Topological order on the lattice:

Dualities and perspectives

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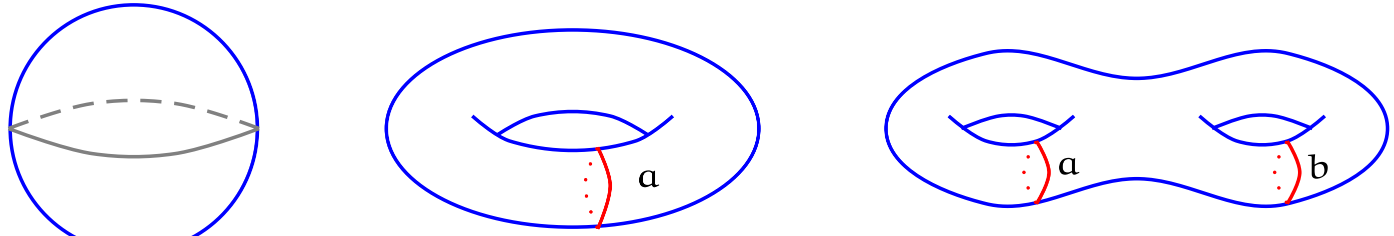
Conclusions.

Topological order

- Topological order: beyond local order parameters.
  Matter in extreme circumstances: quantum Hall effect
  (low T, intense magnetic fields, $2D$ electron gas).

Topological order

- Topological order: beyond local order parameters.
  Matter in extreme circumstances: quantum Hall effect
  (low T, intense magnetic fields, $2D$ electron gas).
- System 'forgets' microscopic structure:
  No local degrees of freedom in the ground level!
  Degrees of freedom dependent on topology.

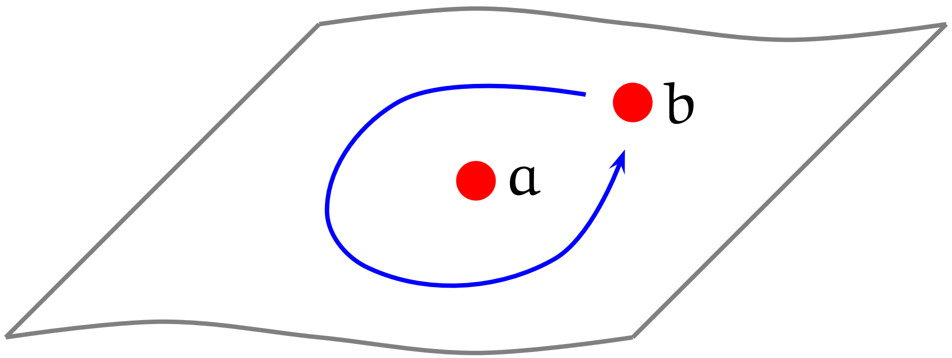


$\mathcal{H}_{\text{sphere}} = \mathbb{C}$ $\quad$ $\mathcal{H}_{\text{torus}} = \mathbb{C}\{|a\rangle\}$ $\quad$ $\mathcal{H}_{\text{breze}} = \mathbb{C}\{|a\rangle \otimes |b\rangle \otimes \ldots\}$

Technically: effective topological quantum field theory.

Topological order

- Particles in topological systems in $2\mathrm{D}$: anyons.



$|\psi\rangle \mapsto U(a, b)|\psi\rangle$

In $\mathrm{D} \geqslant 3$, always $|\psi\rangle \mapsto |\psi\rangle$ (double exchange).

Topological order

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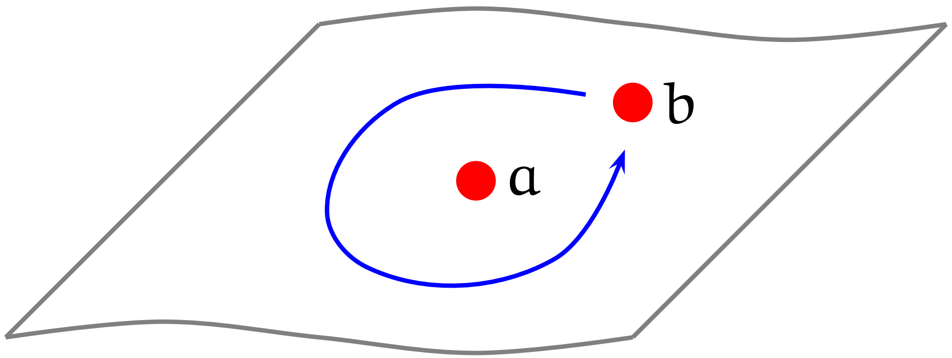

$$|\psi\rangle \mapsto U(a, b)|\psi\rangle$$

In $\mathrm{D} \geqslant 3$, always $|\psi\rangle \mapsto |\psi\rangle$ (double exchange).

- In $\mathrm{D} = 2$, $U(a, b)$ representation of the braid group.
  A way to do quantum computations! (Kitaev '97)
  Computational power given by the TQFT (anyon model).
  Sometimes enough for universal quantum computation.

Topological order

Lattice models with topological order:

- Kitaev '97: Quantum double models based on groups.

  Built in analogy to discrete gauge theories.

  Very well understood algebraically.

