L3 IMAGES

Aims

From this chapter you should develop an understanding and appreciation of what images are and how they are formed. In particular you will learn how lenses work. You will also learn how to solve simple quantitative problems involving image formation by lenses, using both geometrical and algebraic methods.

Minimum learning goals

1. Explain, interpret and use the terms:
   (a) image, object, real image, virtual image,
   (b) plane mirror, lens, convex lens, converging lens, concave lens, diverging lens, positive lens, negative lens, thin lens,
   (c) paraxial approximation, paraxial rays, converging beam, parallel beam, diverging beam,
   (d) principal point, principal axis, principal plane, focal point, focal plane, focal length, power, dioptré,
   (e) lens equation, sign convention,
   (f) object distance, image distance, object position, image position, linear magnification, lateral magnification, longitudinal magnification, virtual object.

2. Explain the nature of images and how they are formed.

3. Describe and explain how lenses and plane mirrors produce images.

4. Describe and apply standard ray-tracing procedures for both thin and thick lenses.

5. Describe and explain those properties of a lens which determine its power. State and apply the lensmaker's formula.

6. Solve simple problems involving image formation and magnification, by single thin lenses or a set of thin lenses with the same principal axis, using both ray-tracing and algebraic techniques. Describe and explain the rules and techniques used in such problems.

3-1 IMAGES

The reflection of your face in a mirror, the view of a small insect under a microscope and the picture on the big screen at the movies are all optical images. They are formed by light rays whose paths have been altered by the action of a mirror or a lens on the light coming from an object.

It is convenient to classify images into two types, called real and virtual. In a real image the light actually goes to (or through) the image. Examples include the image on the cinema screen and the light image formed on the film in a camera. Real images are often formed on screens or other solid surfaces, but a screen is not essential. Even if the cinema screen were to be suddenly whisked away in the middle of a movie, a real image would still exist in the same place, but to see it you would have to leave your seat and go to a place, well behind where the screen used to be, where you could look back towards the projector. Images formed in the eye are also real images.

The light rays which form a virtual image do not actually pass through the image - they only appear to be coming from it. The most familiar example of a virtual image is a "reflection" in a mirror. Although the image is located some distance behind the mirror, the light does not actually go there or come from there. It is an illusion that the light comes from behind the mirror.
The most important image-forming system of all is the human eye. Whenever you see something, real images are being formed on the retinas of your eyes. Sometimes this means that the eye forms an image of an image. This happens, for example, when you see the virtual image of your face in the mirror or an image formed by a lens.

3.2 RAY TRACING

The most basic technique for calculating how images are formed is ray tracing, which is just the construction of diagrams showing where light rays from an object go. Starting from some point on the object, a ray is drawn towards the optical device. Where the ray meets a reflecting or refracting boundary the laws of reflection and refraction give the new direction of the ray. The process continues until the ray does not meet any more surfaces. The whole procedure is then repeated for another ray, starting at the same object point, but heading off in a different direction. In principle, this procedure should be repeated for a large number of rays. Once all the rays have been plotted you can look for a point where they all meet. If such a point can be found then a real image of the object point exists there. If the rays do not actually meet at a point, there is no real image but there may be a virtual image. To look for a virtual image, the rays which finally come out of the optical system are projected backwards to find out if the those projections meet. If they do, then their intersection gives a virtual image point. The tracing of many rays leads to the location of one image point for each object point. To locate the whole image the complete procedure is repeated for other object points, gradually building up a map of the complete image. The complete process is tedious, time-consuming and very accurate. It is one way that professionals can check the design of complex and expensive optical systems. In practice, ray-tracing calculations and plotting are mostly done by computers.

Ray tracing is also the basis of much simpler procedures for calculating the approximate locations of images in simplified optical systems. In some simple cases, exact answers can be found very quickly with little effort. A good example of ray tracing is to locate the virtual image formed by a plane mirror.

3.3 PLANE MIRROR

Figure 3.1 shows the image formed by a flat reflecting surface. The diagram shows just a few of the many rays diverging from the same object point. Each ray is reflected so that its angle of reflection is equal to the angle of incidence. The rays themselves do not meet so there is no real image, but if the reflected rays are projected backwards (shown by gray lines) they do meet. Since light does not really come from that point behind the mirror it is a virtual image point. If more rays are added to the diagram it is found that all the reflected rays diverge from the same virtual image.
point; so each image point is unique. That result indicates that a really flat mirror produces a sharp
to image - does that match your experience?

