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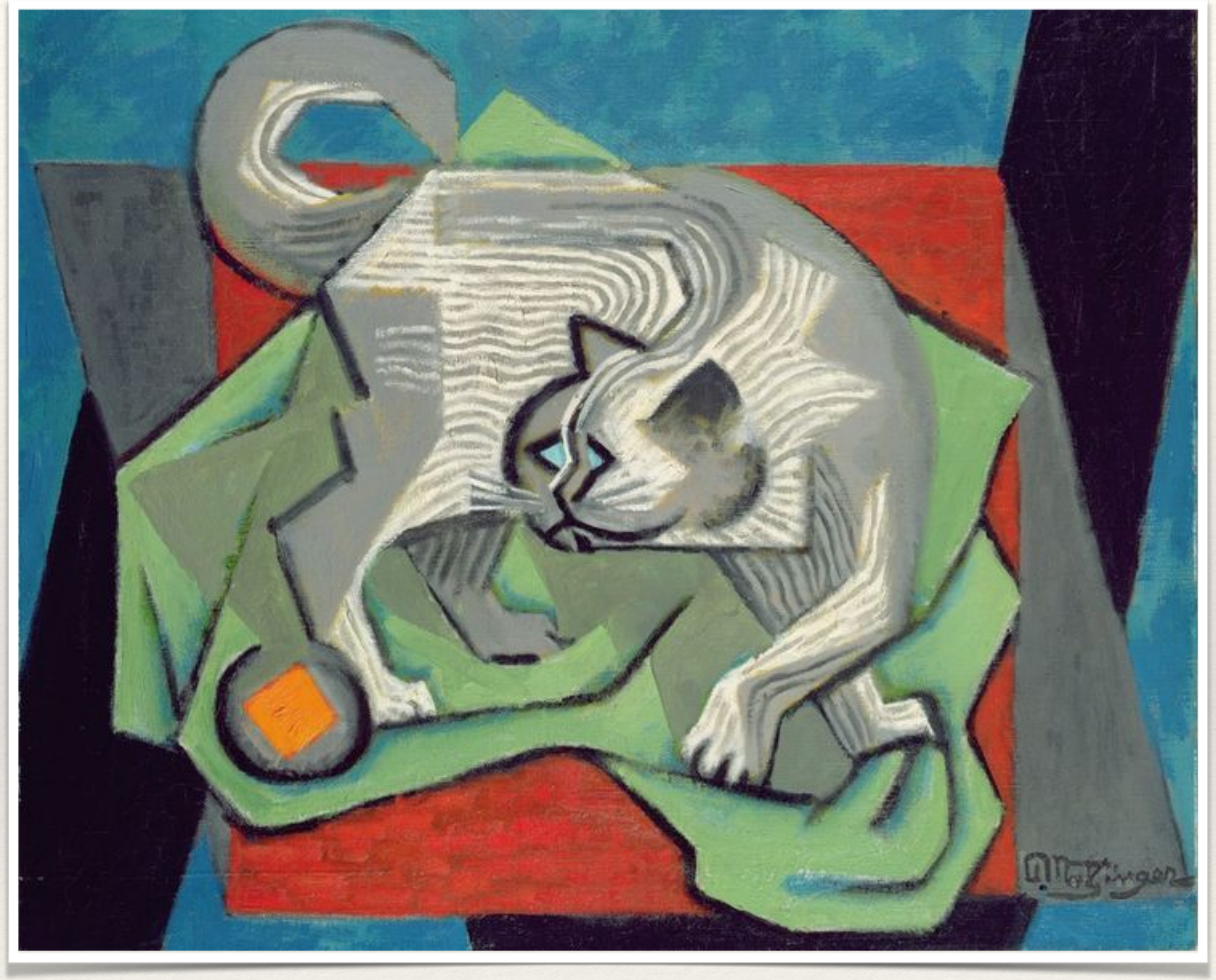
Quantum Mechanics

Lecture 15

Time-dependent perturbation theory;
The interaction picture.



