1. Electrostatic Deflection of the Electron Beam

Electrons emerge from the electron gun with a kinetic energy determined by the accelerating voltage \( V \):

\[
\frac{1}{2} mv^2 = eV
\]

Hence, the longitudinal velocity, \( v \), of the beam is

\[
v = \frac{\sqrt{2eV}}{m}
\] ....(1)

![Figure 5](image)

**DEFLECTION OF ELECTRON BEAM BY A UNIFORM ELECTRIC FIELD**

Now consider a beam of electrons, with velocity \( v \), which enter a region of uniform transverse electric field \( E \). See fig. 5. The transverse force on an electron is \( -eE \). This force produces a transverse acceleration \( a_x \), whose magnitude is given by Newton's second law of motion:

\[
a_x = \frac{eE}{m}
\] ....(2)

This acceleration lasts for a time, \( \tau \), equal to the time spent in the field

\[
\tau = \frac{D}{v}
\] ....(3)

where \( D \) is the length of the deflection region.

The acquired transverse velocity is, therefore,

\[
v_x = a_x \tau = \frac{eE}{m} \cdot \frac{D}{v}
\] ....(4)

from equations (2) and (3).

The electron now travels with uniform velocity, and takes an additional time \( L/v \) to reach the screen. Here, \( L \) is the additional longitudinal distance travelled. During this time it undergoes a transverse displacement, \( x \), (in addition to any displacement which occurred in the deflection region):

\[
x = \frac{L}{v} \cdot v_x
\] ....(5)
Substituting for $v_X$ from equation (4), we get

$$x = \frac{L}{v} \cdot \frac{eE}{m} \cdot \frac{D}{v}$$  \hspace{1cm} \ldots \ldots (6)

Next, using equation (1) to eliminate $v$ we get

$$x = \frac{L D E}{2 v}$$  \hspace{1cm} \ldots \ldots (7)

Now the magnitude of the field $E$ is determined by the potential difference, $V_X$, between the plates and the plate separation, $d$.

$$E = \frac{V_X}{d}$$  \hspace{1cm} \ldots \ldots (8)

From (7) and (8) we get

$$x = \frac{L D V_X}{2 d \sqrt{v}}$$  \hspace{1cm} \ldots \ldots (9)

Note: We have just worked out the transverse deflection which occurs after the electron leaves the deflection region. It can be shown that equation (9) gives the total deflection, $x'$, if we replace $L$ by $L'$, the distance from the middle of the deflection region to the screen.

$$x' = \frac{L' D V_X}{2 d \sqrt{v}}$$  \hspace{1cm} \ldots \ldots (9a)

2. Motion of a Charged Particle Normal to a Uniform Magnetic Field

A particle with mass $m$ and charge $q$ moving with velocity $v$ in a magnetic field $B$ experiences a force

$$F = q v \times B$$  \hspace{1cm} \ldots \ldots (10)

This is a vector equation which tells us that the magnitude of the force is

$$F = q v B \sin \theta,$$  \hspace{1cm} \ldots \ldots (11)

where $\theta$ is the angle between $v$ and $B$, and that the direction of the force is perpendicular to both $v$ and $B$ (in a sense given by the right hand rule).

An electron (charge $-e$) injected into a uniform magnetic field and moving perpendicular to the field experiences a force

$$F = -e v B$$  \hspace{1cm} \ldots \ldots (12)

Now the force remains perpendicular to the velocity and the electron moves in a circular path of radius $R$. The radial (centripetal) acceleration must be

$$a = -\frac{v^2}{R}$$  \hspace{1cm} \ldots \ldots (13)

Now apply Newton's second law of motion:

$$F = ma;$$

$$e v B = m \frac{v^2}{R}.$$  \hspace{1cm} \ldots \ldots (14)

We find that the radius of the electron's path is

$$R = \frac{m v}{e B}$$  \hspace{1cm} \ldots \ldots (14)$$
If we write down the angular velocity, or angular frequency of this circular motion we find a very interesting result:

\[ \omega = \frac{v}{R} \]

\[ \omega = \frac{e}{m} \cdot B \] \hspace{1cm} \text{......(15)}

Note that \( \omega \) is independent of \( v \), the electron's speed! This angular frequency is often called the angular *cyclotron frequency* for the particle in the field, \( B \).

3. Magnetic Deflection of Electrons

Now consider an electron which is fired along the axis of a cathode ray tube with speed \( v \) and then enters a region, of length \( a \), containing a uniform magnetic field, \( B \), which is perpendicular to the axis of the tube. See fig. 6.

![Figure 6](image_url)

**Figure 6**

**DEFLECTION OF ELECTRON BEAM BY A UNIFORM MAGNETIC FIELD NORMAL TO THE VELOCITY**

The electron now moves in a circle of radius \( R \) and is deflected through an angle

\[ \theta = \omega T \], \hspace{1cm} \text{......(16)}

and a linear distance

\[ y_0 = a \tan \left( \frac{\theta}{2} \right) \] \hspace{1cm} \text{......(17)}

before it leaves the region of the field.

