Interacting boson model (IBM)

Historical note: In the 1970’s, there were two basic models of nuclei for describing nuclear spectra:

1. Shell model (diagonalization of very large matrices)
2. Geometrical model (solution of a 5 dimensional differential eq.)

Apart from some simple cases, application of both models required a great deal of computing power, which was not available at that time.

The interacting boson model (1975) overcame this computational bottleneck by providing

- analytical solutions in special cases (e.g. vibrational and rotational)
- simpler numerical solutions in general cases (e.g. diagonalization of matrices of size <100)

Therefore it was received with enthusiasm, especially by the experimentalists who could compare their measurements to theory and test the predictions of the model.
Construction of the IBM

In the last lecture, we have seen that quadrupole vibrations in a spherical nuclei can be described using quadrupole phonon (boson) operators constructed from a 5D harmonic oscillator. While this model gives a simple description of vibrational excitations it has several shortcomings:

1. Because it is based on harmonic potentials, the spectrum is infinite.
2. The phonons are purely phenomenological, that is, they have no connection to the underlying shell model (no microscopic basis).
3. There is no prospect of generalizing this model to other (e.g. deformed or gamma-unstable) nuclei.

The interacting boson model attempts to address all these issues by:

1. Adding a scalar (s) boson to the quadrupole (d) bosons and assuming that the total number of s and d bosons is N (constant).
2. Assuming that s and d bosons are made of pairs of nucleons.
3. Constructing Hamiltonians that allow d-bosons in the ground state.
1. How to create a finite spectrum

Consider a 2D harmonic oscillator with frequencies and energies

\[ \omega_a, \omega_b, \quad \varepsilon_a = \hbar \omega_a, \quad \varepsilon_b = \hbar \omega_b \]

We can describe the spectrum using two scalar boson operators

\[ [a, a^\dagger] = 1, \quad [b, b^\dagger] = 1, \quad \text{and all the rest are zero.} \]

The Hamiltonian and the corresponding eigenstates and energies are given by

\[ H = (a^\dagger a + 1/2)\varepsilon_a + (b^\dagger b + 1/2)\varepsilon_b = (\hat{N}_a + 1/2)\varepsilon_a + (\hat{N}_b + 1/2)\varepsilon_b \]

\[ |N_a, N_b\rangle = \frac{1}{\sqrt{N_a!N_b!}} (a^\dagger)^{N_a} (b^\dagger)^{N_b} |0\rangle \]

\[ E(N_a, N_b) = N_a \varepsilon_a + N_b \varepsilon_b + (\varepsilon_a + \varepsilon_b) / 2, \quad N_a, N_b = 0, 1, 2, \ldots \]

Which yields an infinite spectrum.
To generate a finite spectrum, we impose the condition that the total number of bosons in the system is conserved and given by a constant number $N = N_a + N_b$

We can use this condition to eliminate $N_a$, that is, $N_a = N - N_b$

Assuming $\varepsilon_b > \varepsilon_a$, the ground state becomes a condensate of $N_a$-bosons

$$\left| \text{g.s.} \right> = \left| N_a = N, N_b = 0 \right> = \frac{1}{\sqrt{N!}} \left( a^\dagger \right)^N \left| 0 \right>$$

$$E(N_a = N, N_b = 0) = N\varepsilon_a + (\varepsilon_a + \varepsilon_b)/2$$

The excited states are given by

$$\left| N_b \right> = \sqrt{\frac{(N - N_b)!}{N! N_b!}} \left( b^\dagger a \right)^N \left| \text{g.s.} \right>$$

$$E_{ex}(N_b) = (N - N_b)\varepsilon_a + N_b\varepsilon_b + (\varepsilon_a + \varepsilon_b)/2 - [N\varepsilon_a + (\varepsilon_a + \varepsilon_b)/2]$$

$$= N_b(\varepsilon_b - \varepsilon_a) = N_b\varepsilon, \quad \varepsilon = \varepsilon_b - \varepsilon_a, \quad N_b = 1, 2, \ldots, N$$
Now we apply the same trick to quadrupole bosons, that is, add a scalar s-boson to the d-bosons, and impose the condition that the total number of bosons is conserved, \( N = n_s + n_d \).

