DOING PHYSICS WITH MATLAB

VECTOR ANALYSIS

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cemVectorsA.m

Inputs: Cartesian components of the vector V
Outputs: cylindrical and spherical components and [3D] plot of vector

cemVectorsB.m

Inputs: Cartesian components of the vectors A B C
Outputs: dot products, cross products and triple products

cemVectorsC.m

Rotation of XY axes around Z axis to give new of reference X'Y'Z'.
Inputs: rotation angle and vector (Cartesian components) in XYZ frame
Outputs: Cartesian components of vector in X'Y'Z' frame

This script can be modified to calculate the rotation matrix for a [3D] rotation and give the Cartesian components of the vector in the X'Y'Z' frame of reference.
SPECIFYING a [3D] VECTOR

A scalar is completely characterised by its magnitude, and has no associated direction (mass, time, direction, work). A scalar is given by a simple number.

A vector has both a magnitude and direction (force, electric field, magnetic field). A vector can be specified in terms of its Cartesian or cylindrical (polar in [2D]) or spherical coordinates.

Cartesian coordinate system (XYZ right-handed rectangular: if we curl our fingers on the right hand so they rotate from the X axis to the Y axis then the Z axis is in the direction of the thumb).

A vector \( \vec{V} \) in specified in terms of its X, Y and Z Cartesian components

\[
\vec{V}(V_x, V_y, V_z) \quad \vec{V} = V_x \hat{i} + V_y \hat{j} + V_z \hat{k}
\]

where \( \hat{i}, \hat{j}, \hat{k} \) are unit vectors parallel to the X, Y and Z axes respectively.

![Fig. 1. Specifying a vector in an orthogonal coordinate system.](image)

The polar angle \( \theta \) is the angle down from the Z axis to the vector \( \vec{V} \).

The azimuthal angle \( \phi \) is the angle around from the X axis.

Polar angle \( 0 \leq \theta \leq \pi \)

Azimuthal angle \( 0 \leq \phi \leq 2\pi \) or \( -\pi \leq \phi \leq +\pi \)
Angles can be measured in radians or in degrees where $2\pi \text{ rad} = 360^\circ$

You can use the Matlab functions `rad2deg` and `deg2rad` for the conversions between radians and degrees

\[
\text{deg2rad}(30) \rightarrow 30^\circ = 0.5236 \text{ rad}
\]
\[
\text{rad2deg}(\pi) \rightarrow \pi = 180^\circ
\]

Fig. 2. The unit vectors $\hat{R}, \hat{\theta}, \hat{\phi}, \hat{\rho}$ pointing in the direction of an increase in the corresponding coordinate.

Cartesian components $\vec{V}(V_x, V_y, V_z)$
\[
\vec{V} = V_x \hat{i} + V_y \hat{j} + V_z \hat{k}
\]

Cylindrical components $\vec{V}(V_\rho, V_\theta, V_z)$
\[
\vec{V} = V_\rho \hat{\rho} + V_\theta \hat{\theta} + V_z \hat{k}
\]

Spherical components $\vec{V}(V_R, V_\theta, V_\phi)$
\[
\vec{V} = V_R \hat{R} + V_\theta \hat{\theta} + V_\phi \hat{\phi}
\]
Magnitudes

\[ |\vec{V}| \equiv V \equiv R = \sqrt{V_x^2 + V_y^2 + V_z^2} \]
\[ \rho = \sqrt{V_x^2 + V_y^2} \]

Relationship between coordinates from figure 2

\[ V_x = R \sin \theta \cos \phi \quad V_y = R \sin \theta \sin \phi \quad V_z = R \cos \theta \]
\[ V_x = \rho \cos \phi \quad V_y = \rho \sin \phi \quad V_z = V_z \]
\[ \tan \phi = \frac{V_y}{V_x} \quad \tan \theta = \frac{\rho}{V_z} \quad \cos \theta = \frac{V_z}{R} \]

Spherical coordinates

\[ \hat{R} = \sin \theta \cos \phi \hat{i} + \sin \theta \sin \phi \hat{j} + \cos \theta \hat{k} \]
\[ \hat{\theta} = \cos \theta \cos \phi \hat{i} + \cos \theta \sin \phi \hat{j} - \sin \theta \hat{k} \]
\[ \hat{\phi} = -\sin \phi \hat{i} + \cos \theta \sin \phi \hat{j} \]

Cylindrical coordinates

\[ \hat{\rho} = \cos \phi \hat{i} + \sin \phi \hat{j} \]
\[ \hat{\phi} = -\sin \phi \hat{i} + \cos \theta \sin \phi \hat{j} \]
\[ \hat{z} = \hat{z} \]