![Figure 3.2. Geometry of mirror reflection](image)

It is actually quite easy to show that with a plane mirror all the rays from one image point are
reflected so that they diverge from one virtual image point. Figure 3.2 shows just two reflected rays,
one normal to the mirror and another one. Marking up equal angles \( \theta \) and spotting the similar
triangles in the diagram shows that the distance of the image point from the mirror is equal to the
distance of the object point from the mirror and that the line joining them is perpendicular to the
reflecting surface. Since that result is true for any value of the angle \( \theta \) it is true for all reflected rays.
These diagrams also illustrate why the study of image formation is called geometrical optics.

3-4 REFRACTION AT A CURVED BOUNDARY

Whenever light crosses the boundary between two optical media which have different refractive
indices the light bends or refracts. As a light ray goes from low to high refractive index it bends
towards a line normal to the surface; when it goes from high to low refractive index it bends away
from the normal. See figure 3.3.

![Figure 3.3. Refraction at a curved boundary](image)

The medium with the higher refractive index, \( n \), is shaded. The broken line is normal to the surface.

**Parallel beam**

When we consider light from a very distant object all the rays from one point of the object are
practically parallel to each other and are said to form a **parallel beam** of rays. Since it is often said
that parallel lines "meet at infinity" an object point which produces a parallel beam can be described
as being at infinity. In practice that means that the distance to the object is very large compared with
the other relevant distances. Many aspects of the behaviour of optical systems can be explained in terms of what they do to a parallel beam, or rays from objects at infinity.

Figure 3.4 shows what can happen to a parallel beam of light (coming from an object point at infinity) when it crosses a spherical boundary between two materials with refractive indices \( n' \) and \( n \). In this case the boundary is convex towards the incoming beam and \( n' \) is less than \( n \).

Each ray obeys the law of refraction and the refracted rays form a converging cone of light. If the beam is narrow compared with the radius of curvature of the surface all the rays will pass close to the same point. The rays are said to come to a focus - the beam has been focussed to form a real image of the object point. The focussing effect or degree of convergence produced by the surface depends on its curvature. A flat surface has no curvature so it can produce no convergence - an incident parallel beam will still be parallel after refraction, although it may be travelling in a different direction. The greater the curvature the greater is the converging effect. Strong focussing is also produced by a large contrast in refractive indices. If the refractive indices of the two materials were equal there would be no focussing effect, but a large ratio \( n/n' \) can produce strong focussing provided that there is also some curvature.

An example of focussing by a curved surface is the action of the cornea of the eye (figure 3.5). The cornea has a curved boundary with the air and most of the eye is filled with transparent materials which have fairly uniform refractive index. Therefore most of the refraction of incoming light occurs at the front surface of the cornea. Some refraction also occurs at the two surfaces of the lens of the eye, but the contrast of refractive index there is quite small so the focussing by the lens is weak. The lens is used essentially for fine adjustment in the focussing of visual images. For more about the eye see chapter L7.

3-5 LENSES
A simple lens is an optical device, usually made of glass or clear plastic, which can form images. Most lenses have a circular outline and two curved, often spherical, faces. They form images by
refracting rays of light at both of their surfaces. If you knew the precise shape of a lens and the refractive index of the material in it you could use ray tracing to calculate (or make a computer calculate) the locations of images. In general you would find that lenses do not produce perfectly sharp images and there are two kinds of reason for that. Firstly, diffraction effects (which will, be considered in chapter L5) produce a blurred image for every object point. The effects of diffraction can be minimised, but not eliminated, by using large lenses. Secondly, even if diffraction did not exist there would still be the geometrical restriction that it is not possible to design a lens, or a system of lenses, which produces a unique image point for every possible object point. On the other hand it is possible to design lenses and systems of lenses which produce satisfactory images. The quality of the image can be improved using a more complex design - at greater cost.

Types of lenses
There are two basically different kinds of lens. When used in air, converging lenses, which are thicker in the middle than at the edges, make a parallel beam of light converge. A diverging lens, which can turn a parallel beam into a diverging beam, is thinner in the middle.

Action of a converging lens
Figure 3.6 shows how a converging lens affects a light ray passing through it. The total effect is just that of refraction at two successive curved surfaces.

As the ray passes through the first surface it is refracted towards the normal and it then continues in a new direction through the glass until it arrives at the other side. There the ray goes from glass to air so it is refracted away from the normal and emerges in a new direction. These changes in direction of the ray can be calculated using the law of refraction (Snell's law) so the path of any ray can be traced out in this way. In principle all you need to do to find out if an image exists, and where it is located, is to trace out rays. Ideally we would like to have all rays from one object point arriving at a unique image point. In reality that is impossible to achieve for all object points, but with good design, it can be achieved approximately.

