Honours AQM Practice Midterm

1. 40%. Previously, we have considered a two-level atom interacting with a cavity mode on resonance, meaning that the energy spacing of the atom is equal to the energy of one photon. Another situation of interest is when they are highly detuned, meaning that the energy spacing of the atom $\Delta$ is very different from the energy of a photon $\hbar \omega$.

In such a situation, the relevant interaction Hamiltonian is

$$\hat{H}_{\text{int}} = \hbar \chi (\hat{a}^\dagger \hat{a}) \hat{\sigma}_z,$$

where $\chi$ is an interaction strength constant with units of frequency.

1. Show that the energy eigenstates of this Hamiltonian are

$$|e\rangle |n\rangle, \quad \text{and} \quad |g\rangle |n\rangle, \quad \text{for} \quad n = 0, 1, 2, \ldots$$

and calculate the energy eigenvalue associated with each state.

2. If the cavity field is initially in a coherent state $|\alpha\rangle$ and the two-level atom is initially in the state $|\pm\rangle = \frac{1}{\sqrt{2}} (|g\rangle + |e\rangle)$, determine the state of the combined system (field+atom) at time $t = \pi/(4\chi)$, working entirely in the interaction picture. Express your answer as a linear superposition of coherent states for the field and energy eigenstates for the atom.

3. In a short paragraph (approx 50 words), describe how the state of the atom affects the coherent state.

2. 40%. Consider a two-level atom and a cavity mode on resonance, so that the Jaynes-Cummings interaction Hamiltonian applies.

1. The cavity is initially in the vacuum state $|0\rangle$ and the atom is initially in the excited state $|e\rangle$. If the atom enters the cavity at time $t = 0$ and leaves the cavity (i.e., stops interacting) at time $t = \pi/(4\Omega)$, calculate the final state of the system (field+atom).

2. Subsequent to this interaction, a second atom, initially in the ground state $|g\rangle$, passes through the same cavity. (The cavity/first atom are still in the state calculated in part (a).) This second atom interacts for twice the length of time ($t_2 = \pi/(2\Omega)$). Calculate the final state of the entire system (field + both atoms). Provide a one-sentence interpretation of this result. 

Hint: You will need a separate 2-level Hilbert space for each atom. It may be helpful to label the states of the two atoms in order to distinguish them, i.e., $|g\rangle_1$ for the ground state of the first atom.

3. 20%. Equation (2) in the paper P. Bertet et al., Nature 411, 166 (2001) gives the probability of detecting a particle in one output mode of a Mach-Zender interferometer as

$$P(\phi) = \frac{1}{2} (1 + \text{Re}(\langle \Psi_B_1(b) | \Psi_B_1(a) \rangle \exp(i\phi))).$$

1. What are the requirements on the states $|\Psi_B_1(a)\rangle$ and $|\Psi_B_1(b)\rangle$ in order to achieve maximum visibility of fringes?

2. Provide an interpretation (less than 100 words) of the physical meaning of the overlap $\langle \Psi_B_1(b) | \Psi_B_1(a) \rangle$, and its role in the interference experiment.

3. In a real Mach-Zender interferometer, what parameter (physical quantity) might you vary in order to change the value of $\langle \Psi_B_1(b) | \Psi_B_1(a) \rangle$? Explain your answer.