Cauchy-Schwarz Inequality

\[ |x \cdot y| \leq \|x\| \|y\| \]

Let \( x, y \in \mathbb{R}^n \), then the area of the parallelogram spanned by \( x, y \) is

\[ A = \sqrt{\det(J^T J)} \], where \( J = \begin{bmatrix} x & y \end{bmatrix} \)

In particular, if \( x, y \in \mathbb{R}^2 \), then \( A = |\det(J)| \); if \( x, y \in \mathbb{R}^3 \), then \( A = \|x \times y\| \)

Let \( x, y, z \in \mathbb{R}^3 \), then the volume of the parallelepiped spanned by \( x, y, z \) is

\[ V = |z \cdot (x \times y)| = |\det(J)| \], where \( J = \begin{bmatrix} x & y & z \end{bmatrix} \)

Partial differential

Let \( f(x) = c \) be constant and \( \nabla f \) be continuous, then \( \nabla f \) is orthogonal to the level set.

Let \( f : D \to \mathbb{R} \), \( D \in \mathbb{R}^n \), then \( f \) is differentiable at point \( a \in D \) iff

\[ \exists \frac{\partial f}{\partial x_i} \wedge \lim_{x \to a} \frac{f(x) - T_i(x)}{\|x - a\|} = 0 \]

Mixed derivative test, let \( f : \mathbb{R}^2 \to \mathbb{R} \), then,

\[ \forall (x, y) \in B((a, b), r), \exists f_{xy}, f_{yx} \Rightarrow f_{xy}(a, b) = f_{yx}(a, b) \]

Alternatively, let \( f : D \to \mathbb{R} \), \( D \in \mathbb{R}^n \),

\[ \forall i, j \frac{\partial^2 f}{\partial x_i \partial x_j} \) is continuous at a point \( x = a \) \( \Rightarrow \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i} \) at \( x = a \)

Taylor series

\[ T_1(x - a) = f(a) + \nabla f(a) \cdot (x - a) \]

\[ T_2(x - a) = T_1(x - a) + \frac{1}{2} (x - a)^T H_f(a) (x - a) \]

If \( f : \mathbb{R}^2 \to \mathbb{R} \), then \( T_2 = f + f_{xx}x + f_{xy}y + \frac{1}{2} f_{xx}x^2 + f_{xy}xy + \frac{1}{2} f_{yy}y^2 \)

Taylor Theorem

\[ \forall i, j \frac{\partial f}{\partial x_i} \frac{\partial^2 f}{\partial x_j} \) are continuous at \( x = a \) \( \Rightarrow \begin{cases} f = T_n + R_n \\ \lim_{x \to a} \frac{R_n(x - a)}{\|x - a\|^n} = 0 \end{cases} \]
Product rule
\[ \frac{\partial}{\partial x_i} f \ast g = \frac{\partial f}{\partial x_i} \ast g + f \ast \frac{\partial g}{\partial x_i} \]
where \( \ast \) denotes either scalar multiplication, dot product, or cross product.

Chain rule
\[ \frac{\partial f}{\partial x_i} = \sum \frac{\partial f}{\partial u_j} \frac{\partial u_j}{\partial x_i} \]

Directional derivative
\[ D_v f = \nabla f \cdot v \quad \text{and} \quad D_v^2 f(x) = v^T H_f v, \text{where} \ H_f = \nabla(\nabla f)^T \]

A point \( a \in D \) is a critical point \( \iff \nabla f(a) = 0 \)

A critical point is a local minimum if \( H_f \) is positive definite, a local maximum if \( H_f \) is negative definite, and a saddle if \( H_f \) is indefinite.

For a symmetric matrix \( A_{nxn} : \)
Positive definite \( \iff \exists c > 0, \forall x \in \mathbb{R}^n, x^T A x \geq cx^T x \iff \) all eigenvalues are positive

If \( n = 2 \), then \( A \) is positive definite \( \iff \det A > 0 \land (A)_{11} > 0 \)

Extreme value theorem
A continuous function over a compact domain attains global maximum and global minimum.

Let \( f : \mathbb{R}^n \to \mathbb{R}^n \), then the Jacobian matrix is \( J_f = \nabla f \), i.e. grad the function component-wise get the rows of the matrix. And \( J_{fg} = J_f J_g \)
Line integral

Let \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) be a vector field, let \( C \) be a curve with parameterisation \( \gamma : \mathbb{R} \rightarrow \mathbb{R}^n \), then

<table>
<thead>
<tr>
<th>Flux along a curve</th>
<th>Flux across a curve</th>
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<tbody>
<tr>
<td>( \int_C f \cdot dx = \int_c f \cdot d(\gamma(t)) = \int_c f \cdot \gamma'(t)dt )</td>
<td>( \int_C f \cdot \hat{n}ds = \int_c f \cdot \hat{n}|\gamma'(t)|dt )</td>
</tr>
<tr>
<td>or alternatively</td>
<td>Divergence theorem</td>
</tr>
<tr>
<td>( \int_C f \cdot \tau ds = \int_c f \cdot \tau|\gamma'(t)|dt )</td>
<td>( \int_C f \cdot \hat{n}ds = \int_A \text{div}f dA )</td>
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Green’s theorem

\[ \int_C f \cdot dx = \int_A \text{curl}\ f dA \]

where \( ds^2 = \sum \tau_i dx_i^2 \), \( \tau \) is the unit tangent vector, and \( \hat{n} \) is the unit normal vector

Green’s theorem

Let \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) be a smooth vector field over a piecewise smooth domain \( D \in \mathbb{R}^2 \), then

\[ \int_{\partial D} f \cdot dx = \iint_D \text{curl}\ f dA \]

A domain \( D \) is piecewise smooth if \( \partial D \) is a union of finite many piecewise smooth curves.

Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) be a scalar field, let \( C \) be a curve with parameterisation \( \gamma : \mathbb{R} \rightarrow \mathbb{R}^n \), then

\[ \int_C fds = \int_c f\|\gamma'(t)\|dt \]

Let \( f : \mathbb{C} \rightarrow \mathbb{C}^n \), let \( C \) be a curve with parameterisation \( \gamma : \mathbb{R} \rightarrow \mathbb{C} \), then

\[ \int_C fdx = \int_c f\gamma'(t)dt \]

Double integral

Fubini’s Theorem

\[ \int_D f dA = \int_{x=a}^{x=b} \left( \int_{y=\Phi_1(x)}^{y=\Phi_2(x)} f dy \right)dx = \int_{y=a}^{y=b} \left( \int_{x=\Phi_1(y)}^{x=\Phi_2(y)} fdx \right)dy \]

Transformation formula

Let \( D \in \mathbb{R}^2 \) be closed, \( g : D \rightarrow \mathbb{R}^2 \) has continuous first order derivative, and \( f : g(D) \rightarrow \mathbb{R} \) be continuous, then

\[ \iint_{g(D)} f(x,y)dxdy = \iint_D f(x',y')|\det(J)|dx'dy' \]