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Stochastic properties of the modulational interaction in packets of random waves

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Abstract

The stochastic properties of turbulent plasma waves are taken into account in the consideration of the modulational instability. The limits of applicability of the theory of weak plasma turbulence in the description of the modulational processes are found. It is shown that this theory can be applied for the description of the modulational interaction only for sufficiently wide wave spectra (and/or sufficiently low levels of plasma turbulence) when the modulational interaction of different modes in the packet is small. It is demonstrated that the development of the modulational interaction processes results in a rapid decrease of the wave stochasticity.

1. The basic process in the transition from weak to strong turbulent plasma state is the modulational instability [1–3]. At present, strong turbulence theory uses the concept of turbulence consisting of many collapsing cavities [1–4] as coherent structures which are randomly distributed in space. The self-contraction (or collapse) of Langmuir waves is a phenomenon which is caused by the modulational instability and is similar to the self-focusing effect in optics. The modulational instability develops when waves refract into density wells in a plasma, which are regions of high refractive index. If they exceed a critical intensity, their ponderomotive force pushes aside the plasma, deepening the density well, raising the refractive index, and leading to a further focusing of waves into the region. The strongly turbulent state is characterized by strong phase correlations of the excited modes within coherent objects (e.g., solitons and nonlinearly collapsing wave packets) with a chaotic interaction between these objects.

The theory of weak plasma turbulence [5–7] is now well elaborated; the key feature which allows the construction of this theory is the so-called random phase approximation applied to Gaussian distributed quantities [6]. Thus the theory of weak plasma turbulence is valid for systems with developed stochasticity [6,7] when the wave phases are random, the arbitrary wave motion can be represented as a linear superposition of oscillation modes, and the wave amplitudes slowly change in time due to interaction with other waves and particles of a plasma. Such an approach can be applied for the description of plasma processes only if the wave amplitudes are sufficiently small. As the am-

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plitudes grow, phase correlations become significant. The modulational interaction results in the amplification of phase correlations, and finally the strong turbulent state is established with different (compared to weak turbulence) characteristics. The growth of the wave phase correlations can result in the appearance of coherent structures such as solitons, collapsing wave packets, etc.

The physics of the modulational instability of a single monochromatic pump mode is now fairly well understood [1–4]. However, the modulational interactions of broad wave spectra have peculiar features absent in the case of a single monochromatic pump [8]. For example, each mode in the broad wave packet cannot be modulationally unstable independently of other modes. This is the reason why for the wave packets we study modulational interaction of different modes (but not their modulational instability). As an example, the modulational interaction of two monochromatic pumps has been considered [9–11]; it has been demonstrated that this interaction has some features similar to those of wave packets. In particular, the modulational interaction of two monochromatic pumps as well as of a wave packet is suppressed in comparison with the case of a single monochromatic pump. Furthermore, for the case of random modes, their correlations are amplified because of the modulational interaction.

Although it has been realized sufficiently a long time ago [1–3] that the modulational interaction leads to the amplification of phase correlations and, consequently, to the transition from the state with the developed wave stochasticity to the strong turbulent state, the exact proof that the modulational interaction with necessity results in such a transition has not been presented. Therefore, the corresponding range of parameters where the description of plasma processes based on the theory of weak turbulence is valid, has not been established. Thus, it is necessary to determine the values of the parameters when the effects of phase correlations can be neglected, and to demonstrate that the modulational interaction in the systems with the developed wave stochasticity results in the decrease of the latter.

In the present paper, we find the conditions determining the spectral width of the Langmuir oscillations and their energy density when the modulational interaction can be described on the basis of the theory of weak plasma turbulence. We demonstrate that the development of the modulational interaction diminishes the plasma stochasticity that is connected with the amplification of wave phase correlations. This process corresponds to a transition from the state with developed stochasticity to the strong turbulent plasma state. The limits on the characteristic wave numbers as well as wave the turbulence level are found.

2. We consider Langmuir oscillations with a sufficiently wide spectrum which is concentrated in the region of small wavevectors (the so-called energy-containing region [1]), so that the spectral width Δk in the wavevector space is of the order of the characteristic wavevector $k_{\rm ch}$ in the spectrum.

