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Modulated instability in collisional plasmas

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We describe the modulational instability development in a collisional plasma where the characteristic frequency $\Delta\omega$ of the modulated perturbations is much less than $\nu_{\text{eff}}^{(\text{eq})}$, the inverse equalization time of the electron and ion temperatures.

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We consider a strongly collisional plasma, where the effective collision frequency ν_{eff} is not only much larger than the characteristic frequency $\Delta\omega$ of the modulational perturbations, but also is larger in the presence of a significantly stronger inequality:

$$|\Delta\omega| \ll \nu_{\text{eff}}^{(\text{eq})} = \frac{m_e}{m_i}\nu_{\text{eff}} . \tag{1}$$

Here, $m_{e(i)}$ is the electron (ion) mass. The characteristic frequency $\nu_{\text{eff}}^{(\text{eq})}$ defines the time of equalization of the electron and ion temperatures; thus the inequality (1) means that for the time of modulational instability development (or for the period of modulations) the electron and ion temperatures have time to equalize each other. This problem arises, for instance, when investigating plasma heating by strong laser radiation [1], which is relevant for inertial confinement fusion schemes [2,3].

The pump waves (which are assumed to be electromagnetic) are high frequency, so we have the following inequality:

$$\omega_0 \gg \nu_{\text{eff}} , \tag{2}$$

where ω_0 is the pump frequency. The plasma is assumed to be underdense,

$$\omega_0 \gg \omega_{pe} , \tag{3}$$

where $\omega_{pe}=(4\pi n_e e^2/m_e)^{1/2}$ is the electron plasma frequency.

We start our investigation with the hydrodynamic equations that can be obtained from the kinetic equation with Landau collision integral [4,5]. We consider that, in general, among all the characteristic frequencies of the problem, only the frequencies of the modulated perturbations are in the hydrodynamical regime $|\Delta\omega| \ll \nu_{\text{eff}}$. The other frequencies correspond to the regime of rare collisions. In calculating the latter quantities, we in fact use a collisionless approximation; i.e., with some well-known assumptions, we can use "collisionless hydrodynamics."

Thus we start from the equations for the electron $\mathbf{v}^{(e)}$ and ion $\mathbf{v}^{(i)}$ velocities:

$$m_e n_e(\partial_t + \mathbf{v}^{(e)}\cdot\nabla)v_j^{(e)} = -\nabla_j n_e T_e - \nabla_l \pi_{lj}^{(e)} - e n_e \left[E_j + \frac{1}{c}[\mathbf{v}^{(e)}\times\mathbf{B}]_j\right] + R_j , \tag{4}$$

and

$$m_i n_i(\partial_t + \mathbf{v}^{(i)}\cdot\nabla)v_j^{(i)} = -\nabla_j n_i T_i - \nabla_l \pi_{lj}^{(i)} + e n_i \left[E_j + \frac{1}{c}[\mathbf{v}^{(i)}\times\mathbf{B}]_j\right] - R_j , \tag{5}$$

where we assume that the ions have the charge e. Equations (4) and (5) are completed by the usual continuity equations for the electrons and ions,

$$\partial_t n_{e,i} + \nabla\cdot(n_{e,i}\mathbf{v}_{e,i}) = 0 , \tag{6}$$

as well as the equations of thermal balance

$$\tfrac{3}{2} n_{e,i}(\partial_t + \mathbf{v}^{(e,i)}\cdot\nabla)T_{e,i} + n_{e,i}T_{e,i}\nabla\cdot\mathbf{v}^{(e,i)} = -\nabla\cdot\mathbf{q}^{(e,i)} - \pi_{lj}^{(e,i)}\nabla v_l^{(e,i)} + Q_{e,i} . \tag{7}$$

In Eqs. (4) and (5) the terms containing ∇nT describe the contribution from the pressure of electron and ion gases; $\pi_{lj}^{(e,i)}$ are the tensors of electron and ion viscosity,

$$\pi_{lj}^{(e)} = -0.73\frac{n_e T_e}{\nu_e} w_{lj}^{(e)} , \tag{8a}$$

$$\pi_{lj}^{(i)} = -0.96\frac{n_i T_i}{\nu_i} w_{lj}^{(i)} , \tag{8b}$$

where

$$w_{lj}^{(e,i)} = \nabla_j v_l^{(e,i)} + \nabla_l v_j^{(e,i)} - \tfrac{2}{3}\delta_{lj}\nabla\cdot\mathbf{v}^{(e,i)} .$$

