Experiment 7. Wave Propagation

1 Safety

- This experiment uses microwave radiation. The power is very low (∼ 5 mW) and does not represent a safety hazard.
- If you suspect an item of equipment is not operating correctly, turn it off and, if mains operated, turn off the power at the mains switch, then consult a tutor.

2 Aim

In this experiment you will measure the radiation pattern of a half-wave dipole antenna, observe the directional effect of adding a parasitic reflector, determine the resonant frequencies of a microwave cavity, and examine standing waves in a rectangular waveguide.

Useful general references for this experiment are

- Cheng, D.K. Field and wave electromagnetics, Addison-Wesley, Reading, Mass, 1992 [530.41 31 B]

Extracts from both books are on the computer desktop.
3 Radiation from a half-wave antenna

Antennas are structures designed for radiating and receiving electromagnetic energy in prescribed patterns. These patterns depend on the geometry of the antenna and can take many different forms. The radiation originating from a linear dipole antenna depends critically on its length. The radiation pattern of a half-wave antenna (one whose length is half a radiation wavelength) is studied in this section. A reflector element is then added to enhance the signal in one preferred direction.

3.1 Measuring the radiation pattern

From the theory of radiating antennas one can deduce the expected polar dependence for a half-wave dipole. The radiation intensity (defined as the time average power density per unit solid angle) can be expressed as follows:

\[ U = U_{\text{max}} \left| \frac{\cos\left(\frac{\pi}{2} \cos \theta \right)}{\sin \theta} \right|^2 \]  

where \( \theta \) is the angle between a line along the arms of the antenna and a line joining the middle of the antenna to the detector; see Fig. 7-1.

To record the radiated intensity as a function of angle, \( \theta \), the signal generator is coupled to the dipole antenna through a rotating joint. The fixed receiving antenna also has the same linear dipole geometry as the transmitter and incorporates a built-in diode detector. The output from the detector is fed via a coaxial cable to a sensitive DC current meter (nanoammeter)\(^1\), whose rear output can be fed to a digital voltmeter (DVM). You can read the output signal either from the analogue meter scale or the DVM.

The diode detector has a “square law” dependence on output with respect to the input, i.e., the time averaged forward current is proportional to the square of the input voltage, and since the power is also proportional to voltage squared \( P = V^2 / R \), the measured output current is proportional to the received power.

**Question 1:** We assume that the detector diode responds according to the Shockley model \( I-V \) characteristic:

\[ I = I_s \exp(eV/nkT) - 1 \],

where \( e \) is the electron charge, \( V \) is the input voltage, \( n \) is known as the diode ideality factor (for silicon diodes \( n \approx 1–2 \)), \( k \) is Boltzmann’s constant, \( T \) is the absolute temperature, \( I_s \) is the inherent leakage current of the diode (when \( V \) has a large negative value) and \( I \) is the measured forward current. Assuming that the input voltage is small and has the form \( V = V_0 \sin \omega t \), and that the meter records the average current (because the frequency is so high), prove that the time averaged forward current is proportional to the square of the input voltage. 

**Hint:** Start by expanding Eq. 2 in a Taylor series, assuming that \( V_0 \ll nkT/e \), where \( nkT/e \approx 30 \text{ mV} \) at room temperature.

\(^1\)NOTE: The nanoammeter should be zeroed only on the 0.3µA range by pressing the PRESS to set zero button and adjusting the zero knob. It’s a good idea to check the zero several times during the measurement sequence.
Procedure

1. Measure the length of the half-wave dipole and hence calculate its approximate resonant frequency. Using a Type-N to BNC adaptor, connect a cable from the microwave generator output to the frequency counter (select input C) and set the switch and fine frequency controls on the generator to your calculated frequency.

2. Now connect the transmitting antenna with its rotating joint and angle scale to the output of the microwave generator. With $\theta = 90^\circ$ (Fig. 7-1), set the transmitting and receiving antennas about 40–50 cm apart, at the same height, and ensure they are both at right angles to the line joining them; use the plastic tube holding the receiving antenna for alignment.

3. Test that the radiation is plane polarised by rotating the receiving antenna about the axis of its plug and noting the changes in the reading of the detector current. The maximum reading should occur when the receiving and transmitting antennas are parallel.