Principal axis
Most lenses are symmetrical about an axis or line through the middle of the lens. If you rotate the lens around this axis it looks just the same. In optics that axis of symmetry is called the principal axis of the lens. From now on we think of every lens as having rotational symmetry about its principal axis. That is a fortunate simplification because it allows us to describe and work out the optics of lenses using two-dimensional drawings and constructions. You need to remember, however, that in reality lenses, objects and images are three-dimensional structures. The usual way of drawing lens diagrams is to draw a line across the paper to represent the lens's principal axis (figure 3.7). Rays are drawn on a two-dimensional diagram which represents a slice through the lens and the three-dimensional bundles of rays.
Focal point and focal length

The meaning of the term *converging lens* is illustrated by the lens's action on a parallel beam of rays coming from a distant object point that is located on the principal axis. When the rays come out the other side of the lens they form a converging cone of light (figure 3.8).

An ideal converging lens would refract all the rays in a beam parallel to the principal axis so that they pass through one real image point, which is called the **focal point** of the lens. (In reality the image "point" is always a bit blurred.) The light is said to have been **focussed** by the lens. The distance from the focal point to the middle of the lens is called the **focal length** of the lens.

Paraxial rays

The somewhat blurred image "point" corresponding to a point object can be made sharper by restricting the rays which form the image so that they are close to the principal axis and also by making sure that the angle between any ray and the principal axis is small. Such rays are said to be **paraxial**. Paraxial rays don't have to be parallel to the principal axis but they do have to be close to it. In most of the lens diagrams in this and other texts the angles between the rays and the principal axis are often quite large and the rays may be a fair distance from the axis; so the drawings are not good representations of the paraxial condition. The angles and off-axis distances are generally exaggerated so that you can see the features of the ray diagrams more clearly.

If the distant object point is not located right on the principal axis, but is off to one side, the incoming bundle of parallel rays will be at an angle to the principal axis. If that angle is small the rays satisfy the **paraxial approximation** and the converging lens will produce a reasonably sharp image point, as shown in figure 3.10. In this case the focus or image point is not on the principal axis but it does lie in a plane, called the **focal plane**, which is perpendicular to the principal axis. The focal point of the lens is at the intersection of the focal plane and the principal axis.
Figure 3.10. Focussing by a converging lens
Incoming parallel rays from any small angle come to a focus in the focal plane.

An important feature of all ray diagrams is that if you reverse the directions of all the rays then you get another valid diagram. The light paths are said to be reversible. So, for example, to find out how a lens affects the rays coming from an object point at a lens's focal point, you could just reverse all the rays in figure 3.9. That would give light going from right to left instead of the usual left to right, so for consistency the diagram is reversed left-to-right, which gives figure 3.11.

Figure 3.11. Object point at the focal point of a converging lens
This diagram is like figure 3.9 with the rays reversed. Light paths are reversible.

Thus every lens has two focal points, one on each side. Provided that the lens is immersed in the same medium on both sides, the two focal lengths are equal.

Converging and diverging lenses
Figures 3.9 and 3.10 show how a converging lens makes a parallel beam of light into a converging beam. A converging lens can also make a converging beam into an even more converging beam (figure 3.12). It can also refract a diverging beam into less diverging beam, a parallel beam or even into a converging beam (figure 3.14, below). In summary, a converging lens increases the convergence of any light beam which passes through it.

Figure 3.12. Increasing the convergence of a beam
The beam is focussed before the focal point.

A diverging lens is thinner in the middle than it is at its edge. It bends the parallel beam from a distant point into a diverging cone of rays which (ideally) appear to come from one virtual image point. Here the focal point and the focal plane are on the same side of the lens as the incident light.
Just as a converging lens increases the convergence of a bundle of rays, a diverging lens decreases the convergence - or you could say that it increases the divergence of the rays.

**Power of a lens**

The shorter the focal length of a converging lens the better it is at converging light. This characteristic of a lens, its converging ability, is called **power**, which can be defined formally as the reciprocal of focal length:

\[ P = \frac{1}{f} \quad ... \ (3.1) \]

The SI unit of optical power is the **reciprocal meter** (m\(^{-1}\)). A commonly used alternative name for the unit, is the **dioptre**. For example, if the focal length is +0.500 m, the power is +2.00 m\(^{-1}\) or 2.00 dioptres and if the focal length is -0.040 m, the power is -25 m\(^{-1}\). Converging lenses always have positive values of power and are often known as **positive lenses**. The negative value of power for a diverging lens expresses the fact that it does the opposite of converging; a diverging lens can be called a **negative lens**.

### 3.6 FORMATION OF IMAGES BY THIN LENSES

Figure 3.14 shows how a converging lens refracts the divergent bundle of rays from one point on an object into a bundle of rays which converge onto a real image point. The diagram shows only five of the many rays which could be drawn from the object point.
If all the rays are paraxial they will all come to a focus at the same image point. The actual paths of all these rays could be worked out by ray-tracing. Image points for other object points could be located in the same way. There are some features of this example which are worth noting; for an ideal thin lens and paraxial rays we get the following results.