Further, $\mathbf{R}$ is the friction force between the electrons and ions,

$$\mathbf{R} = \mathbf{R}_u + \mathbf{R}_T , \tag{9}$$

where $\mathbf{R}_u$ is the force of relative friction depending only on the relative velocity of electrons and ions,

$$\mathbf{R}_u = -0.51 n_e m_e \nu_e \mathbf{u} , \tag{10}$$

if $\omega \ll \nu_e$ (we have $\nu_{\text{eff}} \simeq \nu_e$ in the case considered; let us note that if $\omega \gg \nu_e$ holds, then $\mathbf{R}_u \simeq -n_e m_e \nu_e \mathbf{u}$, but in this case some questions arise on the applicability of the hydrodynamical description). Further, $\mathbf{R}_T$ in Eq. (9) is the thermal force

$$\mathbf{R}_T = -0.71 n_e \nabla T_e . \tag{11}$$

In Eq. (7) $\mathbf{q}^{(e,i)}$ is the heat flux,

$$\mathbf{q}^{(e)}=\mathbf{q}_{\mu}^{(e)}+\mathbf{q}_{T}^{(e)}=0.71n_eT_e\mathbf{u}-3.16\frac{n_eT_e}{m_e\nu_e}\nabla T_e\ , \tag{12a}$$

and

$$\mathbf{q}^{(i)}=-3.9\frac{n_iT_i}{m_i\nu_i}\nabla T_i\ , \tag{12b}$$

and Q is the heating power,

$$Q_e=-\mathbf{R}\cdot\mathbf{u}-Q_i\ , \tag{13a}$$

$$Q_i=3\frac{m_e}{m_i}n_e\nu_e(T_e-T_i)\ . \tag{13b}$$

We solve the above equations by expanding in powers of the electric field $\mathbf{E}$. To investigate modulational instability, we have to take into consideration terms up to the third order in the fields, as well as interactions through virtual waves (which are perturbations of the pump electric field on the beat frequency $\Delta\omega$ and in general also the double frequency $2\omega_0$). As a result, for the Fourier component

$$\mathbf{j}_{\mathbf{k}\omega}=\int \mathbf{j}(\mathbf{r},t)\exp(i\mathbf{k}\cdot\mathbf{r}-i\omega t)d\mathbf{r}\,dt/(2\pi)^4 \tag{14}$$

of the high-frequency plasma current density of the third order, we have the following expression containing only the high-frequency fields:

$$\mathbf{j}_{\mathbf{k}\omega}^{(3),\mathrm{HF}}=\int \Sigma_{ijlm}^{\mathrm{eff}}(\mathbf{k},\omega;\mathbf{k}_1,\omega_1;\mathbf{k}_2,\omega_2;\mathbf{k}_3,\omega_3)\mathbf{E}_{\mathbf{k}_1\omega_1,j}^{\mathrm{HF}}E_{\mathbf{k}_2\omega_2,l}^{\mathrm{HF}}E_{\mathbf{k}_3\omega_3,m}^{*\,\mathrm{HF}}\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)$$
$$\times\delta(\omega-\omega_1-\omega_2-\omega_3)d\mathbf{k}_1d\mathbf{k}_2d\mathbf{k}_3d\omega_1d\omega_2d\omega_3\ , \tag{15}$$

where $*$ denotes complex conjugate.