A quick recap

We derived the quantum Hamiltonian for a classical EM field: (Coulomb gauge)

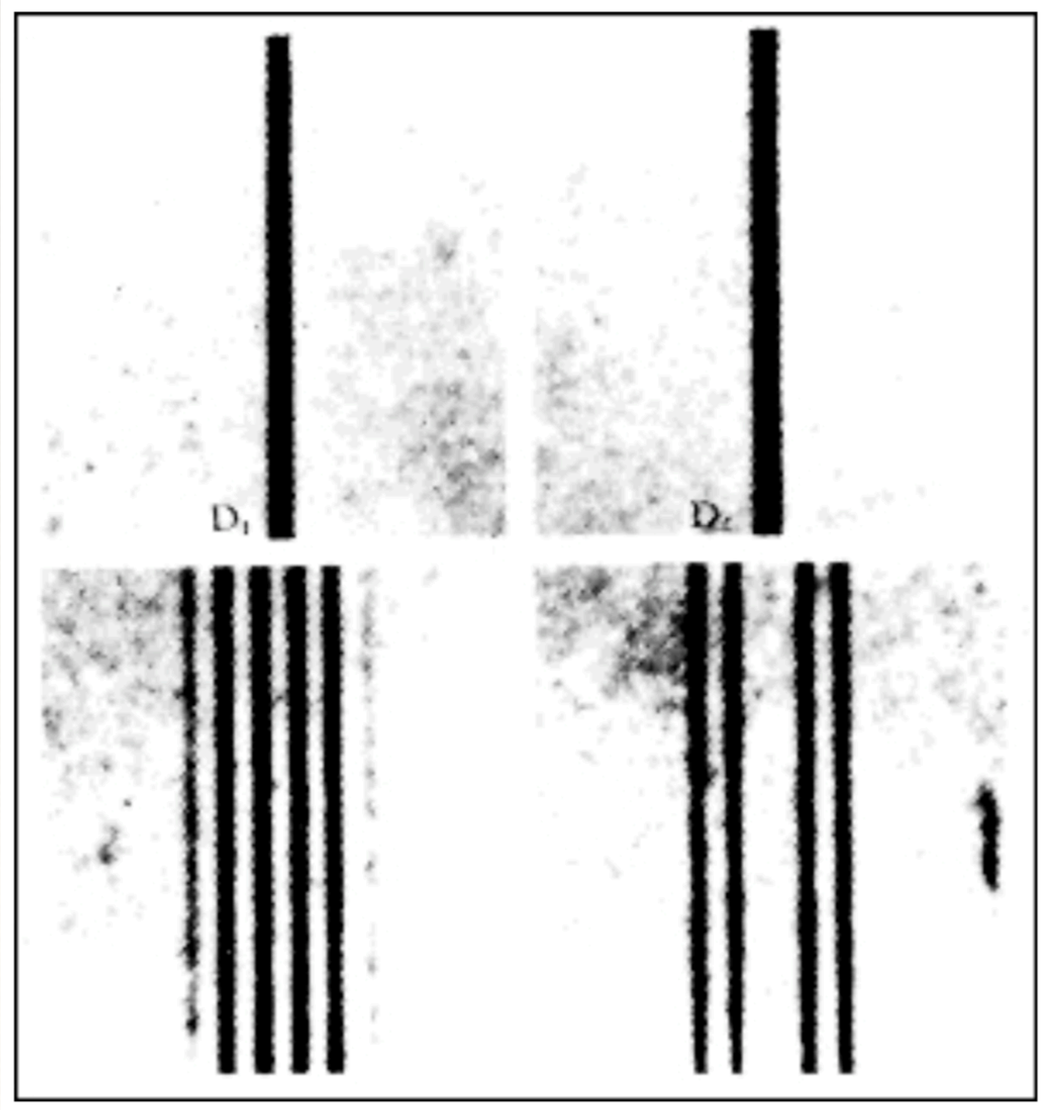
$$H = \frac{-\hbar^2}{2m}\nabla^2 + \frac{i\hbar q}{mc}\mathbf{A}\cdot\nabla + \frac{q^2}{2mc^2}\mathbf{A}^2 + q\phi$$

$$A' = A + \nabla\Lambda$$

$$\phi' = \phi - \frac{1}{c}\frac{\partial\Lambda}{\partial t}$$

And, together with gauge invariance, we derived two phenomena:

Zeeman splitting



$$E_{n,l,m_l} = -\frac{me^4}{2\hbar^2 n^2} + \hbar\omega_L m_l$$

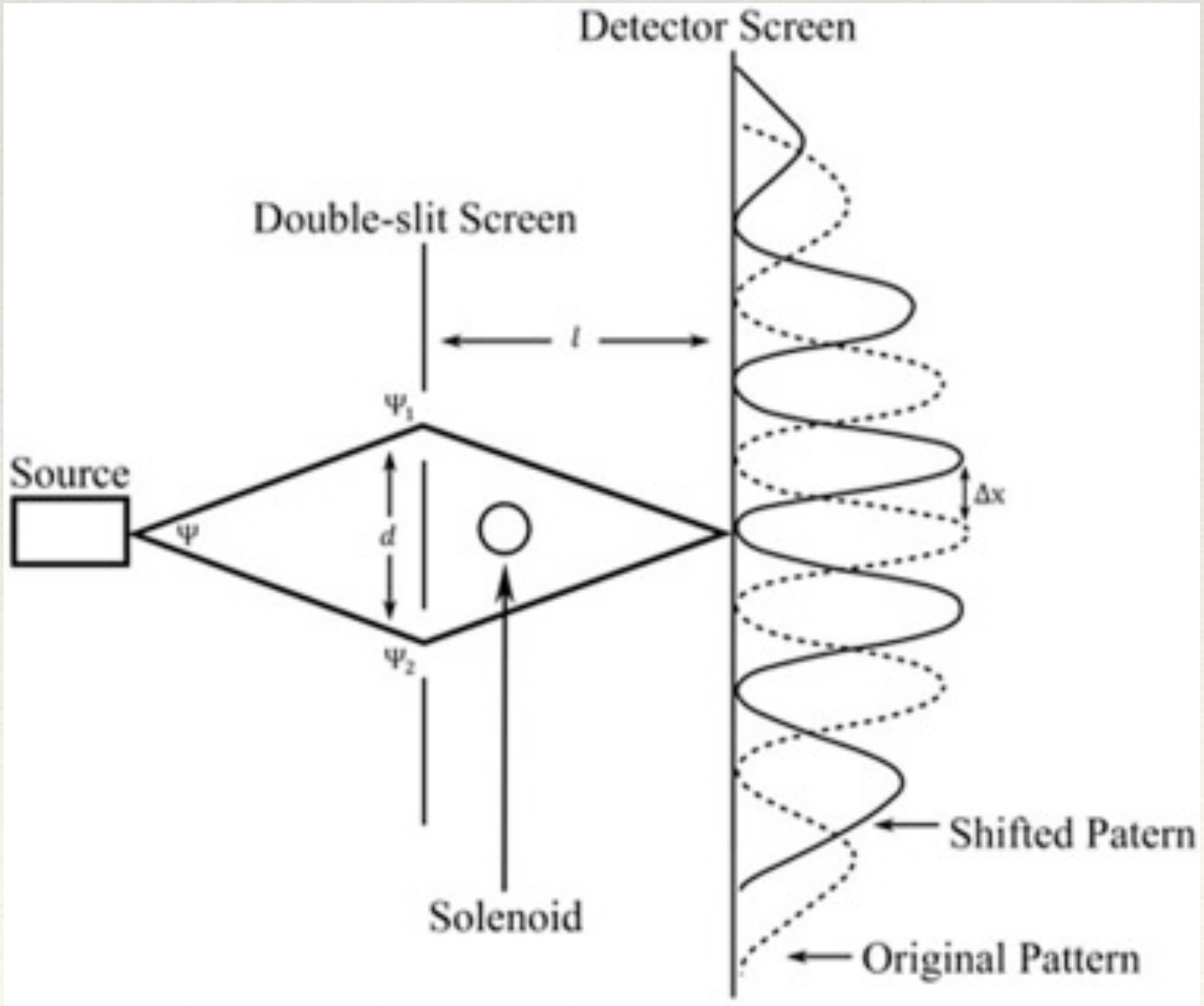
$$\omega_L = \frac{eB}{2mc}$$

Larmor frequency

Aharonov-Bohm effect

Phase shift based on flux through path

$$\Delta\gamma = \frac{e}{\hbar c}\Phi$$



Time-dependent perturbation theory

If we are interested in time dynamics of a system, we need more information from perturbation theory than what the time-independent case gives.

We allow our perturbing Hamiltonian to potentially depend explicitly on time and we make the following ansatz.

total Hamiltonian

initial state,
expanded in unperturbed eigenbasis

Time dynamics,
with c(t) containing the perturbing corrections to the bare evolution

The probability of being in the
*n*th bare state at time *t*.

Using the Schrödinger equation

The Schrödinger equation tells us how the $c_n(t)$ evolve with time.

