**G** is bipartite \(\iff\) **G** contains no odd cycles

**Vertex colouring**

Let \(G = (V, E)\) be a graph such that \(|V| = n\) and \(|E| = k\). Let \(G\) be a connected component of \(G\), then

- \(\chi(G) = 1 \iff k = 0\)
- \(k \neq 0 \Rightarrow \chi(G) \geq 2\)
- \(G = K_n \iff \chi(G) = n\)
- \(G \neq K_n \Rightarrow \chi(G) \leq n - 1\)
- \(\chi(G) = \max\{\chi(G_i)\}\)

**Chromatic polynomial**

\(P_G(t) = \) the number of ways of vertex colouring with \(t\) colours.

Let \(n \geq 1\), then

- \(P_G(t) = \prod_{i=0}^{n-1} (t - s)\)
- \(P_{K_n}(t) = t(t - 1)^{n-1}\)

- if the vertices adjacent to \(v \in V\) are adjacent to each other, then

\(P_G(t) = P_{G_{\sim v}}(t - \deg(v))\)

\(\{v, \{v, w\} \in E\) \Rightarrow \(P_G(t) = P_{G_{\sim v\sim w}}(t) - P_{G_{\sim v\sim w}}(t)\)

**Edge colouring**

- \(\chi'(G) \geq \Delta G\)

- \(\chi'(G) \leq \Delta G + 1\) (Vizing theorem)

- \(H \subseteq G \Rightarrow \chi'(H) \leq \chi'(G)\)

- \(\chi'(G) = \max\{\chi'(G_i)\}\)

- let \(n \geq 2\) then,

\(\chi'(K_n) = \begin{cases} \frac{n}{2} & n \text{ is odd} \\ n-1 & n \text{ is even} \end{cases}\)

- \(G\) is bipartite \(\Rightarrow \chi'(G) = \Delta G\)