- All object points which are at the same distance from the lens produce image points at equal distances from the lens. So the image position can be located using one object point and its image point. (Different object distances give different image distances.)
- If the object is placed further from the converging lens than the focal length its image will be real.
- The orientation of the real image is opposite to that of the object - it is said to be inverted.

**Simplified model of a thin converging lens**

If all the rays are paraxial, detailed ray tracing is not necessary. The function of a lens can be described completely in terms of a geometrical construction in which the lens is represented by a set of points, lines and planes and the location and size of an image can be found by tracing only two rays. Figure 3.15 shows the essential features of the model for a thin lens. The principal plane is a plane perpendicular to the principal axis located centrally within the lens. The first focal plane and the first focal point are located on the side of the lens where the light comes from. Light diverging from a point source in the first focal plane will emerge as a parallel beam. The second focal plane contains all the points where an incoming beam of parallel rays can come to a focus. The focussing properties of the lens are determined by the locations of the principal plane and the focal planes relative to the principal axis.

![Geometrical model for the function of a lens](image)

**Figure 3.15. Geometrical model for the function of a lens**

**Standard ray tracing**

The following procedure produces accurate answers provided that the actual situation is restricted to paraxial rays. To get accurate results you need to make a scale drawing - graph paper helps - but the procedure can also be used to make rough sketches to work out, for example, whether images are real or virtual and whether they are upright or inverted.
• First draw two perpendicular lines to represent the **principal axis** and the **principal plane** of the lens. The **principal point** is at the intersection of those lines. If you like you can include a small sketch above the principal plane to indicate the type of lens. Measure out and mark the positions of the two focal points. (For a converging lens the first focal point is the one on the side where the light comes from; for a diverging lens it is on the other side.)

• Mark the object position, **O**, and draw a line representing the object, perpendicular to the principal axis with one end on the axis.

• Choose an object point somewhere off the principal axis.

• From that object point construct *any two* of the following three rays (figure 3.16).
  1. An incident ray from the object point parallel to the principal axis is refracted so that it intersects the second focal point.
  2. An incident ray which intersects the first focal point is refracted parallel to the principal axis.
  3. An incident ray which intersects the principal point continues undeviated.

• The image point will be at the intersection of the two refracted rays. If those rays actually cross you have a real image. If the rays don't meet you will need to extend them to find where their extensions cross. In that case you have a virtual image. (It is a good idea to show these extended rays in a different style; in diagrams in this book they are printed grey.)

• Draw a line perpendicular to the axis to represent the image.

### Diverging lens

![Figure 3.16. Standard ray tracing for a converging lens](image)

**Figure 3.16. Standard ray tracing for a converging lens**

Note the first focal point is on the far side of the lens.

Figure 3.17 illustrates the construction for a diverging lens. Note that the first focal point is now on the far side of the lens. The rules for constructing rays are exactly the same. Again, any two of the three rays will do. In this case real rays do not intersect, they have to be extended backwards.
The lens equation

Image formation by a thin lens using paraxial rays can also be described by the **lens equation**:

\[
\frac{1}{o} + \frac{1}{i} = \frac{1}{f}.
\]  

Here \(o\) and \(i\) are the distances of the object and the image from the lens (see figure 3.17) and \(f\) is the focal length. This equation is simply the algebraic equivalent of the ray tracing procedures described above and will give exactly the same answers for paraxial rays. The equation works for both converging and diverging lenses provided that a suitable sign convention is used.

**Sign convention**

The lens equation requires the following **sign convention** (or an equivalent one).

- The object distance \(o\) is measured from the object to the lens.
- The image distance \(i\) is measured from the lens to the image.
- Object and image distances are positive if they are measured in the same general direction as that in which the light goes, negative otherwise.
- A negative value of image distance indicates a virtual image and a positive value means that the image is real.
- The focal length, \(f\), is always positive for a converging lens, and negative for a diverging lens.

The convention should be easy to remember because you just follow the light and use the natural sequences: (a) object - lens - image and (b) first focal point - lens - second focal point. Draw a single headed arrow to represent each distance. Then if any arrow points backwards (against the light) the corresponding distance value is negative. The convention means that the object distance is normally positive - for one real object and a single lens it is **always** positive. (It can have a negative value only in a compound optical system where there is an intermediate image which acts as a virtual object for the next component of the system. See chapter L7.)

