THE UNIVERSITY OF SYDNEY

FACULTY OF SCIENCE

PHYS 3048, 3049, 3051, 3073, 3948, 3949, 3951, 3973

SEMESTER 1, 2012 TIME ALLOWED: 2 HOURS

ALL QUESTIONS HAVE THE VALUE SHOWN

INSTRUCTIONS:

There are four sections in this paper, each worth 45 marks:

Section A: BIOPHYSICS
Section B: PLASMA PHYSICS
Section C: HIGH ENERGY PHYSICS
Section D: THERMODYNAMICS

You should attempt the two (2) subjects in which you are enrolled.

You must answer each section in a separate booklet.

You are permitted to take two (2) A4 pages, handwritten on one side only, into the examination room.

Non-programmable calculators are permitted.

The following data may be useful:

speed of light: \( c = 3.00 \times 10^8 \text{ m s}^{-1} \)

fundamental charge: \( e = 1.60 \times 10^{-19} \text{ C} \)

electron mass: \( m_e = 9.11 \times 10^{-31} \text{ kg} \)

Boltzmann constant: \( k_B = 1.38 \times 10^{-23} \text{ J K}^{-1} \)

electron volt: \( 1 \text{ eV} = 1.60 \times 10^{-19} \text{ J} \)

gravitational constant: \( G = 6.67 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2} \)

(Planck’s constant)/\(2\pi\): \( \hbar = 1.05 \times 10^{-34} \text{ J s} \)
SECTION A: BIOPHYSICS

Use a separate booklet. Please write “Section A” on the cover.

1. All students: Give brief physical explanations for the following:

(a) Explain why continuum theories, such as the Poisson-Boltzmann and Poisson-Nernst-Planck equations, fail in nanopores with radii smaller than the Debye length.

(b) Explain why random motion is the dominant form of motion in cells, despite almost all biomolecules having a net charge. How has this affected evolution of single cell animals (e.g., bacteria)?

(c) Explain the physical basis of ion selectivity in potassium channels, that is, how they conduct the K$^+$ ions near diffusion rates but reject the smaller Na$^+$ ions.

(15 marks)

2. All students: The first crystal structure of a sodium channel was determined last year. According to this structure, sodium channels are wider than potassium channels (average radius 4.0 Å) and can accommodate a fully hydrated Na$^+$ ion.

(a) Using the information given below, explain how sodium channels selectively conduct Na$^+$ ions.

(b) Estimate the selectivity ratio, $G(K^+)/G(Na^+)$, assuming that both ions have similar conductivities in bulk but K$^+$ ions must shed a water molecule from their hydration shell in order to fit in the channel.

(c) Approximating the sodium channel with a cylindrical water filled hole in the membrane, and ignoring ion-channel interactions, calculate the conductance of the channel for Na$^+$ ions in symmetric 150 mM NaCl baths.

(d) The calculated conductance in part (c) is 6 times larger than the observed value. Estimate the energy barrier faced by a Na$^+$ ion in crossing the channel (in units of $kT$).

Useful information:
Conductance of a wire of length $L$ and area $A$: $G = gA/L$, where $g$ is its conductivity, and the SI unit for $G$ is siemens (S).
Conductivity of Na$^+$ ions in a 150 mM NaCl solution: $g = 0.5 \Omega^{-1}m^{-1}$.
Thickness of membrane: 50 Å.
Radius of a Na$^+$ ion: 0.9 Å; including its first hydration shell: 3.8 Å.
Radius of a K$^+$ ion: 1.3 Å; including its first hydration shell: 4.3 Å.
Binding free energy of a water molecule to the hydration shell of a K$^+$ ion: $\approx 20 kT$.

(15 marks)
3. **Normal only:** The linearized Poisson-Boltzmann equation in 1D is given by

\[
\frac{d^2 \phi}{dx^2} = \kappa^2 \phi, \tag{1}
\]

where \( \phi \) is the electric potential and \( \kappa \) is the Debye length of the salt solution. For a 1–1 salt solution with concentration \( c_0 \), \( \kappa \) is given by

\[
\kappa = \sqrt{\frac{2e^2 c_0}{\varepsilon_0 \varepsilon_w kT}}, \tag{2}
\]

where \( e \) is the unit charge, \( \varepsilon_0 \) is the electric permittivity and \( \varepsilon_w \) is the dielectric constant of water.