Without loss of generality, we can assume \( \varepsilon_s = 0, \varepsilon_d - \varepsilon_s = \varepsilon > 0 \). Then the ground state will be given by a condensate of \( N \) s-bosons whose energy is zero

\[
|g.s.\rangle = |n_s = N, n_d = 0\rangle = \frac{1}{\sqrt{N!}} (s^\dagger)^N |0\rangle
\]

The excited (degenerate) boson states can be written as

\[
|n_d\rangle = \sqrt{\frac{(N - n_d)!}{N!n_d!}} \left[ d_1^\dagger d_2^\dagger \ldots d_{n_d}^\dagger \right]^{(L)} (s)^{n_d} |g.s.\rangle
\]

\[
E_{ex}(n_d) = \varepsilon n_d, \quad n_d = 1, 2, \ldots, N
\]

Thus there are exactly \( N \) excited phonon states (with many degeneracies)
2. Physical basis of the s and d-bosons

All known even-even nuclei have $J=0^+$ for the g.s. and $J=2^+$ for the first excited state. In the last lecture, we have seen that the pairing interaction provides an intuitive understanding of this fact: pairs of nucleons with the same angular momentum $j$ gain energy when they couple to $J=0^+$ and their first excited states have $J=2^+$. Using these facts, we can represent pairs of fermions as boson operators

$$[a_j^+ a_j^+]^{(0)} \rightarrow s^+, \quad [a_j^+ a_j^+]^{(2)} \rightarrow d^+$$

This fermion $\rightarrow$ boson mapping of the shell model states has been investigated in numerous studies and its validity for low-lying levels of collective nuclei is well established.

As in the shell model, we consider only the active shells so that the number of bosons, $N$ is given by half of the active nucleons (or holes).
3. How to obtain a deformed nucleus

To deform the system we need to introduce d-bosons in the ground state. With only d-bosons in the system, the ground state will always be spherical because it contains no d-bosons. Introduction of s-bosons provides a natural mechanism for achieving that. By including two-body interactions in the one-body Hamiltonian that favour the d-bosons, we can create a g.s. that contains a mixture of s and d-bosons:

\[
\begin{align*}
\text{Spherical} & \quad |g.s.\rangle = \frac{1}{\sqrt{N!}}(s^\dagger)^N |0\rangle \\
\text{Deformed} & \quad |g.s.\rangle = \frac{1}{\sqrt{N!}}(s^\dagger + \beta d_0^\dagger)^N |0\rangle
\end{align*}
\]

This is similar to including higher-order terms in the potential energy in the Bohr-Mottelson model to in order to generate a deformed minimum. We will return to this problem and see how deformation is achieved after introducing the IBM Hamiltonian.
Specific features of the interacting boson model:

- Pairs of nucleons are treated as bosons that occupy only two levels $s$ ($L=0$) and $d$ ($L=2$) (provides a bridge to the shell model).
- Total number of bosons is given by half the number of active nucleons, giving rise to a finite spectrum.
- Unified perspective: able to describe all collective nuclei (vibrational, rotational, transitional) under a single framework.
- The single $s$ boson and 5 components of $d$ bosons transform under the $U(6)$ group. $U(6)$ has three subgroups:
  1. $U(5)$: consists of $d$ bosons only, 5D HO of vibrational nuclei
  2. $SU(3)$: the crucial observation, it describes rotational nuclei
  3. $O(6)$ [$\sim SU(4)$]: new symmetry corresponding to $\gamma$-unstable nuclei
- Dynamical symmetries allow analytical solutions for the spectrum!
- Numerical solutions (via diagonalization of $H$) is relatively easy.
- Proved extremely popular among experimentalists.
Group structure of IBM

First consider the U(5) group associated with the 5D harmonic oscillator. It has $5 \times 5 = 25$ generators given by the bilinear operators

$$T^{(k)}_{\mu} = \left[d^{\dagger} \tilde{d}\right]^{(k)}_{\mu}, \quad k = 0,1,2,3,4 \quad (\tilde{d}_{\mu} = (-1)^{\mu} d_{-\mu})$$

These generators close under commutations to form the U(5) algebra.

