DOIING PHYSICS WITH MATLAB

ELECTROMAGNETIC INDUCTION
FARADAY’S LAW
MUTUAL & SELF INDUCTANCE

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Download and inspect the mscripts and make sure you can follow the structure of the programs.

cemB01.m

Calculations of the induced emfs and current in a square shaped coil due to the changing magnetic flux through the surface of the coil produced by a time varying current in a long straight wire.

cemB02.m

Calculations of the induced emfs and current in a square shaped coil due to the sinusoidal magnetic flux through the surface of the coil produced by a sinusoidal time varying current in a long straight wire.

simpson1d.m  simpson2d.m

Computation of [1D] and [2D] integrals using Simpson’s rule. The functions to be integrated must have an ODD number of the elements.
Faraday’s law is applied to a system of a long straight wire (1) and a square shaped conducting coil (2). A time dependent current $I_1$ in the wire produces a time varying magnetic field $B$ surrounding it. The coil is coupled to the wire by the **mutual inductance** $M$ of the system of wire and coil. The changing magnetic flux $\phi_B$ through the coil induces emf $\varepsilon_1$ around the coil which opposes the change in magnetic flux through it. The coil has a **self-inductance** $L$ and the current in the coil produces its own emf $\varepsilon_2$ to oppose the emf $\varepsilon_1$.

![Diagram of a long straight wire and a square coil](image)

**Fig. 1.** System of long straight wire conductor aligned along the Y axis and conducting square shaped coil aligned in the XY plane and centred on the X axis. The current in the wire is $I_1$ and the induced current in the coil is $I_2$. The side length of the square coil is $s_L$ and the radius of the coil wire is $a$. The closest side of the coil to the wire is the distance $x_1$ and the opposite of the side of the coil is at a distance $x_2 = x_1 + s_L$. The conductivity of the of the coil is $\sigma$ (resistivity $\rho = 1/\sigma$) and the resistance of the coil is $R$. The magnetic field $\vec{B}$ through the coil is parallel to the Z axis and the magnetic flux through the coil is $\phi_B$. 

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Doing Physics with Matlab 2
The magnitude of the magnetic field $B$ at a distance $x$ from the wire is

$$B = \left( \frac{\mu_0}{2\pi} \right) \frac{I_1}{x}$$

If the wire current $I_1$ is in the +Y direction, the magnetic field is in the –Z direction through the coil and in the +Z direction if the wire current $I_1$ is the –Y direction (right hand screw rule).

The magnetic flux $\phi_B$ through the square coil is

$$\phi_B = \int \int \vec{B} \cdot d\vec{A} = \int_{s_1}^{s_2} \int_{-s_{L/2}}^{s_{L/2}} B \ dy \ dx$$

(2) $$\phi_B = \left( \frac{\mu_0}{2\pi} \right) s_L I_1 \log_e \left( \frac{x_2}{x_1} \right)$$

Faraday’s law can be expressed as

$$\varepsilon = -\frac{d\phi_B}{dt} = \oint \vec{E} \cdot d\vec{s} \quad \nabla \times \vec{E} = -\frac{d\vec{B}}{dt}$$

Normally we think of fields created by charges. However, when a magnetic flux through some surface changes with time, then there is also an electric field created to give an emf around the boundary of the surface.

A steady current $I_1$ in the wire produces a constant magnetic flux $\phi_B$ through the coil and the induced emf is zero. Only when the wire current $I_1$ changes with time that the magnetic flux $\phi_B$ changes and a net emf created around the coil (the coil forms the boundary of the surface through which the magnetic flux changes). The induced emf drives a current $I_2$ through the conductive coil. The direction of the induced current $I_2$ in the coil is determined by Lenz’s law.

The induced current $I_2$ gives a magnetic flux that opposes the change in magnetic flux $\phi_B$ produced by the wire current $I_1$. Hence, the direction of the induced current $I_2$ can be determined by using the right hand screw rule. In figure 1, if the current $I_1$ is increasing the magnetic field through the coil is increasing in the –Z direction. The induced current $I_2$ in the coil is in an anticlockwise sense which gives its magnetic field in the +Z direction (opposite to the $B$ field from the wire).
The magnetic flux $\phi_B$ at every point within the coil is proportional to the wire current $I_1$ (equation 2), therefore, we can write

$$ (4) \quad \phi_B = M I_1 $$

where the constant of proportionality $M$ is the **mutual inductance** of the system composed of the coil and long straight wire. The S.I. unit for the mutual inductance is the henry [H].