(a) Solve the linearized Poisson-Boltzmann equation to determine the potential outside a uniformly charged membrane with surface charge density \( \sigma \), using the boundary condition at the membrane surface \( (x = 0) \):

\[
E(0) = -\frac{d\phi(0)}{dx} = \frac{\sigma}{\varepsilon_0 \varepsilon_w} \tag{3}
\]

(b) Calculate the electric field due this potential and hence the force \( F \) acting on a protein with total charge \( q \). Find the force, \( F_0 \), for the case \( c_0 = 0 \) (i.e. no salt in the solution), and plot the ratio \( F/F_0 \) as a function of \( x \) (in units of \( \kappa \)). Comment on the effect of salt solutions on the membrane-protein interactions.

\( 15 \) marks
4. **Advanced only**: For a spherically symmetric system, the linearized Poisson-Boltzmann equation is given by

\[ \frac{1}{r} \frac{d}{d r^2} (r \phi) = \kappa^2 \phi, \]  
(4)

where \( \phi \) is the electric potential and \( \kappa \) is the Debye length of the salt solution. For a 1–1 salt solution with concentration \( c_0 \), \( \kappa \) is given by

\[ \kappa = \sqrt{\frac{2e^2c_0}{\epsilon_0 \epsilon_w kT}}, \]  
(5)

where \( e \) is the unit charge, \( \epsilon_0 \) is the electric permittivity and \( \epsilon_w \) is the dielectric constant of water.

(a) Consider a spherical protein with radius \( a \) and a total charge \( q \) that is uniformly distributed over its surface. Show that the boundary condition to be used at the protein surface \( (r = a) \) is given by

\[ -\frac{d \phi(a)}{dx} = E(a) = \frac{\sigma}{\epsilon_0 \epsilon_w}, \]  
(6)

where \( \sigma \) is the surface charge density. Using this boundary condition, solve the linearized Poisson-Boltzmann equation and determine the potential outside the protein.

(b) Calculate the electric field due this potential and hence the force, \( F \), acting on a similar protein with total charge \( q' \). Find the force, \( F_0 \), for the case \( c_0 = 0 \) (i.e. no salt in the solution), and plot the ratio \( F/F_0 \) as a function of \( r \) (in units of \( \kappa \), assuming \( a = 1/\kappa \)). Comment on the effect of salt solutions on the protein-protein interactions.

(15 marks)
SECTION B: PLASMA PHYSICS
Use a separate booklet. Please write “Section B” on the cover.

5. All students: Explain, in at most a few sentences, what the following plasma physics concepts are and why they are physically relevant.

   (a) The plasma parameter: explain its significance
   (b) Electrical conductivity: comment on its real and imaginary parts
   (c) Longitudinal dielectric function
   (d) Runaway electrons
   (e) The electric drift
   (f) An adiabatic invariant: give an example
   (g) The MHD equations: name them or write them down
   (h) Magnetic tension
   (i) The frozen-in condition
   (j) Fast and slow MHD modes

(15 marks)
6. **All students:**

The longitudinal dispersion relation for waves in a plasma consisting of cold ions and thermal electrons is

\[ K^L(\omega, k) = 0, \quad K^k(\omega, k) = 1 + \chi_i(\omega) + \chi_e^L(\omega, k), \]  

with \( \chi_i(\omega) = -\omega_{pi}^2/\omega^2 \), with \( \omega_{pi} \) the ion plasma frequency, and

\[ \chi_e^L(\omega, k) = \begin{cases} \frac{1}{k^2 \lambda_{De}^2} & \omega \ll kV_e, \\ -\omega_{pe}^2/\omega^2 & \left(1 + \frac{3k^2V_e^2}{\omega^2} + \cdots\right), \quad \omega \gg kV_e, \end{cases} \]  

where \( \omega_{pe} \) is the electron plasma frequency, \( V_e \) is the thermal speed of electrons, and \( \lambda_{De} = V_e/\omega_{pe} \) is the electron Debye length.

