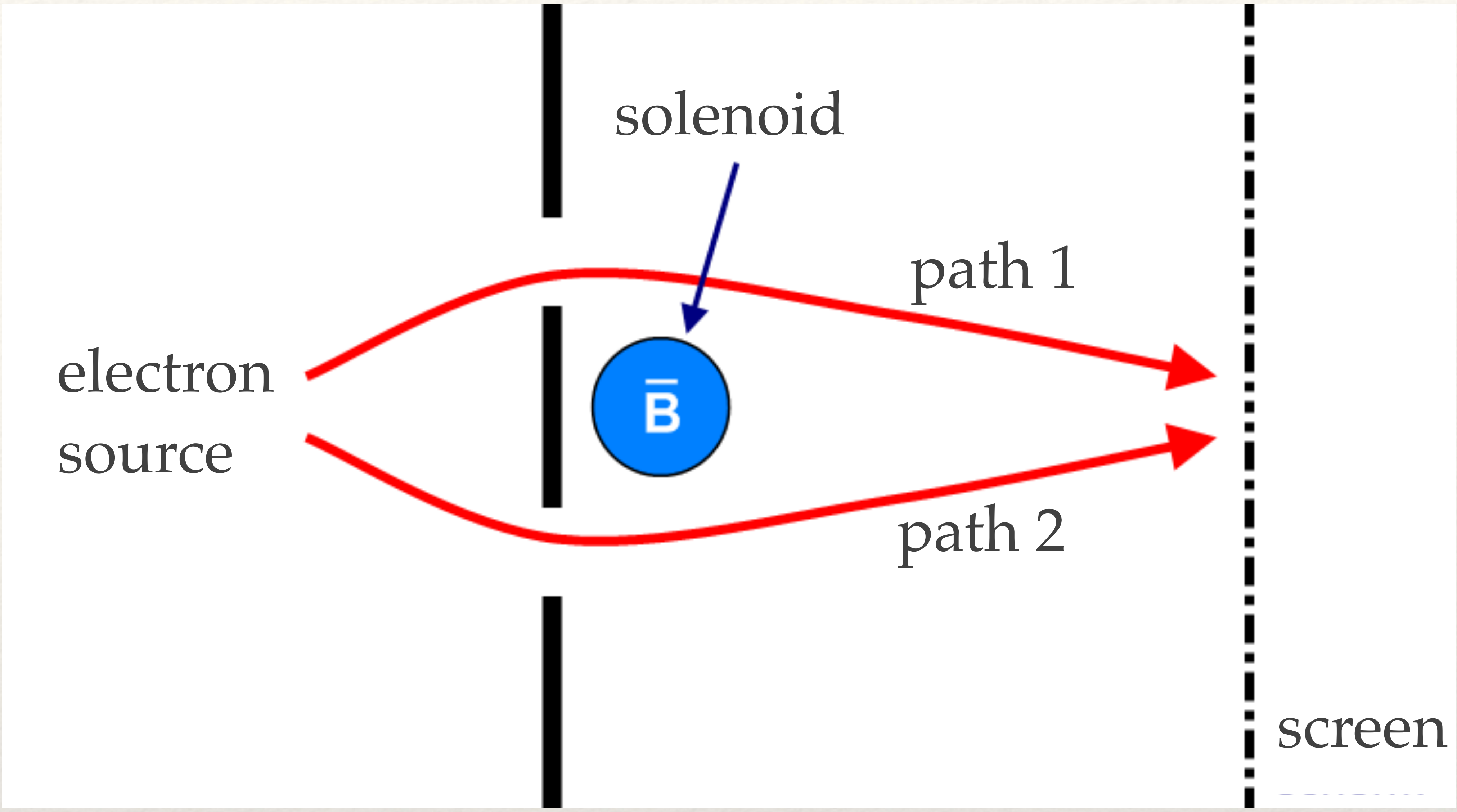
All probabilities are invariant:

Of course, we can have non-zero **A** even if the magnetic field **B** is zero. Consider a solenoid:



Aharonov-Bohm effect

Consider the following double slit experiment:



With each path P, an electron will pick up a different phase factor that depends on the vector potential:

The **phase difference** is observable as interference fringes on the screen!