From equations 1 and 2, the mutual inductance $M$ is

$$ (5a) \quad M = \left( \frac{\mu_0}{2\pi} \right) \int_{s_1}^{s_2} \int_{s_{1/2}}^{s_{1/2}} \frac{dy \, dx}{x} $$

$$ (5b) \quad M = \left( \frac{\mu_0 s_L}{2\pi} \right) \int_{s_1}^{s_2} \frac{dx}{x} $$

$$ (5c) \quad M = \left( \frac{\mu_0 s_L}{2\pi} \right) \log_e \left( \frac{x_2}{x_1} \right) $$

We have three ways of computing $M$. For the surface enclosed by the coil a [2D] grid can be created. Then $M$ from equation 5a can be estimated by calling the function *simpson2d.m* which evaluates the integral for the grid of NxN points where N must be an odd number. The integral in equation 5b can be evaluated by dividing the area of the coil into strips parallel to the Y axis and using the function *simpson1d.m*. $M$ can be found analytically using equation 5c.

**Matlab cemB01.m**

Setting up the [2D] grid

```matlab
% Grid for square coil   xG   yG
x = linspace(x1,x2,N);
y = linspace(y1,y2,N);
[xG, yG] = meshgrid(x,y);
```
Computing the mutual inductance

% Mutual inductance for wire and square coil M:
three ways of calculating
fn = (1./xG);
ax = x1; bx = x2; ay = y1; by = y2;
integral2D = simpson2d(fn,ax,bx,ay,by);
M1 = (mu0 / (2*pi)) * integral2D;

fn = 1./x;
integral1D = simpson1d(fn,ax,bx);
M2 = (mu0 / (2*pi)) * (by-ay) * integral1D;
M3 = (y2-y1)*mu0/(2*pi) * log(x2/x1);
M = M1;

The three ways of computing M give the exact same result.

The current \( I_2 \) in the coil also creates a magnetic field and a magnetic flux through the coil. If this current changes with time, so does the magnetic flux and additional emf exists around the coil. This additional emf influences the current \( I_2 \).

emf generated by current \( I_1 \) in the wire

\[ \varepsilon_1 = M \frac{dI_1}{dt} \]  

(6)

emf generated by current \( I_2 \) in the coil

\[ \varepsilon_2 = \pm L \frac{dI_2}{dt} \]  

(7)

where \( L \) is the constant of proportionality called the self-inductance [henries H].

The self-inductance \( L \) for the square coil of side length \( s_L \) and the coil wire has a circular cross-section with radius \( a \), then

\[ L = 8.0 \times 10^{-7} s_L \left( \log e \left( \frac{s_L}{a} \right) - 0.52401 \right) \]  

(8)

where \( L \) is in henries, \( s_L \) and \( a \) are in meters.
So, the changing current $I_1$ in the wire gives the time varying magnetic flux $\phi_B$ through the coil that induces an electric field that produces an emf $\varepsilon$ in the coil which drives the coil current $I_2$. The emf around the coil is

$$\varepsilon = \varepsilon_1 + \varepsilon_2 \quad \varepsilon_2 < 0$$

If the coil has a resistance $R$, then $\varepsilon = I_2 R$ and we can obtain a differential equation that can be solved to give the coil current $I_2$

$$I_2 R = M \frac{dI_1}{dt} - L \frac{dI_2}{dt} \quad (10)$$

$$\frac{dI_2}{dt} = \left( \frac{M}{L} \right) \frac{dI_1}{dt} - \left( \frac{R}{L} \right) I_2 \quad (11)$$

**Analytical solution of equation 11**

$$\frac{dI_2}{dt} = \left( \frac{M}{L} \right) \frac{dI_1}{dt} - \left( \frac{R}{L} \right) I_2 \quad k_1 = \left( \frac{M}{L} \right) \quad k_2 = \left( \frac{R}{L} \right)$$

$$\frac{dI_2}{dt} = k_1 - k_2 I_2$$

$$dt = \frac{dI_2}{k_1 - k_2 I_2}$$

$$\int_{0}^{t_2} dt = \int_{0}^{t_2} \frac{dI_2}{k_1 - k_2 I_2}$$

$$t = \left[ \frac{-1}{k_2} \log_e \left( k_1 - k_2 I_2 \right) \right] _0^{t_2}$$

$$-k_2 t = \log_e \left( \frac{k_1 - k_2 I_2}{k_1} \right)$$

$$\frac{k_1 - k_2 I_2}{k_1} = \exp \left( -k_2 t \right)$$

$$I_2 = \left( \frac{k_1}{k_2} \right) \left\{ 1 - \exp \left( -k_2 t \right) \right\} \quad (12)$$
**Finite difference solution to equation 11**

A finite difference method can be used to solve equation 11. This approach is better, because you can’t always find an analytical solution.

