THE UNIVERSITY OF SYDNEY

FACULTY OF SCIENCE

PHYS3940/3941
PAPER 1 (ELECTROMAGNETISM ADVANCED)

SEMESTER 1, 2012          TIME ALLOWED: 1.5 HOURS

ALL QUESTIONS HAVE THE VALUE SHOWN

INSTRUCTIONS:
Students should attempt all questions.
Total marks: 80
No written material of any kind may be taken into the examination room.
A formula sheet is included.
Non-programmable calculators are permitted.
**Maxwell’s equations (in general):**

\[
\oint_S \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\varepsilon_0} Q_{\text{enc}} \\
\oint_S \mathbf{B} \cdot d\mathbf{a} = 0 \\
\int_P \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi_B}{dt} \\
\int_P \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}} + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}
\]

\[
\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0} \rho \\
\nabla \times \mathbf{B} = 0 \\
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}
\]

**Maxwell’s equations (in matter):**

\[
\nabla \cdot \mathbf{D} = \rho_f \\
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \cdot \mathbf{B} = 0 \\
\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}
\]

**Auxiliary fields:**

\[
\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \\
\mathbf{H} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{M}
\]

**Linear media:**

\[
\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}, \quad \mathbf{D} = \varepsilon \mathbf{E} \\
\mathbf{M} = \chi_m \mathbf{H}, \quad \mathbf{H} = \frac{1}{\mu} \mathbf{B}
\]

**Potentials:**

\[
\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}
\]

**Lorentz force law:**

\[
\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})
\]

**Energy, momentum & power:**

\[
U = \frac{1}{2} \int \left( \varepsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) d\tau \\
P = \varepsilon_0 \int (\mathbf{E} \times \mathbf{B}) d\tau \\
S = \frac{1}{\mu_0} (\mathbf{E} \times \mathbf{B})
\]

**Second derivatives:**

\[
\nabla \cdot (\nabla \times \mathbf{A}) = 0 \\
\nabla \times (\nabla f) = 0 \\
\n\nabla \times (\nabla \times \mathbf{A}) = \nabla (\mathbf{\nabla} \cdot \mathbf{A}) - \nabla^2 \mathbf{A}
\]

**Binomial theorem:**

\[
(1 + x)^n \approx 1 + nx \quad (x \ll 1)
\]
1. (a) Stokes’ theorem relates the path integral of a vector around a closed loop to the flux of its curl through the loop. Using this theorem, derive the differential form of Faraday’s law \((\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t)\) from the integral form. Include a brief explanation and be sure to define all the terms in the equations.

(b) Suppose the electric field in some region is found to be \(\mathbf{E} = kr^3 \hat{r}\), in spherical coordinates (\(k\) is a constant).

(i) Find the charge density as a function of radius.

(ii) Find the total charge contained in a sphere of radius \(R\), centred at the origin.

(iii) Is this situation physically realistic? Explain briefly.

The divergence in spherical coordinates is:

\[
\nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta v_\theta) + \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi}
\]

(c) In class we showed that the path integral of the magnetic vector potential around any closed loop is equal to the flux of the magnetic field through that loop. Show that this result does not depend on the choice of the gauge.

\((20\text{ marks})\)
2. (a) The continuity equation in differential form is

\[ \nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}. \]  

(i) Show how to convert the continuity equation into integral form. Make sure you give all steps in the argument. Hint: use the divergence theorem, which relates the flux of a vector field through a closed surface to the volume integral of its divergence.

(ii) Briefly describe the physical meaning of the equation you derived.

(b) The one-dimensional Dirac delta function, \( \delta(x) \), is zero everywhere except at the origin, and its integral from minus infinity to infinity is 1. Using these properties, evaluate the following integrals and explain your reasoning:

(i) \( \int_{-5}^{1}(x^2 - x - 1)\delta(x - 3)dx \)

(ii) \( \int_{0}^{3}(x^3 + 7x)\delta(x + 1)dx \)

(c) Suppose the electric and magnetic potentials are as follows:

\[ V(\mathbf{r}, t) = 0 \]  
\[ \mathbf{A}(\mathbf{r}, t) = -\frac{1}{4\pi \epsilon_0} \frac{qt\hat{\mathbf{r}}}{r^2} \]  

(i) Calculate the electric and magnetic fields that would produce these potentials.

(ii) What distribution of charges and currents would produce these potentials?

(20 marks)
3. Suppose that the current in a long ideal solenoid is increasing with time at a constant rate. The radius of the solenoid is $R$.

(a) The figure below shows the solenoid viewed from one end. Make three copies of this figure, with one showing the electric field, one showing the magnetic field and one showing the Poynting vector. Be sure to indicate any regions where each of these is zero.

![Diagram of a solenoid](image)

(b) Show that the magnitude of the electric field inside the solenoid at a distance $r$ from the central axis (where $r < R$) is proportional to $\frac{dB}{dt}$.

(c) Use the result from part (b) to show that the magnitude of the Poynting vector inside the solenoid is

$$ S = \frac{1}{2\mu_0} r B \frac{dB}{dt}. $$

(d) Consider a segment of the solenoid of length $L$.

(i) Calculate the flux of the Poynting vector inwards through the surface of this segment.

(ii) Show that this equals the rate at which energy is deposited in the magnetic field in this volume.

(iii) Explain why the rate at which energy is deposited in the electric field is zero.

(20 marks)
4. A plane electromagnetic wave is travelling through a linear non-conducting medium.

(a) Briefly explain what is meant in this context by:

(i) a plane wave;

(ii) a linear medium.

(b) In class we showed that the intensity of such a wave is given by

\[ I = \frac{1}{2} \varepsilon v E_0^2. \]  

Briefly explain the meanings (and give SI units) of each quantity in this equation \((I, \varepsilon, v\) and \(E_0\)).

(c) We also showed that when the wave strikes the boundary between two such media at normal incidence, and provided both materials have magnetic permeabilities close to \(\mu_0\), the amplitudes of the complex electric fields are related as follows:

\[ \tilde{E}_{0r} = \left( \frac{v_2 - v_1}{v_2 + v_1} \right) \tilde{E}_{0t}, \]  

\[ \tilde{E}_{0\tau} = \left( \frac{2v_2}{v_2 + v_1} \right) \tilde{E}_{0t}. \]

(i) Briefly explain the implication of the minus sign in Equation 5.

(ii) Show that Equations 5 and 6 satisfy conservation of energy.

(20 marks)

THERE ARE NO MORE QUESTIONS.