Matlab

Vector \( \vec{V} = 3\hat{i} + 5\hat{j} - 6\hat{k} \)

row vector \( V = [3 \ 5 \ -6] \)

column vector \( V = [3; \ 5; \ -6] \)

magnitudes \( R = \text{norm}(V) \quad \rho = \sqrt{V(1)^2+V(2)^2} \)

azimuthal angle \( 0 \leq \phi \leq 2\pi \)

\( \text{phi} = \text{atan2}(V(2),V(1)); \)
\( \text{if phi < 0, phi = phi + 2*pi; end} \)

polar angle \( \theta = \text{acos}(V(3)/R); \)
\( \theta = \text{acosd}(V(3)/R); \)
Matlab: changing orthogonal systems

You can also use MATLAB functions to make the conversion between Cartesian, polar, cylindrical, and spherical coordinate systems easy.

**Cartesian components**

\[ V_x \quad V_y \quad V_z \quad Vx \quad Vy \quad Vz \]

**Cylindrical components**

\[ V_\rho \quad V_\phi \quad V_z \quad Vr \quad Vphi \quad Vz \]

**Spherical components**

\[ V_R \quad V_\phi \quad V_\theta \quad VR \quad Vphi \quad Vtheta \]

where the angles \( Vphi \) and \( Vtheta \) are in radians

**[2D]**

\[
\begin{align*}
[Vphi \quad Vrho] &= \text{cart2pol}(Vx, Vy) \\
[Vx \quad Vy] &= \text{pol2cart}(Vphi, Vrho)
\end{align*}
\]

**[3D]**

\[
\begin{align*}
[Vphi \quad Vrho \quad Vz] &= \text{cart2pol}(Vx, Vy, Vz) \\
[Vx \quad Vy \quad Vz] &= \text{pol2cart}(Vphi, Vrho, Vz)
\end{align*}
\]

\[
\begin{align*}
[Vtheta \quad Vphi \quad VR] &= \text{cart2sph}(Vx, Vy, Vz) \\
[Vx \quad Vy \quad Vz] &= \text{sph2cart}(Vtheta, Vphi, VR)
\end{align*}
\]

\[
\begin{align*}
[Vphi \quad Vrho \quad Vz] &= \text{sph2pol}(Vtheta, Vphi, VR) \\
[Vtheta \quad Vphi \quad VR] &= \text{pol2sph}(Vphi, Vrho, Vz)
\end{align*}
\]

**Sample results**

<table>
<thead>
<tr>
<th>( V_x )</th>
<th>( V_y )</th>
<th>( V_z )</th>
<th>( V_\rho )</th>
<th>( V_R )</th>
<th>( V_\phi )</th>
<th>( V_\theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>5</td>
<td>-6</td>
<td>5.831</td>
<td>8.367</td>
<td>1.03 rad</td>
<td>0.80 rad</td>
</tr>
<tr>
<td>3/2</td>
<td>1/\sqrt{2}</td>
<td>\sqrt{2}</td>
<td>1.4142</td>
<td>2.0000</td>
<td>0.52 rad</td>
<td>0.79 rad</td>
</tr>
<tr>
<td>3/2</td>
<td>-1/\sqrt{2}</td>
<td>-\sqrt{2}</td>
<td>1.4142</td>
<td>2.0000</td>
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<td>-\sqrt{2}</td>
<td>1.4142</td>
<td>2.0000</td>
<td>-0.52 rad</td>
<td>0.79 rad</td>
</tr>
</tbody>
</table>

[1.2247] 0.7071 | [1.2247] -0.7071 | 59.04° | -45.82° | 30.00° | -45.00°
Figure (1) gives a [3D] plot of a vector plus a summary of the input values for the Cartesian components and the calculated spherical and cylindrical components of the vector using the mscript \texttt{cemVectorsA.m}.

\begin{itemize}
    \item \textbf{3D VECTOR}
    \begin{align*}
        V_x &= 1.225e+00 \\
        V_y &= 7.071e-01 \\
        V_z &= 1.414e+00 \\
        \text{magnitude} \ V_R &= 2.000e+00 \\
        \text{XY magnitude,} \ V_\rho &= 1.414e+00 \\
        \text{azimuthal angle,} \ V_\phi &= 0.52 \ \text{rad} = 30.00^\circ \\
        \text{polar angle,} \ V_\theta &= 0.79 \ \text{rad} = 45.00^\circ 
    \end{align*}
\end{itemize}

\textbf{Fig. 1.} Figure Window for a vector with inputs as the Cartesian components. \texttt{cemVectorsA.m}
VECTOR ALGEBRA

Addition / Subtraction / Scalar multiplication

To add or subtract vectors, you add or subtract the components. For multiplication of a vector by a scalar, simply multiply each component by the number for the scalar.