Physical significance of the generators

$$T^{(0)}_{0} = \frac{1}{\sqrt{5}} \hat{n}_{d}$$

(number operator, E0 transitions)

$$T^{(1)}_{\mu} = \frac{1}{\sqrt{10}} L_{\mu}$$

(angular momentum operator, M1 transitions)

$$T^{(2)}_{\mu} = Q^{(2)}_{\mu}$$

(quadrupole operator, E2 transitions)

$$T^{(3)}_{\mu} = O^{(3)}_{\mu}$$

(octupole operator, M3 transitions)

$$T^{(4)}_{\mu} = H^{(4)}_{\mu}$$

(hexadecapole operator, E4 transitions)
We have already seen that the 3 operators of \( L \) generate the \( O(3) \) group. Similarly, the 10 operators with \( k=1 \) and 3 close under commutations to form the \( SO(5) \) group. Thus one can write the following chain of subgroups starting with \( U(5) \) and ending with \( O(3) \) and \( O(2) \)

\[
U(5) \supset O(5) \supset O(3) \supset O(2)
\]

\[
\begin{align*}
   n_d & \quad v \quad L \quad M
\end{align*}
\]

(Group labels)

These labels (quantum numbers) help to classify the states uniquely

\[
\begin{align*}
n_d &= 0, 1, 2, \ldots \\
v &= n_d, n_d - 2, \ldots, 1 \text{ or } 0 \\
L &= 2v, 2v - 2, 2v - 3 \ldots, v \oplus 2(v - 3), 2(v - 3) - 2, \ldots \\
M &= -L, -L + 1, \ldots, L - 1, L
\end{align*}
\]

The second series in \( L \) correspond to triplets of bosons coupled to \( J=0 \).
The label counting such triplets is denoted by \( n_\Delta \)
When we add the s-bosons, the group becomes U(6) and there are 11 more operators (total number is 6x6=36=25+11)

\[ s^\dagger s, \quad s^\dagger \tilde{d}_\mu, \quad d^\dagger \mu s \]

here the first is the number operator for the s bosons, \( n_s \). The others have \( k=2 \) and hence add to the quadrupole operator

\[ \hat{Q}_\mu = \left[ s^\dagger \tilde{d} + d^\dagger s \right]^{(2)}_\mu + \chi \left[ d^\dagger \tilde{d} \right]^{(2)}_\mu \]

If we impose the condition that N is conserved, we can eliminate \( n_s \) using

\[ n_s = N - n_d \]

This reduces the number of generators to 35, and the group becomes SU(6). Also the range of the \( n_d \) quantum number (hence the spectrum) becomes finite \( n_d = 0,1,2,\ldots,N \)

Group chain  \[ SU(6) \supset U(5) \supset O(5) \supset O(3) \supset O(2) \]

Labels  \[ N \quad n_d \quad v \quad L \quad M \]
After establishing the $SU(6) \supset U(5)$ chain as the spectrum generating algebra (SGA) for vibrational nuclei, Arima & Iachello looked for other subgroup chains of $SU(6)$. The crucial observation was the recognition that the $SU(6) \supset SU(3)$ chain gave rise to a SGA for rotational spectra.

$SU(3)$ group was previously used in describing nuclei in the sd-shell by Elliott (1958), e.g. rotational spectra in Mg isotopes (but mostly forgotten). [SU(3) was made famous by Gell-Mann in the sixties as a dynamical symmetry for hadrons, and in seventies it assumed a more fundamental role as the gauge group for quantum chromodynamics.]

$SU(6)$ has one more nontrivial subgroup chain, $SU(6) \supset O(6)$ which has not been considered before as a SGA. Arima & Iachello showed that the SGA associated with this chain was that of a gamma-unstable rotor.

