Quantum Phases for Quantum Computation
Renormalization & Symmetry-Protected Topological Order

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Motivation

State of the Art Computers

Classical Intel 6 core processor (Gulftown); \(\sim 10^9\) transistors

Quantum NIST Racetrack ion trap; \(\sim 100\) ions (?)

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No problem: use error correction!
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Trouble: noise and reliability

Pretty soon we’re swimming in it

No problem: use error correction!
Can we solve this problem in hardware?
Motivation

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Classical solution
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Classical solution

Quantum solution
Can we solve this problem in hardware?

Classical solution

Quantum solution
Perhaps some exotic quantum phase of matter? Anyons?
(graphene FQHE, Andrei group Rutgers)
Can we solve this problem in hardware?

Ambitious
(Intel 4004, 1972)

Quantum solution
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Is there something a little easier to build?
Motivation

Yes!

The Haldane phase of spin-1 chains offers several interesting ideas:

- MBQC renormalization
- Holonomic QC from symmetry-protected topological order
Quantum computational renormalization in the Haldane phase

First, the short version

- Can define MBQC model at the AKLT point, in the Haldane phase
- Gate fidelities decay as we move away from AKLT
- But there’s an RG flow towards AKLT, so just measure the block spins!
Quantum computational renormalization in the Haldane phase

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- Can define MBQC model at the AKLT point, in the Haldane phase
- Gate fidelities decay as we move away from AKLT
- But there’s an RG flow towards AKLT, so just measure the block spins!
- That would require multispin measurements, so you could do QC anyway
- Simulate block measurements with single-site measurement & postselection!
- QC ability is a property of the phase, in this sense
AKLT spin-chain

Affleck-Kennedy-Lieb-Tasaki nearest-neighbor Hamiltonian

\[ H_{AKLT} = \sum_{j=1}^{n} (\vec{S}_j \cdot \vec{S}_{j+1}) + \frac{1}{3} (\vec{S}_j \cdot \vec{S}_{j+1})^2 \approx \sum_{j=1}^{n} (P_2)_{j,j+1} \]

- Ground state is unique under periodic BCs or \( n \to \infty \); 4fold degenerate under open BCs and \( n < \infty \)
- Gap to first excited state (conjectured by Haldane, analytic example by AKLT)
- Ground state is a “valence bond solid” (VBS), frustration-free
Chain encodes one logical qubit (think of it at C); $|s\rangle \equiv |J_s = 0\rangle$ & $s = \hat{x}, \hat{y}, \hat{z}$.

$|G_0\rangle = \sum_{\{s_k\}} |s_1, s_2, \ldots, s_n\rangle_B \otimes \left(\sigma_{s_n} \sigma_{s_{n-1}} \cdots \sigma_{s_1}\right)_C |\psi^-\rangle_{AC}$
Chain encodes one logical qubit (think of it at $C$); $|s\rangle \equiv |J_s = 0\rangle$ & $s = \hat{x}, \hat{y}, \hat{z}$.

Initialize: Measure $|0\rangle, |1\rangle$ on end qubit $A$
MBQC with AKLT

\[
|G_2\rangle = |1\rangle_A \otimes |\hat{Z}\rangle_{B_1} \otimes \sum_{s_2, \ldots, s_n} |s_2, \ldots, s_n\rangle_{B} \otimes \sigma_{s_n} \sigma_{s_{n-1}} \cdots \sigma_{s_2} |0\rangle_C
\]

- Chain encodes one logical qubit (think of it at C); \(|s\rangle \equiv |J_s = 0\rangle \) & \(s = \hat{x}, \hat{y}, \hat{z}\).
- Initialize: Measure \(|0\rangle, |1\rangle\) on end qubit \(A\)
- Measuring in the \(|s\rangle\) basis rotates the qubit by \(\pi\) around \(s\)
MBQC with AKLT

\[ |G_3\rangle = |1\rangle_A \otimes |\hat{z}\rangle_{B_1} |\hat{z}'\rangle_{B_2} \otimes \sum_{s} |s_3, \ldots, s_n\rangle_B \otimes \sigma_{s_n} \sigma_{s_{n-1}} \cdots \sigma_{s_3} \sigma_{\hat{z}'\hat{z}} |0\rangle_C \]

- Chain encodes one logical qubit (think of it at C); \(|s\rangle \equiv |J_s = 0\rangle \& s = \hat{x}, \hat{y}, \hat{z}.
- Initialize: Measure \(|0\rangle, |1\rangle\) on end qubit \(A\)
- Measuring in the \(|s\rangle\) basis rotates the qubit by \(\pi\) around \(s\)
- Works for rotated basis \(|s'\rangle\), too, by spherical symmetry
- Combine measurement in different bases to perform arbitrary rotations
- Compound rotations are probabilistic, but heralded
Haldane Phase Renormalization

- Gate fidelity decreases when using non-AKLT ground states
- What to do? Renormalize!
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What to do? Renormalize!