The effective nonlinear third-order plasma response is equal to

$$\Sigma_{ijlm}^{\mathrm{eff}}\simeq\frac{n_0e^4}{5m_e^2T_e}\frac{\nu_e\delta_{ij}\delta_{lm}}{\Delta\omega\omega_1\omega_2\omega_3}\frac{\Delta\mathbf{k}^2v_s^2}{\Delta\omega^2-\Delta\mathbf{k}^2v_s^2}\ , \tag{16}$$

where n_0 is the nonperturbed electron density, $\Delta\mathbf{k}=\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3$, $\Delta\omega=\omega-\omega_1-\omega_2-\omega_3$, and $v_s=(10T_e/3m_i)^{1/2}$ is the speed of sound [under the assumption (1)]. The expression $\Delta\omega^2-\Delta\mathbf{k}^2v_s^2$ in the denominator of the right-hand side of (16) is incorrect when $\Delta\omega^2\sim\Delta\mathbf{k}^2v_s^2$; for this case we have to proceed with the following substitution:

$$\Delta\omega^2-\Delta\mathbf{k}^2v_s^2\rightarrow i\frac{0.51\nu_em_e}{\Delta\omega m_i}\left[\frac{10}{3+2\times1.71}\right]^2\frac{1}{\Delta\mathbf{k}^2v_s^2}\ . \tag{17}$$

To find the characteristic times of the modulated instability development, we have to substitute (15) and (16) [taking into account (17)] into the equation for the high-frequency wave field,

$$\left[\epsilon_{\mathbf{k}\omega}^{t}-\frac{\mathbf{k}^2c^2}{\omega^2}\right]E_{\mathbf{k}\omega,i}^{\mathrm{HF}}=-\frac{4\pi i}{\omega}j_{\mathbf{k}\omega,i}^{(3),\mathrm{HF}}\ , \tag{18}$$

where $\epsilon_{\mathbf{k}\omega}^{t}$ is the usual (linear) transverse dielectric permittivity of a plasma. Then we proceed with the following ansatz:

$$\mathbf{E}_{\mathbf{k}\omega}^{\mathrm{HF}}=\mathbf{E}_0\delta(\mathbf{k}-\mathbf{k}_0)\delta(\omega-\omega_0)+\delta\mathbf{E}_{\mathbf{k}\omega}\ , \tag{19}$$

where $\delta\mathbf{E}_{\mathbf{k}\omega}$ is the modulated perturbation of the pump field $\mathbf{E}_0$, $|\delta\mathbf{E}_{\mathbf{k}\omega}|\ll|\mathbf{E}_0|$.

After linearizing the corresponding equation for $\delta\mathbf{E}_{\mathbf{k}\omega}$ and using an analogous procedure for the complex conjugate field $\mathbf{E}^*$, we have the following dispersion equation for the modulational instability (as a condition of solving a system of two linear equations):

$$1=\frac{i}{5}\frac{\nu_e}{\Delta\omega}\frac{\omega_{pe}^4}{\omega_0^4}\frac{|E_0|^2}{4\pi n_0T_e}\frac{\Delta\mathbf{k}^2v_s^2}{\Delta\omega^2-\Delta\mathbf{k}^2v_s^2}\ . \tag{20}$$

This equation differs from the equation of modulational instability in collisionless plasma [6] by the factor $\nu_e/\Delta\omega$ (ignoring a numerical factor).

Equation (20) has the following solutions.

(a) If $|\Omega|\gg|\Delta\mathbf{k}|v_s$,

$$\gamma_1\equiv\mathrm{Im}\Delta\omega=-\Omega\left[\frac{|\Delta\mathbf{k}|v_s}{\Omega}\right]^{2/3}\ ,$$
$$\gamma_{2,3}=-\tfrac{1}{2}\Omega\left[\frac{|\Delta\mathbf{k}|v_s}{\Omega}\right]^{2/3}(1\mp i\sqrt{3})\ . \tag{21}$$

(b) If $|\Omega|\ll|\Delta\mathbf{k}|v_s$,

$$\gamma_1=-\Omega\ ,$$
$$\gamma_{2,3}=-\frac{\sqrt{3}}{2}\Omega\pm i|\Delta\mathbf{k}|v_s\ . \tag{22}$$

In (21) and (22), we have introduced the notation

$$\Omega\equiv\tfrac{1}{5}\nu_e\frac{\omega_{pe}^4}{\omega_0^4}\frac{|E_0|^2}{4\pi n_0T_e}\ . \tag{23}$$

Let us note that $\gamma=\mathrm{Im}\Delta\omega$; thus the imaginary part of the instability rate describes a real frequency shift.