The sign convention is illustrated by the directions of the arrows in figures 3.16 and 3.17. For the formation of a real image by a converging lens (figure 3.16) \(f\), \(o\) and \(i\) are all positive. In the case of the diverging lens (figure 3.17), \(f\) is negative, \(o\) is positive and \(i\) is negative.

**Example**

Find the image formed by a converging lens of focal length 35 mm when the object is placed 85 mm from the lens.

**Answer**

Rearrange the lens equation to get: \[
\frac{1}{i} = \frac{1}{f} - \frac{1}{o}.
\]

You can either start substituting in this equation, or continue the algebraic manipulation, making the image distance the subject:

\[
\frac{1}{i} = \frac{o - f}{o \cdot f};
\]

so

\[
i = \frac{of}{o - f}.
\]

\[
= \frac{(85 \text{ mm}) \times (35 \text{ mm})}{(85 - 35) \text{ mm}} = 60 \text{ mm}.
\]

Since value of \(i\) is positive the image is real and it is located 60 mm from the lens, or 145 mm from the object.
3-7 MAGNIFICATION

The magnification produced by a lens is defined as the ratio of the image size to the object size. If the sizes are specified in terms of lengths, the corresponding magnification is called a linear magnification whereas sizes described as angles subtended at the lens give angular magnification. Angular magnification will be considered in chapter L7 - here we concentrate on linear magnification. Linear sizes can be measured in different ways. If the dimensions of object and image are specified using measurements perpendicular to the principal axis, the linear magnification is called lateral magnification. It can easily be worked out from a standard ray diagram such as figure 3.18.

The similar triangles (shaded) show how magnification is related to object and image distances.

The magnitude of the lateral magnification is defined as the ratio of the image size to the object size:

$$|m| = \left| \frac{h_i}{h_o} \right|.$$  \hspace{1cm} ... (3.3)

The pair of similar triangles, shaded in figure 3.18, also shows that the magnification is given by the formula:

$$m = -\frac{i}{o}.$$ \hspace{1cm} ... (3.4)

where the minus sign is inserted so that a negative value for magnification indicates an inverted image. In figure 3.18 the real image is inverted and magnification is negative.

Figure 3.19 shows the same situation as figure 3.17; here the virtual image is upright and the magnification is positive. The two shaded triangles can be used to show that the magnification formulas (equations 3.3 and 3.4) are the same as those for the converging lens.
3-8 COMBINATIONS OF LENSES

The general case

To find an image formed by two or more lenses which are not in contact, you can’t just add the powers. But the image can be located by standard ray tracing or by repeated application of the lens equation. Both methods proceed in the same way. The first step is to calculate the image formed by the first lens alone, ignoring the second lens. Then use that image as the object for the second lens. The new object distance is found by subtracting the previous image distance from the lens separation. Then calculate the second image. Repeat the process for each lens in the system.

Example

Two converging lenses with focal lengths of 20.0 mm and 30.0 mm are placed 10.0 mm apart with their principal axes coinciding. An object is located 60 mm along the principal axis from the stronger lens. Find the position and lateral magnification of the image.

Answer

The first step is to calculate the image formed by the first lens alone - call it lens A. Proceed as though the second lens is not there. Rearranging the lens equation gives

$$\frac{1}{i_A} = \frac{1}{f_A} - \frac{1}{o_A} .$$

Substituting $f_A = 20.0 \text{ mm}$ and $o_A = 60 \text{ mm}$ gives $i_A = 30 \text{ mm}$. The result means that there would be a real image 30 mm from lens A. Now imagine that lens B is put in place 10.0 mm beyond lens A and use the image formed by lens A as the object for lens B (figure 3.20).

![Figure 3.20. Distances for calculations with two lenses](image)

The image formed by lens A acts as the object for lens B.

In this example there is a problem: lens B will intercept the light before it gets to the location of image A. The problem can be handled by saying that the object distance is negative and that the object for B is a virtual object.

Now calculate the new object distance from the previous image distance and the separation $L$ between the lenses: $o_B = L - i_A = 10.0 \text{ mm} - 30 \text{ mm} = -20 \text{ mm}$. Use the lens equation again:

$$\frac{1}{i_B} = \frac{1}{f_B} - \frac{1}{o_B} .$$

The final image distance is $i_B = 12.5 \text{ mm}$.

The answer means that the image is real and is located 12.5 mm past lens B, or 22.5 mm from lens A.

The final magnification is the product of the two individual magnifications:

$$m = m_A m_B$$

$$= \left( \frac{i_A}{o_A} \right) \left( \frac{i_B}{o_B} \right)$$

$$= \left( \frac{30 \text{ mm}}{60 \text{ mm}} \right) \left( \frac{12.5 \text{ mm}}{-20 \text{ mm}} \right) = -0.31 .$$
The negative value here indicates that the image is inverted.