\( T \) is the time spent in the field.
If $\theta$ is small, i.e. if the transverse velocity acquired is small compared with $v$, we can make some approximations:

\[
T = \frac{\theta}{v} \quad ;
\]

\[
\tan \left( \frac{\theta}{2} \right) = \frac{\theta}{2} = \frac{uT}{v} = \frac{wa}{2v}.
\]

Hence \[y_0 = \frac{wa^2}{2v} \quad \cdots (18)\]

The electron leaves the field region with a transverse velocity

\[v' = v \sin \theta,\]

which, by the same approximations, can be written

\[v' = a \omega \quad \cdots (19)\]

If the electron now travels a further longitudinal distance, $b$, in a field-free region it acquires an additional transverse displacement $y_1$. Since the longitudinal velocity is still approximately $v$,

\[
\frac{y_1}{b} = \frac{v'}{v} \quad ;
\]

\[y_1 = \frac{u}{v} ab \quad \cdots (20)\]

Hence the total transverse displacement is

\[
y = y_0 + y_1 = \frac{u}{v} (a^2 + ab) \quad \cdots (21)\]

We now draw two conclusions.

Firstly, if we substitute for the electron cyclotron frequency, $\omega$ (equation 15) we see that $y$ is proportional to $B$ and hence, also, to the current $I$ which produces $B$.

Secondly, $y$ is inversely proportional to $v$ and hence, by equation (1), $y$ is inversely proportional to the square root of the accelerating voltage, $V$.

The situation we have just considered is only a crude approximation to our experimental arrangement; $B$ is not uniform and does not extend over a sharply defined region. We now claim that this should not affect the two conclusions just stated, provided that we keep our assumption of small deflection angle. To see this, note that we could divide the non-uniform field into many small regions, each one small enough for $B$ to be regarded as constant in that region, and in each case we would get a small deflection given by equation (16). In each case we would find

\[
\Delta y_0 = I
\]

and \[\Delta y_0 = V^{-\frac{1}{2}}\]

Note that in appendix 1 we found a deflection inversely proportional to the accelerating potential difference. The difference is, of course, that here the magnetic deflecting force is proportional to the electron velocity whereas the electrostatic deflecting force is independent of velocity.
4. **Helical Motion of Electrons**

Consider an electron moving with a component of velocity \( v \), parallel to a uniform magnetic field, \( \mathbf{B} \) (direction \( z \)) and with a transverse component of velocity, \( v' \), perpendicular to \( \mathbf{B} \).

The longitudinal motion is unaffected by the field, but the transverse motion is a circular motion with angular frequency \( \omega \) (the electron cyclotron frequency) and radius \( R \).

From (15) \[ \omega = \frac{e}{m} \cdot B \]

From (14) \[ R = \frac{v'}{\omega} = \frac{mv'}{eB} \] \[ \ldots (22) \]

The combined motion is a helix - the shape of a coiled spring. See fig. 7.

![Figure 7](image)

**HELICAL PATH OF ELECTRONS IN A MAGNETIC FIELD**

4.1 **Pitch of the Helix**

Let \( \phi \) be the total angle through which the electron rotates from the point where it acquires its transverse velocity (the electrostatic deflection plates) to the screen.

\[ \phi = \omega t \]

where \( t \) is the time taken to reach the screen.

The distance, \( L \), from deflection region to screen is given by

\[ L = vt \]

\[ \ldots (24) \]

Eliminate \( t \) from (23) and (24) to get

\[ \phi = \frac{\omega}{v} L \]

\[ \ldots (25) \]
Figure 8
CO-ORDINATES OF SPOT ON THE SCREEN

Note that for a fixed accelerating voltage \( V \) the longitudinal velocity, \( v \), is constant. Since \( \omega \) is proportional to the magnetic field \( B \) (equation 15), \( \phi \) is proportional to \( B \).

Now consider the values of the field required to make 1, 2, 3, ..., revolutions in \( \phi \). It is easy to see that the field required to produce \( n \) revolutions is \( n \) times the field required to produce one revolution.

The longitudinal distance travelled by the electron while it makes one revolution in its transverse motion is called the pitch, \( p \), of the helix. Since the period of the circular motion is \( 2\pi/\omega \), the pitch is

\[
p = 2\pi \frac{V}{\omega}
\]

.....(26)

For constant longitudinal velocity, \( v \), (i.e. constant accelerating voltage) the pitch is inversely proportional to the magnetic field, \( B \).

4.2 Radius of the Helix

Changing the magnetic field alters not only the pitch of the helix but also its radius. We already have an equation, (22), for the radius of the electron's trajectory:

\[
R = \frac{mv}{eB}
\]

If \( v' \) is produced by a deflection voltage applied to, say, the X-plates we can use equation (8) to calculate \( E \) and equation (4) to calculate \( v' \).

\[
R = \frac{D}{d} \frac{V_x}{d}
\]

.....(27)

Thus \( R \) is proportional to \( V_x \) for fixed \( v \) and \( B \).

