THE UNIVERSITY OF SYDNEY

FACULTY OF SCIENCE

PHYS3940/3941
PAPER 1 (ELECTROMAGNETISM ADVANCED)

SEMESTER 1, 2009

TIME ALLOWED: 1.5 HOURS

ALL QUESTIONS HAVE THE VALUE SHOWN

INSTRUCTIONS:
Students should attempt all questions.
Total marks: 80
No written material of any kind may be taken into the examination room.
A formula sheet is included.
Non-programmable calculators are permitted.
Electromagnetism (Advanced) Formula Sheet

Maxwell's equations (in general):
\[
\begin{align*}
\oint_S \mathbf{E} \cdot d\mathbf{a} &= \frac{1}{\varepsilon_0} Q_{\text{enc}} \\
\oint_S \mathbf{B} \cdot d\mathbf{a} &= 0 \\
\int_P \mathbf{E} \cdot d\mathbf{l} &= -\frac{d\Phi_B}{dt} \\
\int_P \mathbf{B} \cdot d\mathbf{l} &= \mu_0 I_{\text{enc}} + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}
\end{align*}
\]
\[
\begin{align*}
\nabla \cdot \mathbf{E} &= \frac{1}{\varepsilon_0} \rho \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \times \mathbf{B} &= \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}
\end{align*}
\]

Maxwell's equations (in matter):
\[
\begin{align*}
\nabla \cdot \mathbf{D} &= \rho_f \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{H} &= \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}
\end{align*}
\]

Auxiliary fields:
\[
\begin{align*}
\mathbf{D} &= \varepsilon_0 \mathbf{E} + \mathbf{P} \\
\mathbf{H} &= \frac{1}{\mu_0} \mathbf{B} - \mathbf{M}
\end{align*}
\]

Linear media:
\[
\begin{align*}
\mathbf{P} &= \varepsilon_0 \chi_e \mathbf{E}, \quad \mathbf{D} = \varepsilon \mathbf{E} \\
\mathbf{M} &= \chi_m \mathbf{H}, \quad \mathbf{H} = \frac{1}{\mu} \mathbf{B}
\end{align*}
\]

Potentials:
\[
\begin{align*}
\mathbf{E} &= -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}
\end{align*}
\]

Lorentz force law:
\[
\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})
\]

Energy, momentum & power:
\[
\begin{align*}
U &= \frac{1}{2} \int \left( \varepsilon_0 \mathbf{E}^2 + \frac{1}{\mu_0} \mathbf{B}^2 \right) \, dt \\
P &= \varepsilon_0 \int (\mathbf{E} \times \mathbf{B}) \, dt \\
S &= \frac{1}{\mu_0} (\mathbf{E} \times \mathbf{B})
\end{align*}
\]
1. (a) Consider two concentric spherical shells, of radii \( a \) and \( b \). Suppose the inner shell carries a charge \( q \) and the outer shell carries \( -q \) (both of them uniformly distributed over the surface). Calculate the energy of the electric field in this configuration.

(b) Show that the electric potential in some empty region of space in static situations is uniquely determined if the potential is specified on the boundary surface. You may assume without proof that solutions to Laplace’s equation have no local maxima or minima.

Briefly comment on the importance of this result.

(c) The continuity equation is

\[ \nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}. \]  

(i) Briefly describe each of the two terms in this equation, and also explain the overall meaning of the equation.

(ii) Show that Maxwell’s equations can be used to derive the continuity equation.

(20 marks)

2. (a) Stokes’ theorem relates the path integral of a vector around a closed loop to the flux of its curl through the loop. In class we used this theorem to derive the differential form of Faraday’s law from the integral form. Here, you should do the reverse: derive the integral form of Faraday’s law from the differential form, starting from the assumption that \( \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \) at every point in space. Include a brief explanation and be sure to define all the terms in the equations.

(b) In class we showed that the rate at which work is done on charges in a volume \( V \) by electromagnetic forces is

\[ \frac{dW}{dt} = \int_V (\mathbf{E} \cdot \mathbf{J}) \, dV. \]

(i) Starting with equation 2, derive Poynting’s theorem:

\[ \frac{dW}{dt} = \frac{dU}{dt} - \oint_S \mathbf{S} \cdot d\mathbf{a}. \]  

(20 marks)
3. (a) (i) Derive an equation relating the path integral of the magnetic vector potential $A$ around any closed loop to the flux of the magnetic field through the loop. Include a brief explanation and be sure to define all the terms in the equations.

(ii) Consider an ideal solenoid of radius $R$ with current $I$ and $n$ turns per unit length. It can be shown that the magnitude of the magnetic field is zero outside and has magnitude $\mu_0 n I$ inside. Calculate $A$ both inside and outside the solenoid (magnitude and direction).

(b) A charged parallel-plate capacitor of area $A$ with distance $d$ between the plates and with uniform electric field $E = E\hat{z}$ is placed in a uniform magnetic field $B = B\hat{z}$, as shown in the figure below.

(i) Find the total electromagnetic momentum in the space between the plates, in terms of $E$, $B$, $A$ and $d$ (magnitude and direction).

(ii) When a resistive wire is connected between the plates, along the $z$-axis, so that the capacitor slowly discharges, the capacitor is observed to move. In what direction does it move? Explain what is happening.

4. (a) In a linear medium the electric polarization is proportional to the total electric field: $P = \varepsilon_0 \chi E$. For such a medium, show (using its definition) that the $D$ vector is also proportional to the total electric field and find the constant of proportionality, $\varepsilon$.

(b) (i) Starting with Maxwell’s equations in vacuum, derive a wave equation for both electric and magnetic fields. What is the speed of the waves?

(ii) Consider plane waves of the form $(E_x, E_y, E_z) = (E_0 \sin(kz - \omega t), 0, 0)$ and $(B_x, B_y, B_z) = (0, B_0 \sin(kz - \omega t), 0)$. Calculate the ratio between the amplitudes of the electric and magnetic fields.

Now consider a material that is non-conducting and responds linearly to both electric and magnetic fields.

(iii) What are Maxwell’s equations inside this material? Express them in terms of $E$ and $B$ only and briefly explain how you arrived at your answer.

(iv) With what speed will waves propagate through the material? Justify your answer.

(c) Suppose the electric and magnetic potentials, as a function of position and time, are:

\[
V(r, t) = 0 \\
A(r, t) = -\frac{1}{4\pi\varepsilon_0} \frac{q t}{r^2} \hat{r}.
\]

(i) Calculate the electric and magnetic fields that would produce these potentials.

(ii) What distribution of charges and currents would produce these potentials?

(25 marks)

THERE ARE NO MORE QUESTIONS.