(a) Show that for \( \omega \gg kV_e \), a solution of (7) gives the dispersion relation \( \omega = \omega_L(k) \) for Langmuir waves, with

\[ \omega_L^2(k) = \omega_p^2 + 3k^2V_e^2 + \cdots, \]  

with \( \omega_p^2 = \omega_{pi}^2 + \omega_{pe}^2 \approx \omega_{pe}^2 \).

(b) Show that for \( \omega \ll kV_e \), a solution of (7) gives the dispersion relation \( \omega = \omega_s(k) \) for ion sound waves, with

\[ \omega_s^2(k) = \frac{k^2v_s^2}{1 + k^2\lambda_{De}^2}, \]  

where \( v_s^2 = \omega_{pi}^2\lambda_{De}^2 \) is the ion sound speed.

(c) Show that the dispersion relation (9), specifically, \( \omega^2 = \omega_p^2 + 3k^2V_e^2 \), implies that the product of the phase and group speeds for Langmuir waves is equal to \( 3V_e^2 \).

(d) Evaluate the group speed for ion sound waves assuming the dispersion relation \( \omega = \omega_s(k) \), with \( \omega_s(k) \) given by (10).

(e) In a cold plasma model for Langmuir waves \( V_e^2 \to 0 \) in (9)], collisions are included by assuming \( \chi_e^L(\omega, k) = -\omega_{pe}^2/\omega(\omega + i\nu_e) \), where \( \nu_e \) is the collision frequency. Explain how collisions cause wave damping and derive the rate of damping for electron plasma oscillations due to \( \nu_e \neq 0 \).

\((15 \text{ marks})\)
7. **Normal only:**

(a) Explain the floating potential, describing how one can determine the floating potential from probe measurements. Estimate the floating potential for an argon discharge plasma with electron temperature 2 eV (the ion mass is \(6.6 \times 10^{26}\) kg).

(b) Explain how an axial current flowing along a straight plasma cylinder can be used to confine, compress, and heat the plasma. The balance of what forces and pressures makes it possible to confine the plasma? Name at least two problems with the plasma confinement in the case considered.

(15 marks)
8. **Advanced only:** One form of the plasma dispersion function is defined by the integral

\[ \phi(z) = -\frac{z}{\sqrt{\pi}} \int_{-\infty}^{\infty} dt \frac{e^{-t^2}}{t-z}. \]  

(11)

The longitudinal part of the response tensor is

\[ K^L(\omega, k) = 1 + \sum_\alpha \frac{\omega^2_{p\alpha}}{k^2 V^2_\alpha} [1 - \phi(y_\alpha) + i\pi^{1/2} y_\alpha \exp(-y^2_\alpha)], \]  

(12)

where \( \alpha \) labels the species of particle, with plasma frequency \( \omega_{p\alpha} = (q^2_\alpha n_\alpha/\varepsilon_0 m_\alpha)^{1/2} \), thermal speed \( V_\alpha \), and \( y_\alpha = \omega/kV_\alpha \).

(a) By differentiating (11) and partially integrating, show that \( \phi(y) \) satisfies

\[ \frac{d\phi(z)}{dz} = \frac{\phi(z)}{z} + 2z[1 - \phi(z)], \]  

(13)

and show that the solution of (13) is

\[ \phi(z) = 2ze^{-z^2} \int_0^z dt \ e^{t^2}. \]  

(14)

(b) Use either (11) or (14) to derive the first two terms in the expansion of \( \phi(z) \) for small \( z \),

\[ \phi(z) = 2z^2 - \frac{4}{3}z^4 + \cdots, \]  

(15)

and the first three terms in the expansion of \( \phi(z) \) for large \( z \):

\[ \phi(z) = 1 + \frac{1}{2z^2} + \frac{3}{4z^4} + \cdots. \]  

(16)

(c) A plasma consists of two thermal distribution of electrons, one cold \( (\alpha = c) \) and the other hot \( (\alpha = h) \), plus ions whose contribution is neglected. Evaluate \( K^L(\omega, k) \) using (12) in the limit \( V_c \rightarrow 0, \omega \gg kV_h \).