The plasma system can be described on the basis of the weak turbulence theory if its stochasticity parameter S is sufficiently large,

$$S \gg 1. \qquad (1)$$

This parameter can be expressed as follows [6],

$$S \sim (\tau_{\rm r}/\tau_{\rm K})^{3/4}. \qquad (2)$$

Here, $\tau_{\rm r} \equiv N\tau_{\rm ac}$ is the recurrence time of the correlation of a plasma particle with a wave packet (with spectral width Δk), N ($\sim \Delta k L/2\pi$) is the number of modes in the wave packet, L is the characteristic plasma length (which can be considered as the characteristic inhomogeneity scale),

$$\tau_{\rm ac} = \frac{2\pi}{|v_{\rm g} - v_\phi|\Delta k} \qquad (3)$$

is the autocorrelation time within which the correlation of a plasma particle with a wave packet (with spectral width Δk) is violated, $v_{\rm g}$ (v_ϕ) is the group (phase) velocity of waves in the packet, and

$$\tau_{\rm K} \sim (k_{\rm ch}^2 D)^{-1/3} \qquad (4)$$

is the time of the irreversible decay of the correlations (i.e., of exponential divergence of trajectories and their subsequent mixing in the phase space). The time $\tau_{\rm K}$ is of the order of the inverse Kolmogorov entropy [6,7]. Finally, in (4) D is the quasilinear diffusion coefficient.

For the Langmuir waves (under the assumption $k_{\rm ch} \sim \Delta k$) we find

$$D \sim \frac{W}{n_0 T_e} \omega_{pe} v_{Te}^2, \qquad (5)$$

where W is the energy density of the waves, n_0 is the unperturbed plasma concentration, T_e is the electron temperature, $\omega_{pe} = \sqrt{4\pi n_0 e^2/m_e}$ is the electron plasma frequency, e, m_e and n_0 are the electron charge, mass and density, respectively, and $v_{Te} = \sqrt{T_e/m_e}$ is the electron thermal speed.

Inequality (1) can be rewritten in the form

$$(k_{ch}L)^{3/4}(k_{ch}r_{De})^{1/2}(W/n_0T_e)^{1/4} \gg 1, \qquad (6)$$

where $r_{De} \equiv v_{Te}/\omega_{pe}$ is the Debye length. We note that the length of the plasma inhomogeneity L in inequality (6) for a developed modulational instability can be estimated as the characteristic length of plasma modulations: $L \sim L_{mod} \sim 1/k_{mod}$, where k_{mod} is the characteristic wavevector of the modulational perturbations.

The characteristic feature of the modulational instability of Langmuir oscillations in the cases of both monochromatic pump and broad wave spectra is (see, e.g., Refs. [1–4,7]) the possibility of its development in the limit of long-wavelength pumping (in the case considered this is the limit $k_{mod} > k_{ch}$). Moreover, this limit is the most interesting one because the development of the modulational interaction with long-wavelength perturbations allows us to explain the well-known paradox of the weak turbulence theory, namely the paradox of the oscillation condensate. Also, this limit corresponds to maximum instability rates [1–3]. Thus, we have

$$k_{ch}L < 1. \qquad (7)$$

However, this condition is in contradiction with inequality (6). Indeed, for Langmuir waves we have $k_{ch}r_{De} < 1$ and $W/n_0T_e < 1$ (note that in the processes of the modulational instability $W/n_0T_e \sim \delta n/n_0 < 1$, see Ref. [4]). Hence, inequality (6) can be fulfilled only if $k_{ch}L \gg 1$ which contradicts condition (7).

We have shown that if $k_{mod} > k_{ch}$, the condition of plasma stochasticity (1) is violated. Note that, in principle, the modulational interaction takes place also for $k_{mod} \ll k_{ch}$. However, the rates of the instability in this case are small. On the other hand, the development of the instability with maximum possible rates immediately leads to a significant diminishing of the stochasticity parameter S. This means that in the process of the modulational instability the stochasticity of the system is rapidly diminished.

We note that inequality (6) imposes severe restrictions on the possible wave numbers of the modulated perturbations. Indeed, the description of the modulational interactions based on the random phase approximation is not valid for the modulational wave vectors satisfying the following inequality,

$$k_{mod} > k_{eff} \sim k_{ch}(k_{ch}r_{De})^{2/3}(W/n_0T_e)^{1/3}. \qquad (8)$$

The r.h.s. of this inequality is always less than k_{ch}: $k_{eff} \ll k_{ch}$.