Because of its novelty, the $SU(6) \supset O(6)$ chain attracted a great deal of attention from both theorists and experimentalists.
General IBM Hamiltonian with one- and two-body terms

\[ H = \varepsilon_d \hat{n}_d - \kappa \hat{Q} \cdot \hat{Q} + \gamma L \cdot L \sum_{k=0,3,4} \alpha_k T^{(k)} \cdot T^{(k)} \]

\[ \hat{Q}_\mu = \left[ s^\dagger \tilde{d} + d^\dagger s \right]_{\mu}^{(2)} + \chi \left[ d^\dagger \tilde{d} \right]_{\mu}^{(2)} \]

\[ T^{(k)}_{\mu} = \left[ d^\dagger \tilde{d} \right]_{\mu}^{(k)} , \quad k = 0,3,4 \]

The Hamiltonian contains 7 parameters, which are too many. The role of the first three terms are fairly well established from fits to nuclear spectra but the last three terms are less well known (apart from splitting the degenerate phonon-multiplets in vibrational nuclei). For arbitrary parameters, the eigenvalue problem is solved numerically by diagonalizing the Hamiltonian in an appropriate basis. (Analytical solutions in the form of a 1/N expansion can also be obtained using angular momentum projection with coherent states.)
Dynamical symmetries

When the IBM Hamiltonian is written in terms of the Casimir operators of one of the subgroup chains of the SU(6) group, the eigenvalue problem can be solved exactly using group theory

\[ SU(6) \supset SU(3) \supset O(3) \quad \text{rotational} \]
\[ SU(6) \supset U(5) \supset O(5) \supset O(3) \quad \text{vibrational} \]
\[ SU(6) \supset O(6) \supset O(5) \supset O(3) \quad \gamma \text{-unstable} \]

A well known example is the O(3) group whose generators are the angular momentum operators

\[ L_\mu = \frac{1}{\sqrt{10}} \left[ d^\dagger \tilde{d} \right]^{(1)}_\mu \]

Casimir operator: \[ L \cdot L = \sum_\mu L_\mu \cdot L_\mu \]

Eigenvalues: \[ L \cdot L |LM\rangle = L(L+1) |LM\rangle \]
1. U(5) limit:

Generators of U(5) are given by the operators

\[ T^{(k)}_{\mu} = [d^\dagger \; \tilde{d}]^{(k)}_{\mu}, \quad k = 0, 1, 2, 3, 4 \]

Subgroups:

Generators of O(5) are given by those with \( k = 1, 3 \)

Generators of O(3) are given by those with \( k = 1 \) (Angular mom. operator)

The Hamiltonian and energy formula are given by

\[
H = \varepsilon_d \hat{n}_d + \alpha \sum_{k=0}^{4} T^{(k)} \cdot T^{(k)} + \beta \sum_{k=1,3} T^{(k)} \cdot T^{(k)} + \gamma L \cdot L
\]

\[
E(N, n_d, \nu, L) = \varepsilon_d n_d + \alpha n_d (n_d + 1) + \beta \nu (\nu + 3) + \gamma L (L + 1)
\]

The two body terms are small and are used to split the degeneracy of multi-phonon states. (One can also obtain analytical expressions for E2 and M1 transitions, etc. but we won’t cover them here.)
Fig. 2.5. An example of a spectrum with U(5) symmetry: $^{110}_{48}\text{Cd}_{62}$, $N = 7$. The theoretical spectrum is calculated using (2.79) and (2.82) with $\epsilon' = 722$ KeV, $c_0 = 29$ KeV, $c_2 = -42$ KeV, $c_4 = 98$ KeV.
2. SU(3) limit:

Generators of SU(3) are given by the operators (prolate shape)

\[ Q_\mu = \left[ s^{\dagger} \tilde{d} + d^{\dagger} s \right]_{\mu}^{(2)} - \frac{\sqrt{7}}{2} \left[ d^{\dagger} \tilde{d} \right]_{\mu}^{(2)}, \quad L_\mu = \frac{1}{\sqrt{10}} \left[ d^{\dagger} \tilde{d} \right]_{\mu}^{(1)} \]