4. With $\theta = 90^\circ$, adjust the frequency carefully to maximise the output signal and note the generator dial reading. Take readings of the detector current (or DVM voltage) every 10$^\circ$ as the transmitter antenna is rotated through 360$^\circ$. Because the beam of the antenna is broad, it is important when recording data to be at least a metre away from the transmitting and receiving antennas to avoid reflections that could affect your measurements.

5. Plot the polar diagram of the transmitting antenna (i.e., received power versus angle) using Radar plot in Excel (or equivalent package in other software).

6. Identify $U_{\text{max}}$ from your experimental data (corresponding to $\theta = 90^\circ$ or $270^\circ$), tabulate the expected theoretical response for each angle and plot in a different colour on the same polar plot as the experimental data.

7. Compare both polar diagrams and comment.

Fig. 7-1 Plan view (schematic) for measuring the directional characteristic of the transmitting antenna.
3.2 Radiation with a reflector element

It is often desirable to concentrate the radiation from an antenna in a particular direction. By adding one or more “parasitic” elements we can improve the front-to-back ratio and hence increase the directionality of the antenna array.

By introducing a reflector element behind the transmitting antenna we can maximise the signal in the forward direction. A current is induced in the parasitic element by the current in the driven element. Its phase lag will depend on the distance $d$ between them and the length $l$ of the reflector. If $l > \lambda/2$ there is a phase change of $180^\circ$ on reflection. The field radiated from the antenna will be the vector sum of the fields from each element, so by selecting the path difference appropriately one can achieve constructive interference in one direction and destructive interference in the opposite direction.

The interference between elements to improve directionality and to enhance the signal in a preferred direction works just as well with a receiving antenna. This is exploited in the design of the ubiquitous Yagi-Uda antennas found on top of most houses for TV or FM reception. As well as a single reflector element (as studied above) these antennas use many director elements in front of the “driven” element which is connected to the TV or FM receiver. Only one reflector element (usually slightly longer in length than the driven element) is needed. However, by optimising the number and separation of the somewhat shorter director elements in front of the driven element, the directionality (and bandwidth) can be markedly improved. The antenna is now sensitive to radiation coming in a narrower cone about its axis, compared to a single dipole or a single dipole plus reflector.

The radiation intensity for a half wave dipole transmitter separated by a distance $d$ from a reflector element with $l > \lambda/2$ is given by:

$$ U = U_{\text{max}} \left| \frac{\cos\left(\frac{\pi}{2} \cos \theta\right)}{\sin \theta} \right|^2 \left| \cos \frac{\psi}{2} \right|^2 $$

where $\psi = kd (\sin \theta + 1) - \pi$ is the phase factor and $k = 2\pi/\lambda$ is the wave number.

**Question 2:** Ideally, how far apart should the two elements be to achieve constructive interference in the forward direction and destructive in the backward direction? Use a physical argument, not mathematics.

**Procedure**

1. Check the validity of your prediction in Question 2 by adjusting the position of the reflector element to find the first two maxima and the first minimum. Measure the distances between transmitter and reflector elements for these three cases. Comment on your observations and on any anomalies you find.

2. By placing the reflector at the position of the first maximum, plot the polar diagram in this configuration. Compare your measured results with the expected response from Eq. 3 by plotting them on the same polar diagram.

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Footnote: At smaller distances the mutual impedance between the two elements introduces an extra phase term dependent on the distance between transmitter and reflector. See Wolff, pp 296–298, in the bench file.
4 Cylindrical resonant cavity

A cylindrical cavity can be visualised as a circular waveguide with two conducting ends on it. Resonant modes for these cavities can be either transverse magnetic (TM) or transverse electric (TE) and are defined by three indices, i.e., $TM_{mnp}$ and $TE_{mnp}$. The first index $m$ gives the number of cycles as the azimuthal coordinate $\phi$ goes through $360^\circ$. The second index $n$ gives the corresponding variation with radius from the cylinder axis to the conducting wall. The third index $p$ gives the number of cycles in the axial ($z$) direction.