The same answers can be obtained using standard ray tracing diagrams drawn to scale. Figure 3.21 shows how that can be done, using separate diagrams for each image, in order to avoid cluttering the construction too much.

![Ray tracing using two lenses](image)

Figure 3.21. Ray tracing using two lenses

The top part of the diagram shows how the intermediate real image is located, by ignoring the presence of lens B. When lens B is in place the intermediate image becomes a virtual object for lens B, so the lower part of the diagram shows virtual rays from the object extended back to lens B. On the front (left) side of the lens those rays can be drawn as real rays. The construction rules are still the same. A ray from the object parallel to the axis is bent to go through the second focal point. Another ray extended back from the object to intersect the first focal point emerges parallel to the axis. Since these two refracted rays are real their intersection gives the location of the real final image point.

**Thin lenses in contact**

When the separation between two thin lenses is small compared with each of their focal lengths calculations like the one above become much simpler - the two lenses can be treated as one! When two thin lenses of powers $P_1$ and $P_2$ are placed in contact the resulting power of the **compound lens** is the sum of the individual powers:

$$P = P_1 + P_2.$$ ... (3.5)

**Example.** The combined focal length of two converging lenses in contact, with individual focal lengths of 50 mm and 35 mm is given by

$$\frac{1}{f} = \frac{1}{50 \text{ mm}} + \frac{1}{35 \text{ mm}} .$$

So

$$f = \frac{50 \text{ mm} \times 35 \text{ mm}}{50 \text{ mm} + 35 \text{ mm}} = 21 \text{ mm}. $$

Note that the power is greater than that of either lens and the focal length is shorter. Now treat the combination as a single lens with focal length 21 mm.

**Example.** When a converging lens with power +5.0 m$^{-1}$ and a diverging lens with power -5.0 m$^{-1}$ are placed in contact, the light emerges unchanged: $P = +5.0 \text{ m}^{-1} + (-5.0 \text{ m}^{-1}) = 0.0 \text{ m}^{-1}$. 
3.9 LENS DESIGN

Types of lenses

Lenses are sometimes described in terms of the shapes, convex, concave or plane, of their two refracting surfaces, as illustrated in figure 3.22.

![Diagram of lens shapes]

**Figure 3.22. Naming some lens shapes**

In all cases the refractive index of the lens is greater than that of the surrounding medium.

Remember that all converging lenses are thicker in the middle than at the edges whereas diverging lenses are thin in the middle.

**Lensmaker's formula**

The focal length of a thin lens with spherical surfaces can be calculated from the curvature of the two faces and the refractive index of the lens material. Figure 3.23 shows a double convex lens made of material with a refractive index $n$ surrounded by a medium of refractive index $n'$. The radii of curvature of the two surfaces are $R_1$ and $R_2$. The power is given by the **lensmaker's formula**:

$$P = \frac{1}{f} = \left( \frac{n}{n'} - 1 \right) \left( \frac{1}{R_1} - \frac{1}{R_2} \right).$$

If the lens is surrounded by vacuum or air, $n'$ will be 1.000 and the formula can be written as

$$P = \frac{1}{f} = (n - 1)(\frac{1}{R_1} - \frac{1}{R_2}). \quad \text{... (3.6)}$$

The sign convention needs to be extended, as follows, to give signs to the values of the radii. Imagine the light coming from a particular direction and label the first surface reached by the light number 1; the other is number 2. Measure each radius from the surface to the centre of curvature. An equivalent statement is that if a surface is convex to the incident light, its radius is positive and if it is concave, the radius is negative.
In the example shown in figure 3.23, imagine the light coming from the left so the radius, \( R_1 \), of the first refracting surface is positive and \( R_2 \) is negative. The negative value of \( 1/R_2 \) will be subtracted from the positive value of \( 1/R_1 \), which must give a positive answer. You will get the same result if you take the light coming in from the right and swap the labels 1 and 2.

**Example**

Find the power and focal length of a biconvex lens made of glass with refractive index 1.53 whose surfaces have radii of curvature 0.250 m and 0.400 m.

**Answer**

Use the lensmaker's formula with \( R_1 = 0.250 \text{ m}; \ R_2 = -0.400 \text{ m}; \ n = 1.53 \).

\[
P = (n - 1) \left( \frac{1}{R_1} - \frac{1}{R_2} \right) = 0.53 \left( \frac{1}{0.250 \text{ m}} - \frac{1}{0.400 \text{ m}} \right) = +3.4 \text{ m}^{-1}.
\]

[Keep 3.445 in the calculator.]

\[
f = \frac{1}{P} = 0.29 \text{ m}.
\]

You can check that you will get the same answer if you swap the sequence and put \( R_1 = 0.400 \text{ m}; \ R_2 = -0.250 \text{ m}. \)

### 3-10 ABERRATIONS

Aberrations are departures from the desired or ideal shape of an image produced by an optical system. They are not necessarily faults in the design of a lens but are due simply to the inability of any lens to produce perfect images from a wide range of objects. In fact the only perfect image-forming optical device is the plane mirror, which is unable to magnify.