Renormalization recipe:
1. From three adjacent spins, extract the $J = 1$ components
2. Discard the one antisymmetric in (1, 3) permutations
3. Thoroughly mix the remaining two until a nice consistency is reached
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Simulate the block spin measurements!
MBQC Renormalization

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- Simulate the block spin measurements!

Suppose we want to do a $\pi$ rotation around $\hat{x}\cos \theta + \hat{y}\sin \theta$.

$$|z, \theta, z\rangle_{123} \propto |\theta\rangle_J |\chi_s\rangle_L + J \neq 1 \text{ component},$$

$$|z, z, z\rangle_{123} \propto |z\rangle_J |0\rangle_L + J\neq 1 \text{ component}.$$

*Buffered* measurement effectively replicates the block spin measurement.
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Buffered measurement effectively replicates the block spin measurement

Success

| Z | θ | Z |

Failure

| Z | Z | Y |
Buffering Works

But needs a lot of postselection

(a) Buffered $\pi/2$ Rotation Fidelity

(b) Buffering Probability (relative to AKLT)
Holonomic quantum computation from symmetry-protected topological order

First, the short version

- Haldane phase possesses SPTO
- Symmetries of SPTO also define qubit encoding, gates
- Architecture inherits some protection from SPTO
SPTO of 1D systems

- Topological order doesn’t exist for 1D systems. All states are $\sim$ product states.
- But in the presence of certain symmetries, distinct phases appear.
- For spin-1 chains $\Rightarrow$ Haldane phase.

What symmetries?
- $\pi$ rotations about orthogonal axes ($D_2$), time-reversal, bond inversion.

What properties?
- Gapped ground state, fourfold degenerate.
- Fractionalized spin-$\frac{1}{2}$ edge modes.
- Nearest-neighbor, two-body couplings.

$H_0 = \sum h_{j,j+1}$.
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note $D_2$ symmetry
Two-qubit gate: CPHASE + $\hat{x} \pi$ rotation

\[ H(t) = t \left[ W^{AB} - h_1^A - h_1^B \right] + H_0^A + H_0^B \]

\[ W = \left[ (S_1^x)^2 - (S_1^y)^2 \right] \otimes S_1^z + S_1^z \otimes \left[ (S_1^x)^2 - (S_1^y)^2 \right] \]
Two-qubit gate: CPHASE + $\hat{x} \pi$ rotation

$$H(t) = t \left[ W^{AB} - h^A_{12} - h^B_{12} \right] + H^A_0 + H^B_0$$

$$W = \left[ (S^x_1)^2 - (S^y_1)^2 \right] \otimes S^z_1 + S^z_1 \otimes \left[ (S^x_1)^2 - (S^y_1)^2 \right]$$

not $D_2$ symmetric, but doesn’t close the gap
Turn off coupling, measure $J_z$

- $+1 \rightarrow |↑\rangle$
- $-1 \rightarrow |↓\rangle$
- $0 \rightarrow R_z(\pi)$
Turn off coupling, measure $J_z$

- $+1 \rightarrow |\uparrow\rangle$
- $-1 \rightarrow |\downarrow\rangle$
- $0 \rightarrow R_z(\pi)$

Need full $SO(3)$ symmetry!
Advantages

▶ Just operate on the boundary spin (don’t consume spins, as in MBQC)
▶ Only 2-body interactions
▶ Don’t need terribly long chains: edge modes well-localized
▶ Don’t even need chains at all: can terminate with spin-1/2s!
   Or convert everything to spin-1/2.
▶ Robust to symmetry-preserving disorder in the couplings:
   Only care about total angular momentum
▶ Gates “immune” to timing errors, intensity fluctuations
▶ Only need a small number of fixed control fields
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- Indications of limited protection against local noise @ low temperatures
  - Rotating bulk spins doesn’t affect the logical state
  - Bigger rotations cost more energy; remove via cooling
  - Rotating boundary spin does affect the logical state
  - Error rates should be suppressed