There is an interesting dependence of the electric and magnetic fields on radius from the axis. For the $TM$ modes the radial component of the magnetic field, the azimuthal and axial components of the electric field all depend on radius as Bessel functions of integral order $J_m$. For $TE$ modes, swap the words electric and magnetic. For both sets of modes the boundary conditions must be obeyed: the electric field must be normal to the walls (i.e., the component parallel to the wall must be zero), and the magnetic field tangential. If the radius of the cavity is $a$ and the height $h$, then the $TM_{mnp}$ modes are given by:

$$E_r = AJ_m'(k_c r) \cos(m\phi) \sin(\pi p z/h)$$  
$$E_\phi = (B/r) J_m(k_c r) \sin(m\phi) \sin(\pi p z/h)$$  
$$E_z = C J_m(k_c r) \cos(m\phi) \cos(\pi p z/h)$$

with $J_m(k_c r) = 0$ at $r = a$.

The $TE_{mnp}$ modes are given by:

$$E_r = (D/r) J_m(k_c r) \sin(m\phi) \sin(\pi p z/h)$$  
$$E_\phi = F J_m'(k_c r) \cos(m\phi) \sin(\pi p z/h)$$  
$$E_z = 0$$

with $J_m'(k_c r) = 0$ at $r = a$.

$A$, $B$, $C$, $D$ and $F$ are constants depending on the wave amplitude, and $k_c$ is defined by

$$k_c^2 = (\omega/c)^2 - k_z^2 = k_0^2 - k_z^2,$$

where $k_z = 2\pi/\lambda_z$ and $\lambda_z = 2h/p$.

Depending on whether the mode is $TM$ or $TE$, $J_m(k_c r)$ or its derivative $J_m'(k_c r)$ will be zero at the cavity wall $r = a$. These conditions will be met if

$$k_c a = x_{mn} \quad (TM)$$  
$$k_c a = x'_{mn} \quad (TE)$$

where $x_{mn}$ is the $n$th zero of $J_m(x)$ and $x'_{mn}$ is the $n$th zero of $J_m'(x)$, where $n = 1, 2, \ldots$. There is more information in the reference book by Ramo, Whinnery & Van Duzer.

Each of the different modes of oscillation for a cavity of a certain size can be associated with a resonant frequency. At resonance the amount of energy absorbed by the cavity is a maximum.
**Question 3:** Prove that the resonant wavelengths associated with the first three resonant modes of oscillation are as follows:

\[
\begin{align*}
\lambda_0 &= 2.613a \quad \text{for the } \text{TM}_{010} \text{ mode} \\
\lambda_0 &= \frac{2\pi}{\sqrt{\left(\pi/a\right)^2 + (1.841)^2}} \quad \text{for the } \text{TE}_{111} \text{ mode} \\
\lambda_0 &= \frac{2\pi}{\sqrt{\left(\pi/a\right)^2 + (2.405)^2}} \quad \text{for the } \text{TM}_{011} \text{ mode}
\end{align*}
\]

To do this you will need to make the parallel component of the electric field zero at the walls of the cavity, knowing that:

\[
J_0(k_c a) = 0 \quad \text{when } k_c a = 2.405 \quad \text{and} \quad J_1'(k_c a) = 0 \quad \text{when } k_c a = 1.841.
\]

Note that \( k_0^2 = k_c^2 + k_z^2 \), where \( k_c \) is the transverse component and \( k_z \) is the axial component of the propagation vector, and the resonant wavelength is \( \lambda_0 = 2\pi/k_0 \).

We can affect the electric field inside the cavity by inserting conducting “needles” through strategically placed holes drilled through the sides or ends. If the insertion of a needle produces a shift in the resonant frequency it must mean that there is a component of electric field parallel to the direction of the needle.

![Fig. 7-2 Experimental arrangement for cavity measurements.](image)

**Procedure**

1. Connect the apparatus as shown in Fig. 7-2, with the frequency counter connected to the output of the microwave generator via a T-piece; select input C on the counter. We
use the BWD function generator to sweep the frequency of the microwave generator with a 1 kHz asymmetrical triangular wave (SYMM and triangle wave buttons pressed); this signal is also connected to the X channel (CH1) of the Agilent DSO1002A digital oscilloscope, while the detector output is connected to the Y channel (CH2).