Equation (20) and its solutions fairly differ from the well-known equations of modulational instability (and their solutions) in a collisionless plasma [6]. The most essential difference is in the cubic character of Eq. (20) regarding the modulation frequency $\Delta\omega$. The transversal character of waves considered in our situation does not play a crucial role; it is possible to demonstrate that the corresponding equation for Langmuir pump waves has the same dependence upon $\Delta\omega$ as (20). Let us also note that the instability rate is increased on the entire interval of the allowed values of $|\Delta\mathbf{k}|$, whose maximum is given by the so-called diffusion condition

$$|\Delta\omega| \gg \frac{|\Delta\mathbf{k}|^2 v_{Te}^2}{\nu_e}, \tag{24}$$

where $v_{Te}=(T_e/m_e)^{1/2}=v_s(3m_e/10m_i)^{1/2}$ is the electron thermal velocity. The condition (24) is in fact closely connected with the inequality (1); see details in [5]. From (21) and (24) we have the following maximum instability rate:

$$|\gamma_{\max}|=\nu_e\left[\frac{\Omega}{\nu_e}\right]^{1/2}\frac{v_s}{v_{Te}}. \tag{25}$$

By deriving (20) we have used an assumption $|\Delta\mathbf{k}| > \omega_{pe}^3\nu_e/\omega_0^3 c$, which gives the minimum value for the wave vector of the modulated perturbations. So the following threshold arises:

$$\frac{\Omega}{\nu_e} > \max\left\{\left[\frac{\omega_{pe}}{\omega_0}\right]^6\left[\frac{v_{Te}}{c}\right]^2, \left[\frac{3m_i}{10m_e}\right]\left[\frac{v_{Te}}{c}\right]^4\left[\frac{\omega_{pe}}{\omega_0}\right]^{12}\right\}. \tag{26}$$

We also stress that from the condition (1) we have

$$\Omega \ll \nu_e . \tag{27}$$

It is worth comparing the modulated instability rate and threshold with other nonlinear processes taking place in laser plasmas. Having in mind the inequality (3), we consider only strongly underdense plasmas. The most effective nonlinear process there is stimulated Raman backscattering (SRS, see [3] for details). Its rate is

$$|\gamma^{\mathrm{SRS}}|=\omega_{pe}\left[\frac{\omega_{pe}}{\omega_0}\frac{|E_0|^2}{4\pi n_0 m_e c^2}\right]^{1/2} \tag{28}$$

as well as the threshold

$$|\gamma^{\mathrm{SRS}}| > \frac{1}{2}\left[\frac{\omega_{pe}}{\omega_0}\right]^2\nu_{\mathrm{eff}} . \tag{29}$$

Comparing (25) and (28), we have

$$\frac{\gamma^{\mathrm{mod}}}{\gamma^{\mathrm{SRS}}}=\frac{10m_e}{3m_i}\frac{\nu_e}{\omega_{pe}}\left[\frac{\omega_{pe}^3}{\omega_0^3}\frac{c^2}{v_{Te}^2}\right]^{1/2}. \tag{30}$$

For the typical laser plasma parameters ($\omega_0\sim10^{15}$ rad/sec, $\omega_{pe}\sim10^{14}$ rad/sec, $\nu_e\sim10^{12}$ rad/sec, $v_{Te}/c\sim10^{-2}-10^{-1}$), the rate of the modulational instability is always less than the γ^{SRS}. But we note that the SRS is a resonant process occurring only in localized regions of the (inhomogeneous) plasma corona determined by corresponding energy and momentum matching conditions. The modulational instability is a nonresonant process (like filamentation of the laser light), and consequently no matching conditions are to be satisfied. Moreover, if we compare the thresholds of the modulational instability and the SRS, we find (we also take into account here the case of high-Z plasma)

$$\frac{\gamma_{\mathrm{thr}}^{\mathrm{mod}}}{\gamma_{\mathrm{thr}}^{\mathrm{SRS}}}\sim10^6\left[\frac{v_{Te}}{c}\right]^4, \tag{31}$$

this expression can be much less than unity, depending on plasma temperature. Thus the modulational instability can occur when the SRS is "switched off."

The considered case of relatively small instability rates [because of condition (1)] is useful to establish the instability threshold. It is also interesting to examine the case of larger rates, when an inequality opposite to (1) takes place. For this situation the modulated instability rate can exceed the rate of the SRS.

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