(d) Show that your result implies that \( K^L(\omega, k) = 0 \) has a solution

\[ \omega^2 \approx \frac{k^2 v^2_{cs}}{1 + k^2 \lambda^2_{Dh}}, \]  

(17)

with \( v_{cs} = \omega_{pc} \lambda_{Dh}, \lambda_{Dh} = V_h/\omega_{ph} \). Waves satisfying the dispersion relation (17) are called electron acoustic waves.

(15 marks)
SECTION C: HIGH ENERGY PHYSICS
Use a separate booklet. Please write “Section C” on the cover.

FORMULAE & USEFUL DATA:

<table>
<thead>
<tr>
<th>Particle</th>
<th>Quark Content</th>
<th>Mass (GeV/c^2)</th>
<th>Spin (ℏ)</th>
<th>Strong Isospin (I, I₃)</th>
</tr>
</thead>
<tbody>
<tr>
<td>p</td>
<td>uud</td>
<td>0.9383</td>
<td>1/2</td>
<td>(1/2, +1/2)</td>
</tr>
<tr>
<td>n</td>
<td>udd</td>
<td>0.9396</td>
<td>1/2</td>
<td>(1/2, −1/2)</td>
</tr>
<tr>
<td>Λ</td>
<td>uds</td>
<td>1.1157</td>
<td>1/2</td>
<td>(0, 0)</td>
</tr>
<tr>
<td>Σ⁺</td>
<td>uus</td>
<td>1.1894</td>
<td>1/2</td>
<td>(1, +1)</td>
</tr>
<tr>
<td>Σ⁰</td>
<td>uds</td>
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<td>1/2</td>
<td>(0, 0)</td>
</tr>
<tr>
<td>π⁺</td>
<td>ud</td>
<td>0.1396</td>
<td>0</td>
<td>(1, +1)</td>
</tr>
<tr>
<td>π⁰</td>
<td>uū or dū</td>
<td>0.1350</td>
<td>0</td>
<td>(0, 0)</td>
</tr>
<tr>
<td>K⁺</td>
<td>uū or dū</td>
<td>0.4937</td>
<td>0</td>
<td>(1/2, +1/2)</td>
</tr>
<tr>
<td>K⁰</td>
<td>dū or ud</td>
<td>0.4977</td>
<td>0</td>
<td>(1/2, −1/2)</td>
</tr>
<tr>
<td>e⁻</td>
<td></td>
<td>0.0005110</td>
<td>1/2</td>
<td></td>
</tr>
<tr>
<td>μ⁻</td>
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<td>0.1057</td>
<td>1/2</td>
<td></td>
</tr>
<tr>
<td>τ⁻</td>
<td></td>
<td>1.7768</td>
<td>1/2</td>
<td></td>
</tr>
<tr>
<td>νₑ, νₑ, νₑ⁻</td>
<td></td>
<td>≈ 0</td>
<td>1/2</td>
<td></td>
</tr>
<tr>
<td>W⁺</td>
<td></td>
<td>80.4</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Z⁰</td>
<td></td>
<td>91.2</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

Cabibbo angle \( \theta_C = 12.7^\circ \)

Relativistic Equations

\[ E^2 = (pc)^2 + (mc^2)^2 \quad E = \gamma mc^2 \quad p = \beta E/c \quad \gamma = (1 − \beta^2)^{-1/2} \quad \beta = v/c \]

\[ E' = \gamma (E - \beta pc) \quad p'_X = p_X \quad p'_Y = p_Y \quad p'_Z = \gamma (pc - \beta E/c) \]

Selected Clebsch-Gordan Coefficients

Note: A \( √ \) sign is to be understood over every coefficient, e.g. for \(-1/2\) read \(-√1/2\).
9. **All students:**

Consider the following diagram, which depicts the octet of baryons with spin $\frac{1}{2}\hbar$. The values in brackets give the approximate mass of the particles with the given symbol in the corresponding row, in units of MeV/$c^2$.

(a) Briefly explain what the quantities $I$, $I_3$ and $Y$ represent.

(b) The mass of each row of particles generally increases from top to bottom of the diagram. Give an explanation for this.

(c) Make a copy of the diagram, and for each particle, indicate on your diagram its quark content.

(d) Are the $\Sigma^+$ and $\Sigma^-$ antiparticles of each other? Explain.