The first derivative is approximated by the finite difference

\[
\frac{dI}{dt} \approx \frac{I[n+1] - I[n]}{\Delta t}
\]

for time steps \( n+1 \) and \( n \)

where \( n = 1, 2, 3, \ldots, N \) (\( N \) odd integer)

\[
I_2[n+1] = I_2[n] + \left( \frac{M}{L} \right) (I_1[n+1] - I_1(n)) - \left( \frac{R \Delta t}{L} \right) I_2[n]
\]

Given the initial values \( (n = 1) \) of \( I_1 \) and \( I_2 \) it is easy to find the values of \( I_2 \) at later times.

**Example 1**

cemB01.m

Wire current varies linearly with time \( I_1 = I_{10} t \) \( I_{10} = 2.0 \) A

\[
\frac{dI_1}{dt} = 2.0 \, \text{A.s}^{-1}
\]

constant (don’t need numerical approximation for derivative)

Square coil

- side length \( s_L = 1.0 \) m
- radius \( a = 1.0 \times 10^{-3} \) m
- copper wire resistivity \( \rho = 1.68 \times 10^{-8} \, \Omega \cdot \text{m} \), resistance \( R = 0.0214 \, \Omega \)
- distance from wire \( x_1 = 0.10 \) m
- self inductance \( L = 5.1070 \times 10^{-6} \) H

Wire and coil

- mutual inductance \( M = 4.7958 \times 10^{-7} \) H
Figure 2 show plots of the wire current $I_1$ and the induced coil current $I_2$. The current $I_2$ is computed by solving equation 11 and the numerical result is identical to the analytical solution provided the time step is smaller enough. The analytical solution gives a saturation value $I_{2,\text{sat}}$ for the current $I_2$ as $t \to \infty$

$$I_{2,\text{sat}} = \frac{k_1}{k_2} = \frac{M}{R} \frac{dI_1}{dt} = 4.484 \times 10^{-5} \text{ A}$$

From the solution given by equation 12, we can define a time constant $\tau$ such that

$$k_2 \tau = 1 \quad \tau = \frac{1}{k_2} = \frac{L}{R} \quad \rightarrow$$

$$\tau = 2.388 \times 10^{-4} \text{ s}$$

$$I_2 = \frac{k_1}{k_2} \left(1 - e^{-1}\right) = 0.6321 I_{2,\text{sat}} = 2.834 \times 10^{-5} \text{ A}$$

The value for the time constant $\tau$ calculated numerically is the same as the analytical value for 5001 grid points and 5001 time steps ($\Delta t = 1.00 \times 10^{-6} \text{ s}$).

The final steady state value $I_{2,\text{sat}}$ does not depend upon the self-inductance $L$, but the time the current takes to reach steady state does.
The self-inductance tends to inhibit changes in the coil current \( I_2 \), and the larger the value of \( L \), the longer the system takes to reach steady-state.

The induced emf \( \varepsilon \) in the coil is due to two components: the mutual inductance of the wire and coil \( \varepsilon_1 \) and the self-inductance of the coil \( \varepsilon_2 \) as described by equations 6, 7, and 9. Figure 3 shows a plot of the emf induced in the coil.

\[
\varepsilon_{1_{\text{sat}}} = 9.591 \times 10^{-7} \text{ V} \\
\varepsilon_{2_{\text{sat}}} = 0 \text{ V}
\]

Can we find the electric field induced by the time varying magnetic flux?

You may think that equation 3 can be used to find the value of \( \vec{E} \)

\[
\varepsilon = -\frac{d\phi_B}{dt} = \oint \vec{E} \cdot d\vec{s}
\]

\[
\nabla \times \vec{E} = -\frac{dB}{dt}
\]

but this can only be done for very symmetrical cases such as when there is circular symmetry. Consider an irregular shape closed loop. An emf is induced in the loop due to an induced electric field whose direction and magnitude at points around the loop vary quite differently. Faraday’s law does not allow us to find anything more than the average magnitude of the electric field, the direction and magnitude depend on the path chosen. The induced emf around
a closed path has meaning whether or not a conductor lies on the path. The electric field is not directly related to the value of $B$ at points on the path taken, it only depends on the rate of change of the magnetic flux within the area enclosed by the loop.