2. Make sure the oscilloscope is in X-Y mode, and the sampling rate is set to 125k samples/second. Resonances will appear as sharp dips on the trace as the frequency is adjusted upwards from \(~2\) GHz. Disregard the more numerous small “bumps”; they are due to weak emissions from the microwave generator at \(2f, 3f, \text{etc.}\).

3. Measure the first three resonant frequencies accurately by progressively reducing the sweep amplitude and adjusting the frequency control so that the resonant dip is centred. If frequency drift is a problem, use the ‘Hold’ button on the frequency counter to freeze the count when the dip is accurately symmetric.

4. Identify the following modes: \(TM_{010}, TE_{111}\) and \(TM_{011}\) by comparing the theoretical and measured resonant frequencies, and use the “needle” technique to trace the field directions and show they are consistent with the detailed patterns in Ramo et al. The cavity radius is \(a = 4.902\) cm and its height is \(h = 7.646\) cm.

5. Using the dependence of the different components of the electric field shown in Eqns 4–9, make neat sketches of the electric field patterns for the three resonant modes \(TM_{010}, TE_{111}\) and \(TM_{011}\). A diagram of the first four Bessel functions is shown in Fig. 7-3.

![Dependence of Bessel functions](image)

**Fig. 7-3** Dependence of Bessel functions \(J_n(x)\) as a function of \(x\).

### 5 TE\(_{10}\) waves in a rectangular guide

A waveguide is a structure that allows a wave to propagate in a chosen direction while confining it inside a prescribed geometry. An open-ended rectangular waveguide will be studied in this section. The metal boundaries impose the condition that the tangential component of the electric field remains zero and thus the \(x\) and \(y\) axes support standing waves. Even though the net propagation of energy only occurs along the \(z\) axis, we also find standing waves along this axis because the waveguide is not “matched” to its environment, i.e., the impedance of the waveguide is not the same as the impedance of free space, causing a reflection at the interface.
By analogy with the cylindrical cavity, the allowed modes of oscillation in a rectangular waveguide are \( TM_{mn} \) and \( TE_{mn} \), where \( TM \) means that the magnetic field is entirely transverse (perpendicular to the \( z \) axis) and \( TE \) that the electric field is transverse. The first index \( m \) gives the number of cycles of variation of the field allowed along the \( x \) axis while the second \( n \) describes the number of cycles allowed along the \( y \) axis. By choosing the frequency of operation appropriately we can select just the \( TE_{10} \) oscillation mode.

There is a linear relationship between the inverse of the square of the free-space wavelength (\( \lambda \)) and the inverse of the square of the guide wavelength (\( \lambda_z \)), given by

\[
\frac{1}{\lambda^2} = \frac{1}{\lambda_z^2} + \frac{1}{4a^2}
\]  

(13)

where \( a \) is the guide width in the \( x \) direction. \( \lambda_c = 2a \) is called the cut-off wavelength because this is the maximum wavelength that the guide can support (for the \( TE_{10} \) mode).

![Experimental arrangement for waveguide measurements.](image)

**Procedure**

1. Connect the microwave generator to the conical antenna at the end of the waveguide and to the frequency counter. Power picked up by the sliding probe passes through an adaptor into a detector and then to the DC nanoammeter (see Fig. 7-4).

2. Starting with a frequency of 2.5 GHz, move the probe carriage back and forth to locate the positions where the DC nanoammeter reads a minimum or maximum. Because these minima and maxima are generated by standing waves in the \( z \) direction, the average separation between successive minima or maxima is equal to half the guide wavelength, i.e., \( \lambda_z/2 \). Repeat this for frequencies from 3.0 to 5.0 GHz in steps of 0.5 GHz.

3. For each setting, record the frequency and the mean separation between successive minima and maxima, and plot a graph of \( 1/\lambda^2 \) against \( 1/\lambda_z^2 \).

4. Is the slope of the graph consistent with Eq. 13? Use the intercept to calculate the guide width, \( a \), and compare your estimate with a direct measurement made with the digital vernier caliper.