The Hamiltonian and energy formula are given by

\[ H = -\kappa \left( Q \cdot Q + \frac{3}{8} L \cdot L \right) + \gamma L \cdot L \]

\[ E(N, \lambda, \mu, L) = -\kappa \left( \lambda^2 + \mu^2 + \lambda \mu + 3\lambda + 3\mu \right) + \gamma L(L+1) \]

Labels: \((\lambda, \mu) = (2N,0), (2N - 4,2), \ldots, (0, N)\) or \((2, N - 1)\)

\[ \oplus (2N - 6,0), (2N - 10,2), \ldots, \]

\[ K_{\text{min}} = \min \{\lambda, \mu\}, \quad K_{\text{max}} = \max \{\lambda, \mu\}, \quad K = 0, 2, \ldots, K_{\text{min}} \]

\[ L = K, K + 1, \ldots, K_{\text{max}}, \quad \text{if} \quad K \neq 0 \]

\[ L = 0, 2, 4, \ldots, K_{\text{max}}, \quad \text{if} \quad K = 0 \]
Fig. 2.6. An example of a spectrum with SU(3) symmetry: $^{156}_{64}\text{Gd}_{92}, N = 12$. The theoretical spectrum is calculated using (2.84) with $\gamma = 3.8$ KeV, $\delta = -20.1$ KeV.
3. O(6) limit:

Generators of O(6) are given by the operators

\[ Q_\mu = \left[ s^\dagger \tilde{d} + d^\dagger s \right]^{(2)}_\mu, \quad T^{(k)}_\mu = \left[ d^\dagger \tilde{d} \right]^{(k)}_\mu, \quad k = 1,3 \]

Generators of the subgroups O(5) and O(3) are the same as in U(5)

The Hamiltonian and energy formula are given by

\[ H = \alpha \left( Q \cdot Q + \sum_{k=1,3} T^{(k)} \cdot T^{(k)} \right) + \beta \sum_{k=1,3} T^{(k)} \cdot T^{(k)} + \gamma L \cdot L \]

\[ E(N, \sigma, \tau, L) = \alpha \sigma (\sigma + 1) + \beta \tau (\tau + 3) + \gamma L (L + 1) \]

Because of the common subgroup structure, the spectrum is similar to that of U(5). O(6) is in between the vibrational and rotational limits.

Labels:

\[ \sigma = N, N - 2, \ldots, 1 \text{ or } 0 \]

\[ \tau = \sigma, \sigma - 1, \ldots, 0 \]

\[ L = 2\tau, 2\tau - 2, 2\tau - 3, \ldots, \tau \oplus 2(\tau - 3), 2(\tau - 3) - 2, \ldots \]
Fig. 2.7. An example of a spectrum with O(6) symmetry: $^{196}_{78}$Pt, $N=6$. The theoretical spectrum is calculated using (2.92) with $A = 171$ KeV, $B/6 = 50$ KeV, $C = 10$ KeV.
Fig. 2.4. Regions of the periodic table where examples of dynamic symmetries have been found: (I) U(5); (II) SU(3); (III) O(6).
SCHEMATIC EVOLUTION OF STRUCTURE
NEAR CLOSED-SHELL → MID SHELL

$E_4/E_2 < 2$

$E_4/E_2 \sim 2-2.2$

$E_4/E_2 \sim 2.7$

$E_4/E_2 \sim 3.33$

$2^+$

$2^+$

$0^+$

$2^+$

$4^+$

$0^+$

$2^+$

$4^+$

$2^+$

$6^+$

$2^+$

$2^+$

$4^+$

$2^+$

$0^+$

"Shell Model" nucleus

Vibrator

Transitional

Rotor
Simplest IBM Hamiltonian

The most general Hamiltonian with one- and two-body terms contains many terms. However a simple H containing only two terms gives an adequate representation for the spectra of most collective nuclei

\[ H = \varepsilon_d \hat{n}_d - \kappa \hat{Q} \cdot \hat{\Omega} \]

\[ \hat{n}_d = \sum_{\mu} d_{\mu}^\dagger d_{\mu} \quad \text{(number operator)} \]

\[ \hat{Q}_\mu = \left[ s_{\mu}^\dagger \tilde{d} + d_{\mu}^\dagger s \right]_{\mu}^{(2)} + \chi \left[ d_{\mu}^\dagger \tilde{d} \right]_{\mu}^{(2)} \quad \text{(quadrupole operator)} \]