For example: consider the vectors in Cartesian coordinates

\[
\hat{A} (1,2,3) \quad \hat{B} (-1, -3, -5) \quad \hat{C} (2,4,-3)
\]
\[
\vec{V} = 3 \hat{A} + \hat{B} - 2 \hat{C}
\]
\[
\vec{V} = (3A_x + B_x - 2C_x) \hat{i} + (3A_y + B_y - 2C_y) \hat{j} - (3A_z + B_z - 2C_z) \hat{k}
\]
\[
\vec{V} = (3-1-4) \hat{i} + (6-3-8) \hat{j} - (9-5+6) \hat{k}
\]
\[
\vec{V} = -2 \hat{i} - 5 \hat{j} + 10 \hat{k}
\]
\[
\vec{V} = (-2,-5,10) \quad V_x = -2 \quad V_y = -5 \quad V_z = 10
\]

Matlab Command Window

A = [1 2 3]
B = [-1 -3 -5]
C = [2 4 -3]
V = 3*A+B-2*C \rightarrow V = [-2 -5 10]
Dot product (scalar product) of two vectors

\[ \vec{A} \cdot \vec{B} = AB \cos \theta = A_x B_x + A_y B_y + A_z B_z \]

where \( \theta \) is the angle between the two vectors when they are placed tail to tail

\[ \vec{A} \cdot \vec{B} \quad \text{scalar} \]

\[ \vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A} \quad \text{commutative} \]

\[ \vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C} \quad \text{distributive} \]

Geometrically \( \vec{A} \cdot \vec{B} \) is the product of \( \vec{A} \) times the projection of \( \vec{B} \) along \( \vec{A} \) or the product of \( \vec{B} \) times the projection of \( \vec{A} \) along \( \vec{B} \)

\[ \vec{B} \]
\[ \theta \]
\[ \vec{A} \]

Angle between the two vectors \( \cos \theta = \frac{\vec{A} \cdot \vec{B}}{AB} = \frac{A_x B_x + A_y B_y + A_z B_z}{AB} \)

Law of cosines

\[ \vec{A} = \vec{B} + \vec{C} \quad \vec{C} = \vec{A} - \vec{B} \]

\[ \vec{C} \cdot \vec{C} = (\vec{A} - \vec{B}) \cdot (\vec{A} - \vec{B}) = \vec{A} \cdot \vec{A} + \vec{B} \cdot \vec{B} - 2 \vec{A} \cdot \vec{B} \]

\[ C^2 = A^2 + B^2 - 2AB \cos \theta \]

Matlab function \( \text{dot}(A,B) \) where A and B are row vectors of the same length.
Cross product or vector product of two vectors

\[ \vec{A} \times \vec{B} = AB \sin \theta \, \hat{n} \]

where \( \theta \) is the angle between the two vectors placed tail to tail and \( \hat{n} \) is a unit vector that is normal to the plane defined by the two vectors and whose direction is determined by the right-hand rule (fingers curl from \( \vec{A} \) to \( \vec{B} \) then extended thumb points in direction of \( \hat{n} \)).

\[ \vec{A} \times \vec{B} = -\vec{B} \times \vec{A} \] non-commutative

\[ \vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C} \] distributive

\[ \vec{A} \parallel \vec{B} \quad \Rightarrow \quad \theta = 0 \quad \vec{A} \times \vec{B} = 0 \]

\[ \vec{A} \times \vec{A} = 0 \]

\[ \vec{A} \perp \vec{B} \quad \Rightarrow \quad \theta = \pi / 2 \, \text{rad} \quad |\vec{A} \times \vec{B}| = AB \]

The cross product is the vector area of the parallelogram having \( \vec{A} \) and \( \vec{B} \) on adjacent sides.