To see how \( R \) depends on accelerating voltage note that \( v = \sqrt{\frac{V}{2m}} \) (equation 1) and to see how it depends on solenoid current, \( I \), note that \( B \propto I \).
4.3 Locus of the Spot on the Screen

What is the shape of the path traced out by the spot on the screen as $B$ is varied while the accelerating and deflection voltages are kept constant?

To work this out, note that both the pitch and the radius of the helix vary as $B$ is altered. Firstly, we work out the position of the spot on the screen for a given value of $B$. The angle, $\phi$, through which the electron has rotated since leaving the deflection region is given by equation (25):

$$\phi = \frac{\omega}{\nu} L$$

The radius of the helix, i.e., the distance of the spot from the axis of the helix (not the axis of the tube) is given by equation (22):

$$R = \frac{V_1}{\omega}$$

Now eliminate $\omega$ to get

$$R = \frac{V_1 L}{\nu \phi}$$

Figure 9

**Locus of Spot as Longitudinal Magnetic Field is Varied**

$$r = k \sin \frac{\phi}{\nu}$$
This equation gives the relationship between $R$ and $\phi$ for the spot on the screen. This relationship is easier to understand if we change our co-ordinate system from $R, \phi$ to $r, \theta$ as shown in figure 8. From the geometry of this figure we can write the co-ordinate transformations:

$$r = 2R \sin \theta \quad \ldots (29)$$

$$\theta = \frac{\phi}{2} \quad \ldots (30)$$

After substitution of these transformations in equation (28) we get

$$r = \frac{L v'}{V} \cdot \frac{\sin \theta}{\theta} \quad \ldots (31)$$

For constant $v'$ and $v$, this is the equation of a spiral called a *cochleoid*. Its shape is sketched in fig. 9.

Note that the spot returns to the same point several times. Can you use this result to explain how an axial magnetic field can focus a beam of electrons?

5. Magnetic Fields Produced by Solenoids

The magnetic field produced by a current in a solenoid is, in general, difficult to calculate. However, for the simple case of a thin solenoid (one whose *winding* is much thinner than the coil diameter) there is a well known expression for the field at the points on the axis of the solenoid. (See figure 10)

![Figure 10: Magnetic Field of a Thin Solenoid](image)

**MAGNETIC FIELD OF A THIN SOLENOID**

$$B = \frac{\mu_0}{2} n I (\cos \theta_1 - \cos \theta_2) \quad \ldots (32)$$

where $\mu_0$ = permittivity of free space

$$= 4\pi \times 10^{-7} \text{ weber/ampere metre}$$

$n$ = number of turns per unit length of solenoid.

For points on the axis inside a long solenoid,$$
\theta_1 \approx 0, \quad \theta_2 \approx 180^\circ \quad \text{so that}
$$

$$B \propto \mu_0 n I \quad \ldots (33)$$
For thick solenoids these results do not hold. The geometry of the solenoid is shown in figure 11, and correction factors for $B$ are given in table 1. $B$ is the corrected field and $B_c$ is the field value obtained from equation (33)*.

Note that when two solenoids are used to produce a transverse field their separation is about equal to the length of one solenoid. The correction factor to be applied to the single solenoid field is then about 0.9.

![Diagram of a thick short solenoid with dimensions](image)

**Figure 11**
A THICK SHORT SOLENOID

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<th>$B/B_c$</th>
<th>$z/s$</th>
<th>$B/B_c$</th>
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</table>

**Table 1**
Correction Factors for Axial Fields of Thick Solenoids

$2s = 4$ in , $d = 3\frac{1}{4}$ in , $D = 4\frac{1}{2}$ in

See figure 11.

* Data based on that given in Berkeley Physics Laboratory, Part A, page A3-33.
6. Magnetic Field of the Helmholtz Coils

The field produced on the axis of a single circular coil is

\[ B = \frac{\mu_0 N I a^2}{2 \left( a^2 + z^2 \right)^{3/2}} \] .......(34)

where

- \( a \) = radius of the coil
- \( z \) = distance along the axis from the centre of the coil
- \( N \) = number of turns in the coil

If two such coils are separated by a distance equal to their radius, \( a \), it is easy to show that the resultant field at a point on their common axis, mid-way between the coils is

\[ B = \frac{\mu_0 N T}{a} \left( \frac{a}{r} \right)^{3/2} \] .......(35)

This field changes very little in the region of space between the coils. You should be able to estimate for yourself how the field varies along the coil axis. For points off the axis the calculation is nasty. Actual measurements show however, that at a distance \( a/3 \) from the axis in the mid-plane \( B \) is down by less than \( 2\% \), and at \( a/2 \) it is down by less than \( 5\% \).

7. Calculations of \( e/m \)

Some of the measurements made in this experiment can be used to obtain an approximate value for \( e/m \), the charge to mass ratio of the electron. If you explore the various equations given in these appendices you should be able to invent several methods of getting \( e/m \).

For example if you use the beam tube apparatus, equations (22) and (1) together with an expression for the field of the Helmholtz coils can be used.

If you can measure the pitch of the helical path in the longitudinal magnetic field, equations (15) and (26) can be used.