- When \( \kappa \approx 0 \), this Hamiltonian describes vibrational nuclei.
- Increasing the value of \( \kappa \) leads to a permanent deformation of the nucleus corresponding to rotational nuclei (prolate if \( \chi < 0 \)).
- If \( \chi = 0 \) in the Hamiltonian, \( \gamma \) is undetermined (\( \gamma \)– unstable nuclei).
Geometrical interpretation of IBM

Unlike fermions which avoid each other, bosons like to condense. Thus we can write the ground state of the system as a condensate of bosons (coherent state)

$$|N, \alpha_\mu\rangle = \frac{1}{\sqrt{N!}} (b^\dagger)^N |0\rangle, \quad b^\dagger = s^\dagger + \sum_\mu \alpha_\mu d^\dagger_\mu$$

Transforming to the body fixed frame, we can write for $\alpha_\mu$

$$\alpha'_0 = \beta \cos \gamma, \quad \alpha'_2 = \alpha'_{-2} = \frac{1}{\sqrt{2}} \beta \sin \gamma, \quad \alpha'_1 = \alpha'_{-1} = 0$$

The condensate boson becomes

$$b^\dagger = s^\dagger + \beta \left[ \cos \gamma d^\dagger_0 + \frac{\sin \gamma}{\sqrt{2}} (d^\dagger_2 + d^\dagger_{-2}) \right]$$

In particular when $\gamma = 0$, $b^\dagger = s^\dagger + \beta d^\dagger_0$ (axial symmetry)

Thus $\beta$ controls the transition from spherical to deformed nuclei.
Energy surface of the simple H using the boson condensate with $\gamma = 0$

$$E(N, \beta) = \frac{\langle N, \beta | H | N, \beta \rangle}{\langle N, \beta | N, \beta \rangle} = \frac{\langle 0 | b^N H (b^\dagger)^N | 0 \rangle}{\langle 0 | b^N (b^\dagger)^N | 0 \rangle}$$

$$= \frac{N \varepsilon_d}{(1 + \beta^2)} \frac{\partial}{\partial b^\dagger} \frac{\partial}{\partial b} \hat{n}_d - \frac{N(N-1)k}{(1 + \beta^2)^2} \left( \frac{\partial}{\partial b^\dagger} \right)^2 \left( \frac{\partial}{\partial b} \right)^2 Q \cdot Q$$

$$= N \varepsilon_d \frac{\beta^2}{(1 + \beta^2)} - N^2 k \frac{\beta^2}{(1 + \beta^2)^2} \left( \bar{\chi}^2 \beta^2 + 4 \bar{\chi} \beta + 4 \right) \quad \bar{\chi} = -\sqrt{\frac{2}{7}} \chi$$

Where the only the leading order terms in N are kept (large N expansion)

Differentiating w.r.t. $\beta$, one can find the minima in the energy surface.

1. $\beta = 0$, is always a minimum point, where $E_1 = 0$
2. A second minimum exists for $k > 0$ with energy $E_2$
3. As $k$ is increased the absolute minimum shifts from $E_1$ to $E_2$
**Detour: boson calculus***

First consider the 1D HO. The boson operators satisfy

\[ [b, b^\dagger] = 1 \]

\[
\left[ b, (b^\dagger)^N \right] = \left[ b, b^\dagger (b^\dagger)^{N-1} \right] = [b, b^\dagger] (b^\dagger)^{N-1} + b^\dagger \left[ b, (b^\dagger)^{N-1} \right]
\]

\[
= (b^\dagger)^{N-1} + b^\dagger \left[ b, (b^\dagger)^{N-1} \right]
\]