(e) The $\Lambda$ and $\Sigma^+$ have quite similar lifetimes of $2.63 \times 10^{-10}$ s and $1.48 \times 10^{-10}$ s respectively, whilst the $\Sigma^0$ has a lifetime of $7.4 \times 10^{-20}$ s. Give an explanation for the large variation in lifetimes between this last particle and the first two.

(f) Draw a Feynman diagram for the following decay mode of the $\Sigma^+$: $\Sigma^+ \rightarrow p + \pi^0$.

(g) Predict one possible decay mode for the $\Xi^-$, and draw a Feynman diagram for it.

The table of particle properties at the beginning of this section may be useful.

(15 marks)
10. **All students:**

A $D^0$ particle at rest is observed to decay according to $D^0 \rightarrow K^- + \pi^+$. 

(a) Calculate the momentum of the $K$ meson, in units of GeV/c.

(b) Which of the two emitted particles will have the greater velocity, the $K$ meson or the pion? Briefly explain your answer.

(c) Draw a Feynman diagram for the decay, clearly labelling all particles, and state which type of interaction is responsible.

(d) Is the above decay more or less likely than the decay $D^0 \rightarrow K^+ + \pi^-$? Explain your answer, and predict the ratio of probabilities for the two decays.

The mass of the $D^0$ particle is 1.8648 GeV/$c^2$, and it has quark content $c\bar{u}$. The table of particle properties at the beginning of this section may also be useful.

*(15 marks)*
11. **Normal only:**

(a) Briefly describe, with the aid of a labelled diagram, the principle of Wu and collaborators’ experiment studying the nuclear beta decay of $^{60}$Co. Comment on the significance of the result that they obtained.

(b) Consider the following two scattering processes, which proceed predominantly via the strong interaction:

\[
\begin{align*}
\bar{K}^0 + p & \rightarrow \Sigma^+ + \pi^0 \\
\bar{K}^0 + p & \rightarrow \Sigma^0 + \pi^+
\end{align*}
\]

Using strong isospin considerations, work out the relative probability of the two processes occurring, briefly explaining your reasoning.

(c) Here is a further potential scattering process involving a $\bar{K}^0$:

\[
\bar{K}^0 + p \rightarrow \Lambda + K^+
\]

Can this process occur in nature? If it can, briefly justify your answer. If not, give one reason why not.

(d) The reactions in part (b) show two ways in which a $K^0$ interacts in matter. Now suppose a $\bar{K}^0$ decays to pions only. Would it decay to two pions or three pions? Explain your answer.

The tables of particle properties and Clebsch-Gordan coefficients at the beginning of this section may be useful.

(15 marks)
12. **Advanced only:**

(a) Consider the following two *scattering processes* involving neutral kaons, which proceed predominantly via the strong interaction:

\[
\bar{K}^0 + p \rightarrow \Sigma^+ + \pi^0 \\
\bar{K}^0 + p \rightarrow \Sigma^0 + \pi^+
\]

Using strong isospin considerations, work out the relative probability of the two processes occurring, briefly explaining your reasoning.

(b) Explain how the following *decay* of a neutral kaon, and a related decay, can be used to establish an absolute convention for the meaning of matter versus antimatter.

\[
K_L^0 \rightarrow \pi^− + e^+ + \nu_e
\]

(c) Briefly discuss how a matter-antimatter asymmetry might develop in the Universe, despite symmetric conditions at the Big Bang.

(d) A free neutron is unstable to the weak interaction. Briefly describe an extension to the Standard Model in which a free neutron might be unstable in a new way, and give an example of a possible decay mode of the neutron in that case.

The tables of particle properties and Clebsch-Gordan coefficients at the beginning of this section may be useful.

*(15 marks)*
SECTION D: THERMODYNAMICS

Use a separate booklet. Please write “Section D” on the cover.

13. **All students:** Explain briefly (less than 50 words each) what is meant by each of the following:

   (a) A microstate.
   
   (b) The partition function.
   
   (c) The chemical potential.
   
   (d) The quantum volume.
   
   (e) Osmotic pressure.

