CHECKLIST

- Harmonic motion, periodic motion, simple harmonic motion (SHM), amplitude $A$ (m), cycle, period $T$ (s), frequency $f$ (Hz), angular frequency $\omega$ (rad.s$^{-1}$)

$$T = \frac{1}{f} \quad \omega = 2 \pi f = 2 \pi / T \quad (\text{Eq. 10.11, 10.12})$$

- displacement $x$ (m), velocity $v_x$ (m.s$^{-1}$) and acceleration $a_x$ (m.s$^{-1}$) in SHM
- displacement amplitude $A$ (m), velocity amplitude $A\omega$ (m.s$^{-1}$), acceleration amplitude $A\omega^2$ (m.s$^{-2}$)
- phase $\phi$ (rad), initial phase $\varepsilon$ (rad), in-phase, out-of-phase

- SHM: $x$ vs $t$, $v$ vs $t$ and $a$ vs $t$ graphs
  SHM: acceleration proportional to displacement

$$x = A \cos(\omega t + \varepsilon) \quad (\text{Eq. 10.13})$$
$$v_x = -A\omega \sin(\omega t + \varepsilon) \quad (\text{Eq. 10.16})$$
$$a_x = -A\omega \cos(\omega t + \varepsilon) = -\omega^2 x \quad (\text{Eq. 10.18})$$

- Elastic restoring force $F_e$ (N)

- SHM: conservation of energy, total energy $E$ (J), kinetic energy $KE$ (J) and potential energy $PE_e$ (J)

$$E = KE + PE \quad KE = \frac{1}{2} m v^2 \quad PE = \frac{1}{2} k x^2$$

- Oscillating mass-spring system
  Newton’s Second Law \quad $\sum F = m \ddot{a}$

natural frequency $\omega_o$ (rad.s$^{-1}$)

$$\omega_o = \sqrt{\frac{k}{m}} \quad f_o = \frac{1}{2\pi} \sqrt{\frac{k}{m}} \quad T = 2\pi \sqrt{\frac{m}{k}} \quad (\text{Eq. 10.19 – 10.21})$$
NOTES

Vibratory or oscillatory motion – motion that repeats itself is referred to as **harmonic motion**.

Simple Harmonic Motion (SHM)

Vibratory motion at a constant single frequency and with constant amplitude.

Describing the motion

Define coordinate system, origin (equilibrium point) and observer (OBS).

\[
\omega = \frac{d\theta}{dt} = \frac{2\pi}{T} \quad \text{angular frequency}\)
\]

\[
T = \frac{1}{f} \quad \text{period T (s)}
\]

\[
f = \frac{1}{T} \quad \text{frequency f (Hz)}
\]

\[
\omega = 2\pi f \quad \text{angular frequency}\)
\]

\[
T = \frac{1}{f} = \frac{1}{T} \quad \text{period T (s)}
\]

\[
f = \frac{1}{T} \quad \text{frequency f (Hz)}
\]

\[
\omega = 2\pi f \quad \text{angular frequency}\)
\]

\[
\omega = \frac{2\pi}{T} \quad \text{angular frequency}\)
\]

\[
A \quad \text{amplitude}
\]

\[
x_{\text{max}} \quad \text{displacement}
\]

\[
v \quad \text{velocity}
\]

\[
a \quad \text{acceleration}
\]

\[
F_{c} \quad \text{Force}
\]

\[
\text{Origin 0}
\]

\[
\text{equilibrium position}
\]

\[
-x_{\text{max}} \quad \text{displacement}
\]

\[
+x_{\text{max}} \quad \text{displacement}
\]

\[
\text{velocity}
\]

\[
\text{acceleration}
\]

\[
\text{Force}
\]

\[
\text{Origin 0}
\]

\[
\text{equilibrium position}
\]

\[
\text{displacement}
\]

\[
\text{velocity}
\]

\[
\text{acceleration}
\]

\[
\text{Force}
\]
Displacement

\[ x(t) = x_{\text{max}} \cos(\omega t + \varepsilon) \]
\[ x(t) = x_{\text{max}} \cos(2 \pi f t + \varepsilon) \]
\[ x(t) = x_{\text{max}} \cos(2 \pi t / T + \varepsilon) \]

Velocity

\[ v(t) = \frac{dx(t)}{dt} = -\omega x_{\text{max}} \sin(\omega t + \varepsilon) = -v_{\text{max}} \sin(\omega t + \varepsilon) \]

slope of the \( x(t) \) vs \( t \) graph \( \Rightarrow v(t) \)

Acceleration

\[ a(t) = \frac{dv(t)}{dt} = -\omega^2 x_{\text{max}} \cos(\omega t + \varepsilon) = -a_{\text{max}} \cos(\omega t + \varepsilon) \]

slope of the \( v(t) \) vs \( t \) graph \( \Rightarrow a(t) \)

\[ a(t) = -\omega^2 x(t) \]

Amplitude

- displacement amplitude \( x_{\text{max}} \equiv A \) (max value of displacement from origin)
- velocity amplitude \( v_{\text{max}} = \omega x_{\text{max}} \) (max value of velocity)
- acceleration amplitude \( a_{\text{max}} = \omega^2 x_{\text{max}} \) (max value of acceleration)

Phase (rad) all angles must be measured in radians (rad) – calculator in rad mode.

- phase angle \( (\omega t + \varepsilon) \)
- initial phase angle \( (t = 0) \) \( \varepsilon \)
**Example:** A SHM oscillating system of a block and spring is shown below in several different positions. For each situation, indicate if the position $x$, velocity $v$, acceleration $a$ and force $F$ are $+$ or $-$ or $0$.

<table>
<thead>
<tr>
<th>position</th>
<th>velocity</th>
<th>acceleration</th>
<th>force</th>
</tr>
</thead>
<tbody>
<tr>
<td>moving right</td>
<td>$-x_{\text{max}}$</td>
<td>$0$</td>
<td>$+x_{\text{max}}$</td>
</tr>
<tr>
<td>moving left</td>
<td>$-x_{\text{max}}$</td>
<td>$0$</td>
<td>$+x_{\text{max}}$</td>
</tr>
<tr>
<td>moving left</td>
<td>$-x_{\text{max}}$</td>
<td>$0$</td>
<td>$+x_{\text{max}}$</td>
</tr>
<tr>
<td>moving right</td>
<td>$-x_{\text{max}}$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>Stopped for an instant</td>
<td>$-x_{\text{max}}$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>Stopped for an instant</td>
<td>$-x_{\text{max}}$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>moving right</td>
<td>$-x_{\text{max}}$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
</tbody>
</table>
Oscillating Block – Spring System

Consider a block on a frictionless surface that is attached to a spring. When the block is disturbed from its equilibrium position, it will oscillate naturally without being driven by some external source of energy. Ignoring any frictional effects or damping, the block attached to spring will vibrate back-and-forth with a single frequency. The restoring force acting on the block always is directed back to the equilibrium position. When the block reaches the equilibrium position it will be moving with its maximum velocity and overshoots. The mechanical energy of the block-spring system is conserved.

\[
\text{total energy } \quad E = PE + KE = \text{constant}
\]

This type of motion is referred to as \textit{simple harmonic motion} (SHM).

Define coordinate system, origin (equilibrium point) and observer (OBS)

\[
\begin{align*}
\text{equilibrium} & \quad F_e = 0 \quad x = 0 \\
\text{compressed} & \quad F_e(t_1) \\
\text{stretched} & \quad x(t_2) \\
\end{align*}
\]

Restoring force \( F_e(t) = -k x(t) \)

Newton’s Second Law and Hooke's Law applied to the vibrating block-spring system (assumption \( x \) is not too large)

\[
\Sigma \vec{F} = m \ddot{a}
\]

\[
\vec{F}_e = -k \ddot{x} = m \ddot{a} \quad \Rightarrow \quad \ddot{a} = -\frac{k}{m} \dddot{x}
\]

For SHM

\[
\ddot{a} = -\omega^2 \dddot{x}
\]

Therefore, the natural angular frequency \( \omega_0 \) – the specific frequency at which the physical system will oscillate all by itself once set in motion is
natural angular frequency (rad.s⁻¹) \[ \omega_0 = \frac{k}{\sqrt{m}} \]

natural (linear) frequency (Hz) \[ f_0 = \frac{1}{2\pi} \sqrt{\frac{k}{m}} \]

period (s) \[ T = 2\pi \sqrt{\frac{m}{k}} \]

Energy Considerations

Total energy of block and spring system \( E \) (J)
Kinetic energy of block \( KE \) (J)
Potential energy of block-spring system \( PE \) (J)

\[ E = K(x, t) + U(x, t) \]

\( E = \text{constant (assuming no loss of mechanical energy)} \)

\[ KE = \frac{1}{2} m v^2 \]

\[ PE = \frac{1}{2} k x^2 \]

\[ E = \frac{1}{2} m v^2 + \frac{1}{2} k x^2 \]

\[ E = \frac{1}{2} m v_{\text{max}}^2 = \frac{1}{2} k x_{\text{max}}^2 \]

\[ E = \frac{1}{2} k x_{\text{max}}^2 = \frac{1}{2} k x^2 + \frac{1}{2} m v^2 \]

\[ \Rightarrow \quad v(x) = \omega \sqrt{x_{\text{max}}^2 - x^2} = \omega x_{\text{max}} \sqrt{1 - \frac{x^2}{x_{\text{max}}^2}} = v_{\text{max}} \sqrt{1 - \frac{x^2}{x_{\text{max}}^2}} \]
**Example:** Complete the table below:

<table>
<thead>
<tr>
<th></th>
<th>$t$</th>
<th>$x$</th>
<th>$V$</th>
<th>$a$</th>
<th>$KE$</th>
<th>$PE$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$A$</td>
<td>0</td>
<td>$-\omega^2 A$</td>
<td>0</td>
<td>$\frac{1}{2} k A^2$</td>
<td></td>
</tr>
<tr>
<td>$\frac{T}{4}$</td>
<td>$T/4$</td>
<td>$-A$</td>
<td>0</td>
<td>$+A$</td>
<td>$-A$</td>
<td>0</td>
</tr>
<tr>
<td>$\frac{T}{2}$</td>
<td>$T/2$</td>
<td>$-A$</td>
<td>0</td>
<td>$+A$</td>
<td>$-A$</td>
<td>0</td>
</tr>
<tr>
<td>$3T/4$</td>
<td>$3T/4$</td>
<td>$-A$</td>
<td>0</td>
<td>$+A$</td>
<td>$-A$</td>
<td>0</td>
</tr>
<tr>
<td>$T$</td>
<td>$T$</td>
<td>$-A$</td>
<td>0</td>
<td>$+A$</td>
<td>$-A$</td>
<td>0</td>
</tr>
</tbody>
</table>
**Problem**
A spring is hanging from a support without any object attached to it and its length is 500 mm. An object of mass 250 g is attached to the end of the spring. The length of the spring is now 850 mm.

(a) What is the spring constant?

The spring is pulled down 120 mm and then released from rest.

(b) Describe the motion on the object attached to the end of the spring.

(c) What is the displacement amplitude?

(d) What are the natural frequency of oscillation and period of motion?

Another object of mass 250 g is attached to the end of the spring.

(e) Assuming the spring is in its new equilibrium position, what is the length of the spring?

(f) If the object is set vibrating, what is the ratio of the periods of oscillation for the two situations?

**Solution**

**Setup**

<table>
<thead>
<tr>
<th></th>
<th>equilibrium</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_0$</td>
<td>500 mm = 0.500 m</td>
<td>$x_{1\text{max}}$</td>
<td>$x_{2\text{max}}$</td>
</tr>
<tr>
<td>$m_1$</td>
<td>250 g = 0.250 kg</td>
<td>$m_2 = 0.500$ kg</td>
<td>$L_2 = \text{? m}$</td>
</tr>
<tr>
<td>$L_1$</td>
<td>850 mm = 0.850 m</td>
<td>$f_1 = \text{? Hz}$</td>
<td>$x_{2\text{max}} = A_2 = \text{? m}$</td>
</tr>
<tr>
<td>$f_1$</td>
<td>$T_1 = \text{? s}$</td>
<td>$T_2 / T_1 = \text{?}$</td>
<td></td>
</tr>
</tbody>
</table>

Hooke’s Law & SHM: $F = k x$, $f_0 = \frac{1}{2\pi} \sqrt{\frac{k}{m}}$
Action

(a) spring constant

\[ m \cdot g = k \cdot s \quad \Rightarrow \quad k = \frac{m \cdot g}{s} \]

\[ s_1 = L_1 - L_0 = (0.850 - 0.500) \quad m = 0.350 \quad m \]
\[ k = m_1 \cdot g / s_1 = (0.250)(9.8)/(0.35) \quad N.m^{-1} = 7.0 \quad N.m^{-1} \]

(c) amplitude: spring pulled down 120 mm

\[ x_{1_{\text{max}}} = A_1 = 0.120 \quad m \]

(d) frequency and period (does not depend upon amplitude)

\[ f_1 = \frac{1}{2\pi} \sqrt{\frac{k}{m}} = \frac{1}{2\pi} \sqrt{7/0.25} \quad \text{Hz} = 0.84 \quad \text{Hz} \]
\[ T_1 = \frac{1}{f_1} = 1.2 \quad s \]

(e)

\[ m \cdot g = k \cdot s \quad \Rightarrow \quad s = \frac{m \cdot g}{k} \]
\[ s_2 = m_2 \cdot g / k = (0.500)(9.8) / (7.0) \quad m = 0.70 \quad m \]
\[ L_2 = s_2 + L_0 = 0.70 + 0.50 = 1.20 \quad m \]

(f)

\[ T_1 = 2\pi \sqrt{\frac{m_1}{k}} \quad T_2 = 2\pi \sqrt{\frac{m_2}{k}} \]
\[ \frac{T_2}{T_1} = \sqrt{\frac{m_2}{m_1}} = \sqrt{\frac{0.500}{0.250}} = \sqrt{2} = 1.4 \]