(repeat N times)

\[
= N (b^\dagger)^{N-1} = \frac{\partial}{\partial b^\dagger} (b^\dagger)^N
\]

In general:

\[
[b, f(b, b^\dagger)] = \frac{\partial}{\partial b^\dagger} f(b, b^\dagger), \quad [b^\dagger, f(b, b^\dagger)] = \frac{\partial}{\partial b} f(b, b^\dagger)
\]

Since s and d bosons are orthogonal, this will also hold for them.

For the condensate, bosons are composite, so we have

\[ [b, b^\dagger] = 1 + \beta^2 \]

\[
[b, f(b, b^\dagger)] = \left\{ \frac{\partial}{\partial s^\dagger} + \beta \frac{\partial}{\partial d_0^\dagger} \right\} f(b, b^\dagger)
\]
Plot of the energy surface for various values of $N\kappa/\varepsilon_d$.
Fig. 2.19. Typical features of the transitional class A. Excitation energies.
General solution
For a general IBM Hamiltonian, the eigenvalue problem can be solved by numerical diagonalization.
One has to create a complete set of basis states (usually the U(5) basis) and evaluate the matrix elements of the Hamiltonian in this basis

\[ H_{\alpha \beta} = \langle \alpha | H | \beta \rangle \]

Next one solves the eigenvalue problem

\[ |H_{\alpha \beta} - \epsilon I| = 0 \]

and determines the eigenvalues and the corresponding eigenvectors.
There are several codes which can perform this task.
These codes can also calculate the E2 and M1 transition strengths between the states and their dipole and quadrupole moments
Fig. 6.41. Observed and calculated energies in $^{168}$Er. The $J = 4^+, 5^+, 6^+$ levels of the $K = 4$ band predicted near 1.6 MeV are not observed below 2 MeV (Warner, 1980).
Going beyond the IBM with sd bosons

• Introduce separate bosons for protons and neutrons (IBM-2) (important because the p-n interactions cause deformation)
• f-boson to describe octupole excitations (spdf-IBM)
• g-boson to describe hexadecapole excitations (sdg-IBM)
• Couple a single particle to describe odd-even nuclei (IBFM) (predicts boson-fermion symmetries – supersymmetry)
• Couple two particles for quasi-particle excitations (~2 MeV)

Finally, IBM inspired other boson models in molecular and particle physics. The former is known as the vibron model. The dipole degree of freedom is represented with p bosons, and addition of s bosons to generate a finite spectrum leads to the SU(4) group. It generates a vibration-rotation spectrum similar to that of Morse potential.
IBM-2 (proton-neutron IBM)

IBM-2, is motivated by the shell model that forms the microscopic basis of the bosons. The proton and neutron pairs are mapped into separate s and d bosons. There are $N = N_\pi + N_\nu$ bosons represented by the operators

$$\left( s^\dagger_\pi, d^\dagger_\pi \right), \left( s^\dagger_\nu, d^\dagger_\nu \right)$$

Each set has an SU(6) group, and the total system is described by

$$U_\pi (6) \otimes U_\nu (6)$$

with the generators given by

$$T^{(k)}_\mu = T^{(k)}_{\pi \mu} + T^{(k)}_{\nu \mu}$$

Simplest Hamiltonian

$$H = \varepsilon_{\pi d} \hat{n}_{\pi d} + \varepsilon_{\nu d} \hat{n}_{\nu d} - \kappa \hat{Q}_\pi \cdot \hat{Q}_\nu$$

Which recognizes the fact that deformation is caused by p-n interactions. IBM-2 gives better results in systematic studies of series of nuclei, e.g. series of isotopes.
Fig. 4.1. (a) Schematic representation of the shell-model problem for $^{118}_{54}\text{Xe}_{64}$; (b) The boson problem which replaces the shell-model problem for $^{118}_{54}\text{Xe}_{64}$. 