Determinant form

\[ \vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = (A_y B_z - A_z B_y)\hat{i} + (A_z B_x - A_x B_z)\hat{j} + (A_x B_y - A_y B_x)\hat{k} \]

Matlab function \texttt{cross(A,B)} returns the cross product of the vectors A and B. A and B must be 3 element vectors.
**Triple Products**

Examples of triple products of three vectors

\[
\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})
\]

\[
\vec{A} \cdot (\vec{C} \times \vec{B}) = \vec{B} \cdot (\vec{A} \times \vec{C}) = \vec{C} \cdot (\vec{B} \times \vec{A})
\]

\[
\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{A} \cdot \vec{C}) - \vec{C} \cdot (\vec{A} \cdot \vec{B})
\]

\[
\vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C}
\]

Calculations of triple products are given in the following examples. You should compare the numerical outputs with the above relationships.
Inputs vectors
\[
A = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 4 & 5 \end{bmatrix}
\]

Outputs

**Magnitudes of vectors**
\[
\text{Amag} = \text{norm}(A) \quad \text{Bmag} = 1.4142 \quad \text{Cmag} = 6.7082
\]

**Dot products**
\[
\text{AdotB} = \text{dot}(A,B) \quad \text{AdotB} = 1 \quad \text{BdotA} = 1 \quad \text{BdotC} = 9 \quad \text{CdotA} = 7
\]

**Cross products**
\[
\text{AB} = \text{cross}(A,B) \quad \text{AB} = \begin{bmatrix} -1 & -1 & 1 \end{bmatrix} \quad \text{BA} = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \quad \text{BC} = \begin{bmatrix} 1 & 2 & -2 \end{bmatrix} \quad \text{CA} = \begin{bmatrix} 4 & 3 & -4 \end{bmatrix}
\]

**Cross products: magnitudes**
\[
\text{ABmag} = \text{norm}(\text{AB}) \quad \text{ABmag} = 1.7321 \quad \text{BAmag} = 1.7321 \quad \text{BCmag} = 3 \quad \text{CAmag} = 6.4031
\]

**Angles between vectors**
\[
\text{ABangle} = \arcsin\left(\frac{\text{norm}(\text{AB})}{\text{Amag} \times \text{Bmag}}\right) \quad \text{ABangle} = 1.0472 \text{ rad} = 60.0000 \text{ deg} \quad \text{BAangle} = 1.0472 \text{ rad} = 60.0000 \text{ deg} \quad \text{BCangle} = 0.3218 \text{ rad} = 18.4349 \text{ deg} \quad \text{CAangle} = 0.7409 \text{ rad} = 42.4502 \text{ deg}
\]

**Triple products**
\[
\text{AdotBC} = -1 \quad \text{AdotBC} = \text{dot}(A,\text{cross}(B,C)) \quad \text{BdotCA} = -1 \quad \text{CdotAB} = -1 \quad \text{AdotCB} = 1 \quad \text{BdotAC} = 1 \quad \text{CdotBA} = 1
\]
\[
\text{AcrossBC1} = \begin{bmatrix} -2 & 3 & 2 \end{bmatrix} \quad \text{cross}(A,\text{cross}(B,C)) \quad \text{AcrossBC2} = \begin{bmatrix} -2 & 3 & 2 \end{bmatrix} \quad B . \text{dot}(A,C) - C . \text{dot}(A,B) \quad \text{AcrossBC} = \begin{bmatrix} -2 & 3 & 2 \end{bmatrix} \quad \text{AcrossBC} = \text{cross}(A,\text{cross}(B,C)) \quad \text{ABcrossC} = \begin{bmatrix} -9 & 7 & -2 \end{bmatrix} \quad \text{ABcrossC} = \text{cross}(\text{cross}(A,B),C)
\]
Example  Find the angle between the face diagonals of a cube

\[ \vec{A}(1,0,1) \quad \vec{B}(0,1,1) \]

\( \Rightarrow \)

The angle between the two vectors can be found from the cross product of the two vectors

\[ \vec{A} \times \vec{B} = AB \sin \theta \hat{n} \]

\[ \sin \theta = \frac{|\vec{A} \times \vec{B}|}{|\vec{A}| |\vec{B}|} \]

Run the mscript \texttt{cemVectorsB.m}

\[ A = [1 \ 0 \ 1] \quad B = [0 \ 1 \ 1] \]

angle is \( \theta = 1.0472 \text{ rad} = 60.0000 \text{ deg} \)

\( \Leftarrow \)
**Example**  Find the components of the unit vector $\hat{n}$ perpendicular to the shaded regions formed by the vectors $\vec{A}$ and $\vec{B}$

\[ \vec{C} = \vec{A} \times \vec{B} = -12 \hat{k} = \left| \vec{C} \right| \hat{n} \]

\[ n = \frac{\vec{C}}{\left| \vec{C} \right|} = \frac{-12 \hat{k}}{12} = -\hat{k} \]

\[ n_x = 0 \quad n_y = 0 \quad n_z = -1 \]
**Example** Find the components of the unit vector $\hat{n}$ perpendicular to the shaded regions formed by the points R, P, Q.