**Spherical aberration**

When a wide parallel beam is focussed by a converging lens with spherical surfaces the focal length for rays which go through the outside margins of the lens is shorter than that for paraxial rays. See figure 3.24. Only rays which are close to the principal axis and nearly parallel to it are brought to a focus at a well-defined focal point \( F_p \). So when a wide beam is used parts of the image will be out of focus.

![Figure 3.24. Longitudinal spherical aberration](image)

Marginal rays come to a focus \( F_m \) before the focus of the paraxial rays \( F_p \).

There are three ways to minimise spherical aberration.

1. **Use a lens whose curved surfaces have specially computed non-spherical shapes.** Such a lens is expensive to make and works only for a limited range of object positions. However, the aberrations can be minimised if the rays are bent approximately the same amount by both surfaces of the lens. So if you have a plano-convex lens spherical aberration will be less if you put the curved face rather than the plane face towards the light.

2. **Correct the aberrations due to a positive lens by placing next to it a negative lens whose aberrations are in the opposite sense.** This technique is not very efficient as you end up with a lens of very little power.
3. Restrict the angles of incidence by using only the central section of the lens and placing the object close to the principal axis. All rays will then be paraxial rays, so the lens equation works for all object and image points. The disadvantage is that you don't get much light through the lens, so the image is not very bright.

**Chromatic aberration**

The focal length of a simple lens depends on the refractive index of the material but that varies with the frequency of light. So there are different focal points for different frequencies. Consequently the magnification will also change for different frequencies. The image of an object illuminated with white light will be surrounded by a coloured halo.

There are two ways to minimise chromatic aberration.

1. Make the lens from a material whose dispersion, the variation in refractive index with wavelength, is small.
2. Make an **achromatic doublet** which consists of a converging lens and a diverging lens in contact. The dispersion in the two kinds of glass is different. A typical combination is a positive lens made of crown glass and a negative lens made of flint glass. An achromatic doublet can completely eliminate chromatic aberration for only two frequencies of light, but if those two frequencies are suitably chosen, then the aberration for other frequencies is reduced. Although an achromatic doublet consists of a positive and a negative lens, the combination still has some power.
Magnifying glass

Beg, borrow or buy a cheap magnifying glass. You can use it to observe many of the things described in this chapter and you can measure its focal length.

Measuring focal length

You can easily measure the focal length of a converging lens by forming a real image of a distant object on a small screen such as a piece of paper. (You may need a support to prop up the screen.) On a bright day in a room without too much light, hold the lens so that it faces a bright scene outside the room. Place your screen parallel to the principal plane of the lens and move the the lens until you get a sharp image of the outside scene on the screen. Measure the distance from the lens to the screen to get an estimate of the focal length. Strictly, you have measured the image distance, but if the object distance is large by comparison, then the image distance is practically equal to the focal length. A variation on the method is to put a piece of paper on the floor under an unshaded light globe and turn the light on. Put the lens between globe and paper and move it until you see a sharp image of the globe's filament on the paper.

On a sunny day you can use the lens to focus an image of the sun on to a piece of paper and measure the focal length. If you leave the image there long enough you can char the paper or even set it alight! That will show how the energy flux from the sun is being concentrated in the image. NEVER look through the lens at the sun - in fact you should never look directly at the sun at all.

Virtual and real images

Look through the magnifying glass at an object such as a pencil. Start with the pencil closer to the lens than the focal length. You should see a virtual, upright image. Gradually move the object further from the lens and note what happens to the image. There will be a position, or a region, where you lose the image. How far from the lens is the pencil when that happens? You would expect it to happen when the object is in the focal plane. Where do you expect the image to be?

If you keep moving the pencil further away you will find another, inverted, image. Although it is not formed on a screen it is a real image. To demonstrate the real image in space more forcefully make an image of a bright object such as a light globe on a piece of translucent paper. Look at the image from behind the paper. Now slowly slide the paper sideways out of the light path, letting part of the image go off the edge of the paper. You can still see the image there in space. The fact that the image could be picked up by the paper shows that it is real.

Combining lenses

If you can scrounge another magnifying glass, measure its focal length. Predict what the combined focal length will be if you put the two lenses in contact. Do it and check your prediction experimentally.

Making a lens

Find a clear bottle and look at things through it. Then fill it with water and look at the same things again. The images you see will be distorted but they are nevertheless images. Use the bottle of water to produce real images of bright objects on a piece of paper. Hence estimate the focal length of the cylindrical bottle-lens.
QUESTIONs

Q3.1 What is the size of the smallest plane mirror you can use to see yourself from head to toe without moving your head or the mirror?

Q3.2 When will a lens give a lateral magnification of +1.00? When will it give a magnification of -1.00? In each case what kind of lens can you use and where do you put the object?

Q3.3 Use the ray-tracing method to locate the image of an object placed in front of a converging lens between the focal point and the lens.

Q3.4 Consider any converging lens. Use the lens equation to demonstrate each of the following results.

(i) An object at an infinite distance from the lens gives a real image in the focal plane on the other side.
(ii) An object at distance 2f gives a real image at a distance 2f on the other side.
(iii) An object in the focal plane gives an image at an infinite distance.
(iv) An object between the focal plane and the lens gives a virtual image on the same side of the lens.

Q3.5 Consider any diverging lens. Use the lens equation to show that for all positions of the object, the image is virtual and is on the same side of the lens.

Q3.6 You can always tell whether a lens will be converging or diverging just by comparing its thickness in the middle with that at the edge. The sign convention used with the lensmaker's formula should give the same result. Check that this is so in the following three examples.

(a) \[ r = 1.0 \text{ m} \]
(b) \[ r = 1.0 \text{ m} \]
(c) \[ r = 0.50 \text{ m} \]

Q3.7 a) Calculate the focal length of the lens in Q3.6(a) when it is immersed in water. The refractive index of water is 1.33.

b) Will an air filled plastic bag used underwater by a skindiver serve as a converging or a diverging lens?

Q3.8 The definition of power of a lens matches intuitive ideas of powerful or less powerful lenses. For example suppose that we have two converging lenses. One needs to be held 0.10 m from a wall to focus a distant scene on the wall. The other needs to be held 0.20 m from the wall. The first is bending the parallel rays more strongly - it is the more powerful lens. Sketch the path of a number of rays to illustrate image formation in these two cases. Calculate the powers of the two lenses.

Q3.9 Suppose that a crown glass lens has a focal length of 0.100 m for a typical frequency of red light. What is its focal length for violet light? Refer to the graph of refractive index as a function of wavelength, figure 2.19.

Q3.10 The angular size of the sun seen from earth is about 0.01 rad. A magnifying glass with a focal length of 150 mm and a diameter of 65 mm is used to produce an image of the sun on a card. What is diameter of the image?

Q3.11 Two lenses with powers of 5.0 m\(^{-1}\) and 4.0 m\(^{-1}\) are arranged so that their principal axes coincide. Light from an object 35 mm high goes first to the 5 dioptre lens. Find the position, nature (real or virtual, upright or inverted) and the size of the image when the object is placed 0.80 m from the first lens,

(a) when the lenses are 0.60 m apart and

(b) when they are 0.10 m apart.
Q3.12 Calculate, by successive applications of the lens formula \( \frac{1}{o} + \frac{1}{i} = \frac{1}{f} \), the position of the image formed by the two-lens system below. The object is at infinity.

To object at infinity

\[ f_1 = 0.40 \text{ m} \quad f_2 = 0.60 \text{ m} \]

\[ -0.20 \text{ m} \]

Discussion Questions

Q3.13 Explain why a lens has chromatic aberration, whereas a mirror does not.

Q3.14 A lens can produce really sharp images only if all the object points are in one plane - i.e. if they are all at about the same distance from the principal plane of the lens. A plane mirror produces sharp images for all objects at once - no matter how far away they are. Explain. How does the lateral magnification produced by a plane mirror depend on the object distance?

Q3.15 Can you devise an arrangement of mirrors which allows you to see the back of your head? Make a sketch and show some rays.

Q3.16 A converging beam of light strikes a plane mirror. Is the image real or virtual? Explain.

Q3.17 Under what conditions does a converging lens produce a real image? Can such a real image ever be upright? Explain.

Q3.18 Which way up is the image on the retina of your eye? Is that a problem?

Q3.19 Under what conditions is the lateral magnification of a lens infinite? Does an image exist if the magnification is infinite?

Q3.20 Can a diverging lens be used to produce a real image? Explain and discuss.

Q3.21 Can a virtual image be viewed on a screen? Can you photograph a virtual image? Discuss.

APPENDIX

Paraxial Rays

Paraxial rays must be close to the principal axis and also they must make small angles with the principal axis. The criterion for a small angle is that \( \tan \alpha \approx \sin \alpha \approx \alpha \). The following table illustrates how good the approximation is.

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