Moreover, condition (6) allows us to determine the limits on the energy density of Langmuir oscillations and their spectral width Δk when the description of the modulational instability on the basis of the theory of weak turbulence is still correct. The characteristic length of the wavevector of modulational perturbations k_{mod} is of the order [1–4]

$$k_{mod}^2 r_{De}^2 \sim W/n_0T_e. \qquad (9)$$

Substituting $k_{mod} \sim 1/L$ from Eq. (9) in inequality (6) we find

$$\Delta k \sim k_{ch} \gg \frac{1}{r_{De}} \left(\frac{W}{n_0T_e}\right)^{1/10}. \qquad (10)$$

We see that the description of the modulational interaction on the basis of the theory of weak plasma turbulence is valid only for very wide spectra, or for very low turbulent levels in a plasma,

$$\frac{W}{n_0T_e} \ll (\Delta k r_{De})^{10}. \qquad (11)$$

In real plasma processes, this inequality can easily be violated.

3. Let us illustrate how the stochasticity of the system is diminished in the process of the modulational instability development. The stochasticity parameter is usually considered as a measurement of island overlap in velocity space [6]. As it is well known, stochasticity ensues when, in order of magnitude, adjacent islands overlap – that is, when the island width (the width of the trapping region) exceeds the velocity spacing between adjacent resonances. The latter can be presented as $\delta V \approx \delta k(v_g - v_\phi)/k_{ch}$, where δk, the wavevector spacing between adjacent resonances, is inversely proportional to the characteristic inhomogeneity scale ($\delta k \sim 2\pi/L$).

As it has been mentioned above, the modulational instability results in self-contraction of Langmuir waves in some regions of a plasma. This process is accompanied by a deepening of the density wells in these regions. Moreover, in these regions the characteristic inhomogeneity scale is decreasing. During the process of self-contraction the phase correlations between different Langmuir waves increase strongly. The decrease of the inhomogeneity scale leads to an increase of the velocity spacing between adjacent resonances. The extent of the island overlap diminishes and, consequently, the stochasticity of the system decreases also. We note that this decrease of the plasma stochasiticity is closely connected with the increase of phase correlations in the wave packet in the modulational instability development. This process of the amplification of wave phase correlations is significant for sufficiently high levels of the wave energy density in plasmas when we have modulational interaction of different modes in the packet. For weak turbulent levels (determined by inequalty (11)) we have the modulational instability of every mode which does not affect significantly any other modes.

We emphasize that inequality (6) (following from the condition $S \gg 1$) determines in the situation considered an *upper* limit on the level of the energy density of waves W. This seemingly contradicts an impression which can arise when considering this inequality for constant plasma inhomogeneity length. However, the self-consistent consideration of the modulational processes results in the necessity to take into account the change of this parameter that, in turn, allows us to give the *upper* limit on W.

4. To summarize, we have demonstrated that in processes of modulational interaction, the stochasticity of the system is significantly diminished. This effect is closely connected with the amplification of wave phase correlations that finally leads to the establishment of the strong turbulent state in a plasma. The increase of wave phase correlations is valid even for wave spectra which can be considered initially as random ones. We have determined the limits when one can study the effects of the modulational interaction on the basis of the theory of weak plasma turbulence. Such a study is correct only for sufficiently wide wave spectra, or for sufficiently low levels of turbulence in a plasma (when inequalities (10) and (11) are satisfied). Moreover, the wave numbers of modulated perturbations should be significantly smaller than the spectral width, see inequality (8). In fact, the latter condition means that the increase of correlations of different modes in the wave packet will be small only in the case when the effective number $N_{\text{eff}} \sim k_{\text{eff}}L/2\pi \sim k_{\text{eff}}/k_{\text{mod}}$ of modes (which modulationally interact with each other) exceeds unity. In other words, inequality (8) gives us the condition when instead of the modulational interaction of every mode which does not affect significantly any other modes in the wave packet we have the modulational interaction of different modes in the packet. Only in the former case the system's stochasticity has the level which justifies use of the random phase approximation and, consequently, the weak turbulence theory.

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