Schrödinger eq. $i\hbar\,|\dot{\psi}(t)\rangle = H\,|\psi(t)\rangle$

Ansatz state $|\psi(t)\rangle = \sum_n c_n(t)\mathrm{e}^{-iE_n^{(0)}t/\hbar}\,|E_n^{(0)}\rangle$

Perturbation series

We have derived a set of coupled differential equations for determining the evolution equations of the new amplitudes.

$$i\hbar\dot{c}_f(t) = \sum_n c_n(t)\mathrm{e}^{i\left(E_f^{(0)}-E_n^{(0)}\right)t/\hbar}\langle E_f^{(0)}\,|\,H_1\,|\,E_n^{(0)}\rangle$$

We now expand $c_n(t)$ as a perturbation series:

As with time-independent perturbation theory, we now equate terms at each order in λ to find self-consistent equations for the perturbative corrections.

0th order and initial conditions

Collecting terms at 0th order, we find

$$i\hbar\left(\dot{c}_f^{(0)} + \lambda\dot{c}_f^{(1)} + \ldots\right) = \sum_n \left(c_n^{(0)} + \lambda c_n^{(1)} + \ldots\right) \mathrm{e}^{i\left(E_f^{(0)} - E_n^{(0)}\right)t/\hbar} \langle E_f^{(0)} | \lambda H_1 | E_n^{(0)} \rangle$$

0th order equation:

We need boundary conditions, so assume we initialize as follows.

Assume we start in an unperturbed eigenstate:

initial state:

This implies:

1st order conditions

Collecting terms at 1st order, we find

$$i\hbar\left(\dot{c}_f^{(0)} + \lambda \dot{c}_f^{(1)} + \ldots\right) = \sum_n \left(c_n^{(0)} + \lambda c_n^{(1)} + \ldots\right) \mathrm{e}^{i\left(E_f^{(0)} - E_n^{(0)}\right)t/\hbar} \langle E_f^{(0)} | \lambda H_1 | E_n^{(0)} \rangle$$

1st order equation:

This can be integrated to obtain:

Schrödinger picture

We have been accustom to thinking of the state vector evolving in time:

$$|\psi_S(t)\rangle = U_S(t)\,|\psi_S(0)\rangle \qquad U_S(t) = \exp\left(-iHt/\hbar\right) \quad \text{(if H is time-independent.)}$$

We will call this the "Schrödinger picture" and label states and operators considered in this picture by a subscript S.

In the Schrödinger picture, time-evolution obeys:

$$i\hbar\dot{U}_S(t) = HU_S(t)$$

and expectation values (of time-independent operators) obey:

$$\frac{d}{dt}\langle\psi_S(t)\,|\,O_S\,|\,\psi_S(t)\rangle = \frac{i}{\hbar}\langle\psi_S(t)\,|\,[H, O_S]\,|\,\psi_S(t)\rangle$$

Heisenberg picture

In contrast to the Schrödinger picture, in the Heisenberg picture the operators evolve and the states remain fixed.

Here we have defined:

Heisenberg operator time evolution obeys:

Interaction (Dirac) picture

The Schrödinger and Heisenberg pictures are "active" or respectively "passive" views of quantum evolution. The interaction picture combines features of both in a convenient way for time-dependent perturbation theory. Define:

Time evolution in the interaction picture proceeds as:

Operator evolution in the interaction picture

We thus have:

$$|\psi_I(t)\rangle = e^{iH_0 t/\hbar}|\psi_S(t)\rangle \qquad i\hbar|\dot{\psi}_I(t)\rangle = e^{iH_0 t/\hbar}H_1 e^{-iH_0 t/\hbar}|\psi_I(t)\rangle$$

Evolution of expected values of operators proceeds as:

Evolution of expected values of operators proceeds as:

This suggests defining:

Unitary evolution operator

In the interaction picture, H_I depends on time, complicating time evolution.

$$\dot{O}_I = \frac{i}{\hbar}[H_0, O_I] \qquad H_I = e^{iH_0t/\hbar}H_1e^{-iH_0t/\hbar} \qquad i\hbar\,|\dot{\psi}_I(t)\rangle = H_I\,|\psi_I(t)\rangle$$

We can integrate the evolution equation as follows:

To get a perturbative expression, we can iterate this:

Amplitude evolution

We can now derive the evolution equations for the $c_n(t)$ amplitudes.

Using the perturbative expansion for U_I(t), we find:

This looks familiar! We can therefore see: