Guest lecture by Dr. Arne Grimsmo

Quantum Mechanics

Lecture 18

Harmonic oscillator redux:
Coherent states;
Quantum phase space.

Simple harmonic oscillator

The simple harmonic oscillator is one of the most important models in all of physics. Let's give a lightning review of the basics.

$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2 \qquad [\hat{x}, \hat{p}] = i\hbar$$

To solve the Hamiltonian, we introduce **creation and annihilation operators**:

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} + \frac{i}{m\omega} \hat{p} \right) \qquad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} - \frac{i}{m\omega} \hat{p} \right) \qquad [\hat{a}, \hat{a}^\dagger] = 1$$

Inverting these equations for position and momentum, we find:

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} \left(\hat{a} + \hat{a}^\dagger \right) \qquad \hat{p} = -i\sqrt{\frac{m\omega\hbar}{2}} \left(\hat{a} - \hat{a}^\dagger \right)$$

Simple harmonic oscillator

In terms of $\hat{a}^\dagger, \hat{a}$ or the **number operator** $\hat{N}$, the Hamiltonian becomes

$$H = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) = \hbar\omega\left(\hat{N} + \frac{1}{2}\right)$$

The eigenstates and energies are given in terms of the **number states**:

$$H|n\rangle = E_n|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle$$

The creation and annihilation operators act on the number states as follows:

$$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle\,, \quad \hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle\,, \quad \hat{a}^\dagger\hat{a}|n\rangle = \hat{N}|n\rangle = n|n\rangle$$

They are sometimes called **raising and lowering operators** because of this.

Simple harmonic oscillator

The number states form a complete orthonormal basis:

$$\langle m | n \rangle = \delta_{mn} \qquad \sum_{n=0}^{\infty} |n\rangle\langle n| = 1$$

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle , \quad c_n = \langle n | \psi \rangle$$

We can create them by applying the raising operator to the vacuum

$$|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}} |0\rangle$$

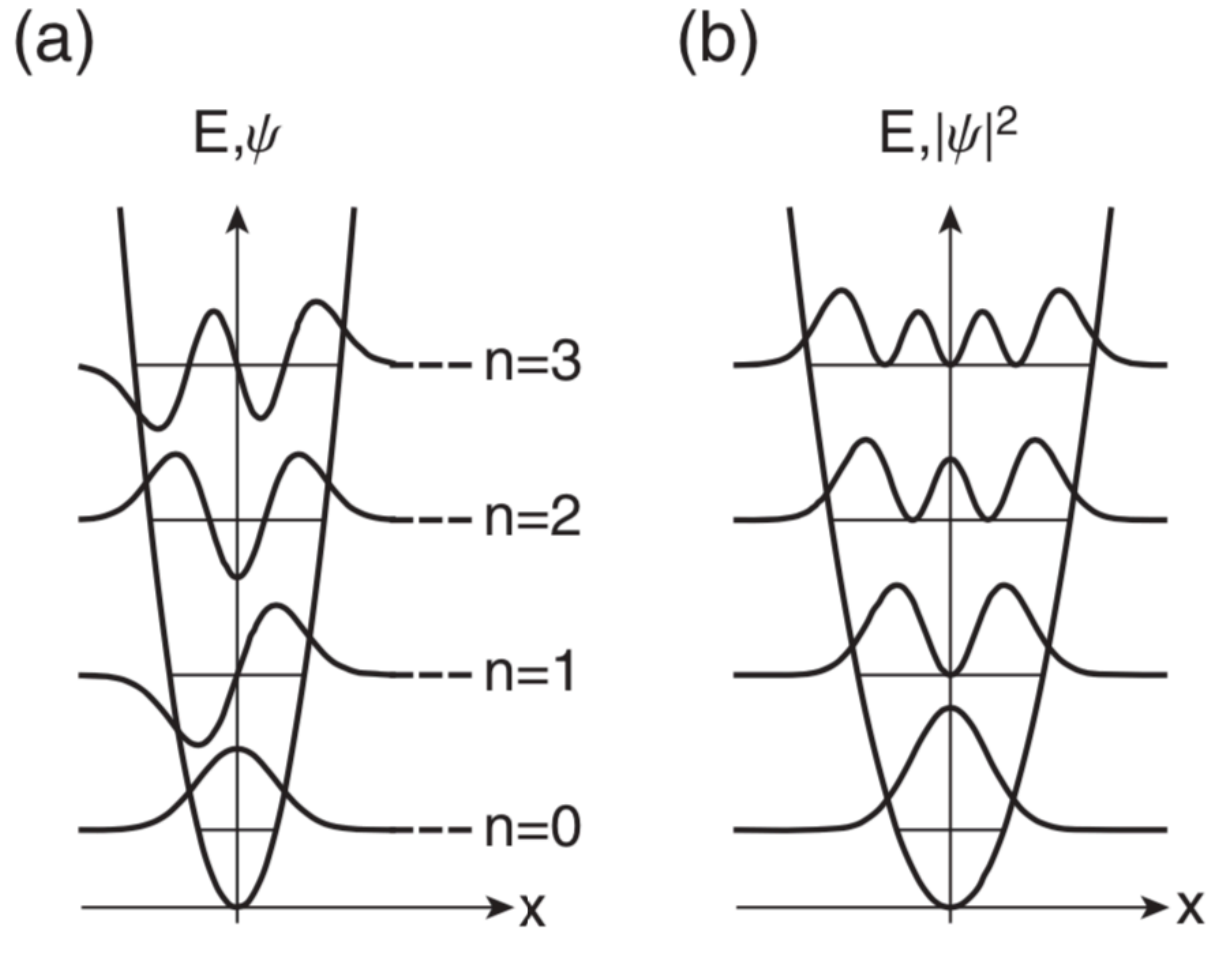

FIGURE 9.6 Energy eigenstate (a) wave functions and (b) probability densities of the harmonic oscillator.

Number state uncertainty

The uncertainty in position or momentum is easy to compute with $\hat{a}^\dagger, \hat{a}$:

$$(\Delta \hat{x})^2 = \langle n | \hat{x}^2 | n \rangle - \langle n | \hat{x} | n \rangle^2$$

Recall: $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} \left(\hat{a} + \hat{a}^\dagger \right)$

We have:

Similarly, we have:

Uncertainty and the large-n limit

A similar calculation for p shows that

$$(\Delta\hat{x})^2 = \frac{\hbar}{m\omega}\left(n + \frac{1}{2}\right) \qquad (\Delta\hat{p})^2 = m\omega\hbar\left(n + \frac{1}{2}\right)$$

As we expect from Heisenberg, even in the ground state there is uncertainty:

Thus, the number states cannot directly correspond to a classical limit with a well-defined mass on a spring. Perhaps we should have expected this, since they are eigenstates and have no dynamics. But it begs the question:

What are the "most classical" states of the harmonic oscillator?

Coherent states

The coherent states are defined as eigenstates of the annihilation operator:

In the number basis, they look like:

The eigenvalue property follows easily:

Coherent state time evolution

The coherent states are normalized, but **not** orthogonal:

However, they are nearly orthogonal when α or β have large magnitude.

Coherent states evolve in time as follows: $|\alpha(t)\rangle = \mathrm{e}^{-i\omega t/2}|\alpha \mathrm{e}^{-i\omega t}\rangle$

Coherent states remain coherent states under time evolution! Only α changes.

Coherent state expected values

The expected values of position and momentum change with time for a coherent state as they would for a classical mass on a spring.

$$\langle \hat{x}(t) \rangle = \sqrt{\frac{\hbar}{2m\omega}} 2\,|\,\alpha\,|\cos(\omega t - \phi)\,, \quad \langle \hat{p}(t) \rangle = -\sqrt{\frac{\hbar m\omega}{2}} 2\,|\,\alpha\,|\sin(\omega t - \phi)\,, \quad \alpha = |\,\alpha\,|\,\mathrm{e}^{i\phi}$$

Coherent states have minimal uncertainty

Coherent states have the minimal uncertainty allowed by quantum mechanics.

Classical vs. quantum phase space

The coherent states suggest the following analogy with classical mechanics.

In classical phase space, point particles evolve along trajectories labeled by coordinates $(x(t), p(t))$.

In quantum phase space, a *distribution* evolves along trajectories labeled by *expected values*, $\left(\langle \hat{x}(t) \rangle, \langle \hat{p}(t) \rangle\right)$.

Simple harmonic oscillator trajectory in classical phase space.

Coherent state distribution evolving as a trajectory in “quantum phase space”.

The Wigner function (non-examinable)

One way to make sense of quantum phase space is with the *Wigner function*. Starting from a wave function ψ, we transform it as follows:

This formula can be inverted to yield

Thus, the Wigner function is a **faithful** representation of a quantum state. It allows us to visualize states and dynamics in phase space, as we will see next lecture.