   \(15 \text{ marks}\)

14. **All students:** Consider a fuel cell that uses methane (“natural gas”) as fuel. The reaction is

\[
\text{CH}_4 + 2\text{O}_2 \rightarrow 2\text{H}_2\text{O} + \text{CO}_2
\]

(a) Use the data from Table 1 to determine the values of \(\Delta H\) and \(\Delta G\) for this reaction, for one mole of methane. Assume that the reaction takes place at room temperature and atmospheric pressure.

(b) How much electrical work can you get out of the fuel cell, for each mole of methane, assuming ideal performance?

(c) Still assuming ideal operation, how much heat is absorbed or expelled by the chemicals during the reaction, for each mole of methane fuel? (Be sure to say which direction the heat flows.)

(d) If this reaction occurred at a higher temperature, how would you expect your answer to part (b) to change? Explain why.

   \(15 \text{ marks}\)

<table>
<thead>
<tr>
<th>Substance</th>
<th>(\Delta_f H) (kJ)</th>
<th>(\Delta_f G) (kJ)</th>
<th>(S) (J/K)</th>
<th>(C_P) (J/K)</th>
</tr>
</thead>
<tbody>
<tr>
<td>CH(_4)</td>
<td>-74.81</td>
<td>-50.72</td>
<td>186.26</td>
<td>35.31</td>
</tr>
<tr>
<td>O(_2)</td>
<td>0</td>
<td>0</td>
<td>205.14</td>
<td>29.38</td>
</tr>
<tr>
<td>CO(_2)</td>
<td>-393.51</td>
<td>-394.36</td>
<td>213.74</td>
<td>37.11</td>
</tr>
<tr>
<td>H(_2)O</td>
<td>-285.83</td>
<td>-237.13</td>
<td>69.91</td>
<td>75.29</td>
</tr>
</tbody>
</table>

Table 1: Data for a methane fuel cell. All values are for one mole of material at room temperature and atmospheric pressure. The enthalpy and Gibbs free energy of formation, \(\Delta_f H\) and \(\Delta_f G\), represent the changes in \(H\) and \(G\) upon forming one mole of the material, starting with elements in their most stable pure states.
15. **Normal only:**

(a) (i) For a system of fermions at room temperature, compute the probability of a single-particle state being occupied if its energy is 0.005 eV less than the chemical potential, $\mu$.

(ii) For a system of fermions at room temperature, at what value of the energy is the probability of a single-particle state being occupied precisely $\frac{3}{4}$?

(iii) For a system of bosons at room temperature, compute the average occupancy of a single-particle state if its energy is 0.005 eV greater than the chemical potential, $\mu$.

(b) (i) In no more than 50 words, explain the *ultraviolet catastrophe*.

(ii) In no more than 50 words, explain how quantum statistical mechanics solves this catastrophe.

*(15 marks)*
16. **Advanced only:**

In addition to the cosmic background radiation of photons, the universe is thought to be permeated with a background radiation of neutrinos ($\nu$) and antineutrinos ($\bar{\nu}$), currently at an effective temperature of 1.95 K. There are three species of neutrinos, each of which has an antiparticle, with only one polarization state (spin) for each particle or antiparticle. Assume that the chemical potential $\mu$ of all three species of neutrino and antineutrino in the cosmic background is zero.

(a) Neutrinos have extremely small masses, and therefore are highly relativistic. Using box quantization in three dimensions, show that the energy of the quantum state labelled by $\vec{n} = (n_x, n_y, n_z)$ is

$$\varepsilon = \frac{hc n}{2L},$$

where $n = |\vec{n}|$.

(b) Given that neutrinos are fermions, show that the total energy density (energy per unit volume) of the neutrino-antineutrino background radiation is

$$\frac{U}{V} = \frac{7\pi^5 (kT)^4}{5(hc)^3}.$$

You will need the following definite integral:

$$\int_0^\infty \frac{x^3}{e^x + 1} dx = \frac{7\pi^4}{120}.$$

(c) In no more than 50 words, compare this result with the total energy density of the photon background radiation. Why are they similar/different?

(d) Calculate an expression for the number of neutrinos per unit volume of the neutrino-antineutrino background radiation. How does this number depend on the temperature?

"\(15 \text{ marks}\)"

**THERE ARE NO MORE QUESTIONS.**