Let $\vec{A}$ be the vector pointing from R to P and $\vec{B}$ be the vector pointing from R to Q. Then the vectors are $\vec{A}(3,0,-2)$ and $\vec{B}(0,4,-2)$.

$$\vec{A} \times \vec{B} = AB \sin \theta \; \hat{n}$$

**Matlab Command Window**

```
A = [3 0 -2]  B = [0 4 -2]
C = cross(A,B)  C = [8  6  12]
Cmag = norm(C)  Cmag = 15.6205
n = C./Cmag
n = [0.5121  0.3841  0.7682]
```

The Cartesian components of the vector $\hat{n}$ are (0.5121, 0.3841, 0.7682).
TRANSFORMATION OF COORDINATES DUE TO ROTATION

What is the change in the components of a vector due to a rotation of the coordinate system from $X\ Y\ Z$ to $X'\ Y'\ Z'$?

The transformation matrix $\mathbf{R}$ due to a rotation uses the following notation

$$X \& X' \rightarrow 1 \quad Y \& Y' \rightarrow 2 \quad Z \& Z' \rightarrow 3$$

$\theta_{11}$ is the angle between axes $X'$ and $X$

$\theta_{32}$ is the angle between axes $Z'$ and $Y$

$$\mathbf{R} = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix} \quad \text{where} \quad R_{mn} = \cos(\theta_{mn}) \quad m = 1,2,3 \quad n = 1,2,3$$

$$\mathbf{V}(V_x, V_y, V_z) \quad \text{in the XYZ coordinate system}$$

$$\mathbf{V}'(V'_x, V'_y, V'_z) \quad \text{in the X'Y'Z' coordinate system}$$

$$\mathbf{V}' = \mathbf{R} \mathbf{V}^T \quad \text{where} \quad \mathbf{V}^T = \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$$

The mscript `cemVectorsC.m` can be modified to calculate the rotation matrix and the components of the vector in the $X'Y'Z'$ frame of reference.
Rotation of the XY axes around the Z axis

Consider the vector
\[ \vec{V} = V_x \hat{i} + V_y \hat{j} + V_z \hat{k} \]
in the unprimed frame of reference.

What will be its components in the primed frame of reference that is rotated by an angle \( \theta \) in an anticlockwise direction in the XY plane?

Angles between the axes XYZ and X'Y'Z'

\[
\begin{align*}
\theta_{11} &= \theta & \theta_{12} &= 90^\circ - \theta & \theta_{13} &= 90^\circ \\
\theta_{21} &= \theta + 90^\circ & \theta_{22} &= \theta & \theta_{23} &= 90^\circ \\
\theta_{31} &= 90^\circ & \theta_{32} &= 90^\circ & \theta_{33} &= 0^\circ
\end{align*}
\]

Transformation of vector components

\[
\begin{pmatrix}
V'_{x} \\
V'_{y} \\
V'_{z}
\end{pmatrix}
= 
\begin{pmatrix}
\cos(\theta) & \cos(90^\circ - \theta) & \cos(90^\circ) \\
\cos(\theta + 90^\circ) & \cos(\theta) & \cos(90^\circ) \\
\cos(90^\circ) & \cos(90^\circ) & \cos(0^\circ)
\end{pmatrix}
\begin{pmatrix}
V_x \\
V_y \\
V_z
\end{pmatrix}
\]

\[
\begin{pmatrix}
V'_{x} \\
V'_{y} \\
V'_{z}
\end{pmatrix}
= 
\begin{pmatrix}
\cos(\theta) & \sin(\theta) & 0 \\
-sin(\theta) & \cos(\theta) & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
V_x \\
V_y \\
V_z
\end{pmatrix}
\]

\[ V'_{x} = \cos(\theta)V_x + \sin(\theta)V_y \]
\[ V'_{y} = -\sin(\theta)V_x + \cos(\theta)V_y \]
\[ V'_{z} = V_z \]
Example

$V(2, 3, 0)$ in $XYZ$ frame of reference

$XYZ$ rotated by $30^\circ$ anticlockwise in $XY$ plane to give the $X'Y'Z'$ frame

What are the components of $V$ in the $X'Y'Z'$ frame?

⇒

Run the mscript `cemVectorsC.m`

Inputs: $V = [2 \ 3 \ 0]$  \( \text{theta} = 30 \)

Output displayed in Command Window: $Vdash = [3.2321 \ 1.5981 \ 0]$