We can find an average value for the electric field $E_{avgL}$, the current around the closed coil from the line integral form of Faraday’s law

$$
\mathbf{\varepsilon} = -\frac{d\Phi_B}{dt} = \oint \mathbf{E} \cdot d\mathbf{s} = E_{avgL} \int ds = 4s_L E_{avgL}
$$

(15)

$$
E_{avgL} = \frac{\varepsilon}{4s_L}
$$

The numerical value of $E_{avgL}$ for the parameters of Example 1 when a steady state situation has been reached is

$$
E_{avgL} = 2.398 \times 10^{-7} \text{ V.m}^{-1} \quad s_L = 1.0 \text{ m} \quad \varepsilon_{sat} = 9.592 \times 10^{-7} \text{ V}
$$

We can also find the value of the average electric field $E_{avgJ}$ from equation 16

$$
\mathbf{j} = \sigma \mathbf{E}
$$

(16)

where the electric field $\mathbf{E}$ drives the current density $\mathbf{j}$ through a material with conductivity $\sigma$.

$$
E_{avgJ} = \frac{\rho I_{2sat}}{\pi a^2} = 2.398 \times 10^{-7} \text{ V.m}^{-1} \quad I_{2sat} = 4.708 \times 10^{-5} \text{ A} \quad a = 1.000 \times 10^{-3} \text{ m}
$$

for copper $\rho = \frac{1}{\sigma} = 1.68 \times 10^{-8} \text{ \Omega.m}$
Changing input parameters

$L \rightarrow 2L = 1.021 \times 10^{-5} \text{ H}
\tau(L) = 2.388 \times 10^{-4} \text{ s}
\rightarrow \tau(2L) = 4.775 \times 10^{-4} \text{ s}

The only change is that it takes longer to reach the steady state situation

$R \rightarrow 2R = 0.043 \Omega
\tau(R) = 2.388 \times 10^{-4} \text{ s}
\rightarrow \tau(2R) = 1.194 \times 10^{-4} \text{ s}

$I_{2\text{sat}}(R) = 4.484 \times 10^{-5} \text{ A}
\rightarrow I_{2\text{sat}}(2R) = 2.242 \times 10^{-5} \text{ A}

No change in emfs

$x_i \rightarrow x_i / 2 = 0.05 \text{ m}
M(x_i) = 4.796 \times 10^{-7} \text{ H}
\rightarrow M(x_i / 2) = 6.089 \times 10^{-7} \text{ H}

I_{2\text{sat}}(x_i) = 4.484 \times 10^{-5} \text{ A}
\rightarrow I_{2\text{sat}}(x_i / 2) = 5.693 \times 10^{-5} \text{ A}

emf_{sat}(x_i) = 9.591 \times 10^{-7} \text{ V}
\rightarrow emf_{sat}(x_i / 2) = 12.18 \times 10^{-7} \text{ A}

No change in time constant $\tau$

Reducing the area of the coil, reduces the magnetic flux and hence reduces the magnitude of the current induced in the coil.
Example 2: Sinusoidal variation in magnetic flux  cemB02.m

The induced current $I_2$ in the square shaped coil is produced by a time varying sinusoidal current $I_1$ in the long straight wire. You can vary the frequency $f$ of the sinusoidal current $I_1$ in the wire to investigate how the induced current $I_2$ depends upon the frequency $f$ of the changing magnetic flux through the coil.

wire I1: frequency  $f = 50.0$ Hz
coil: max current I2 = 63.22 mA

coil: max emf = 1.35 mV
coil: max emf1 = 1.36 mV
coil: max emf2 = 0.10 mV
wire I1: frequency $f = 200.0$ Hz
coil: max current $I_2 = 243.00$ mA

coil: max emf $= 5.19$ mV
coil: max emf$_1 = 5.42$ mV
coil: max emf$_2 = 1.56$ mV
wire I1: frequency $f = 1000.0$ Hz

coil: max current $I_2 = 703.84$ mA

coil: max emf $= 15.01$ mV

coil: max emf1 $= 27.12$ mV

coil: max emf2 $= 22.58$ mV

As expected, the greater the frequency (the greater the rate of change in the magnetic flux and the larger the induced currents in the copper coil.)