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  Describe all doubled topological phases on the lattice.
  Intuition: Renormalisation group fixed points.
  Technically: unitary braided modular tensor categories.
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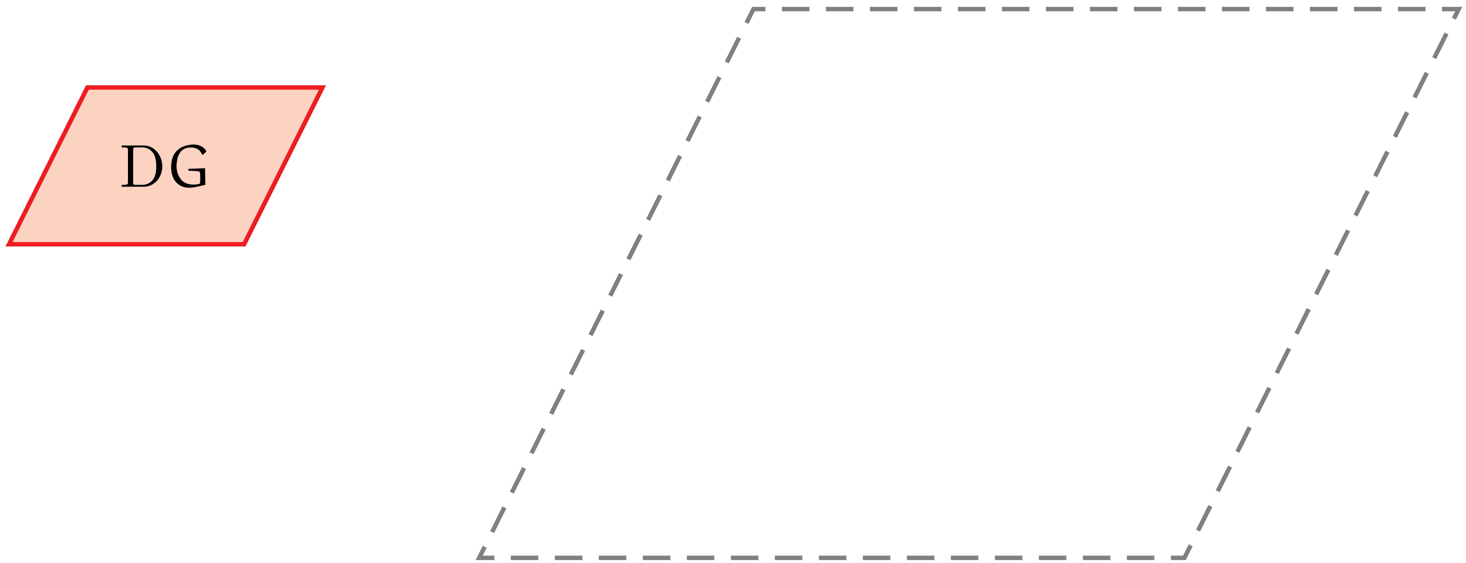
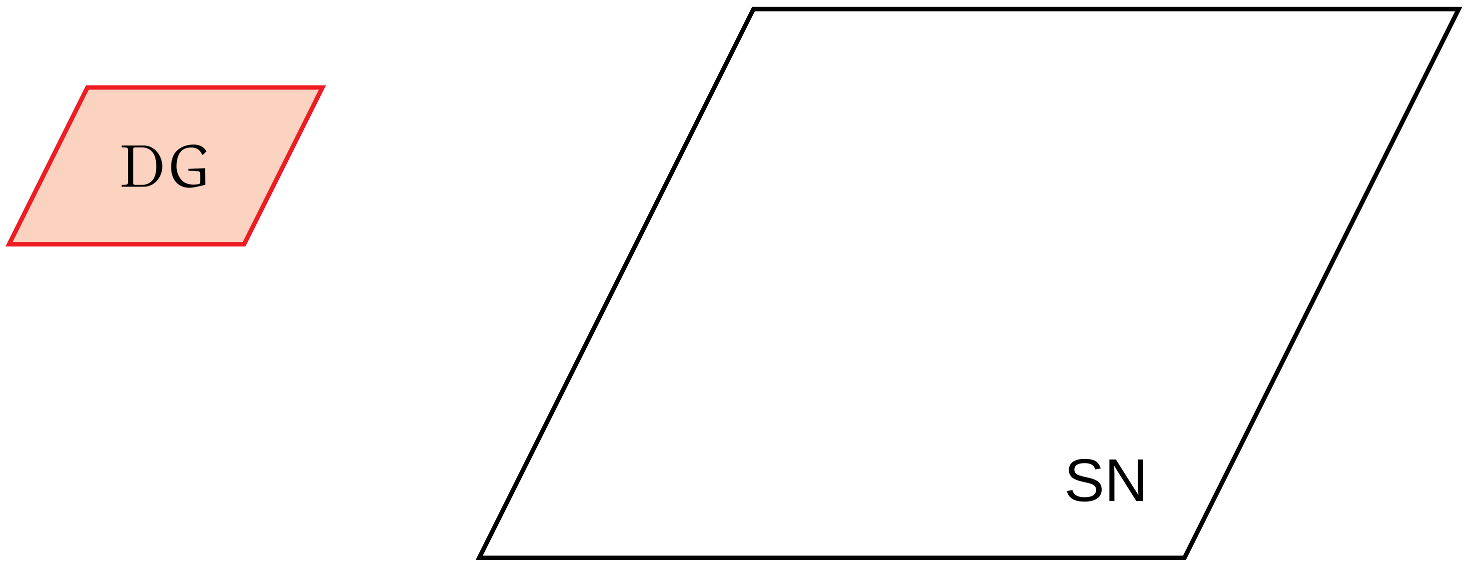
  Technically: unitary braided modular tensor categories.
  Anyons not so well understood locally...
- And many more...

Topological order



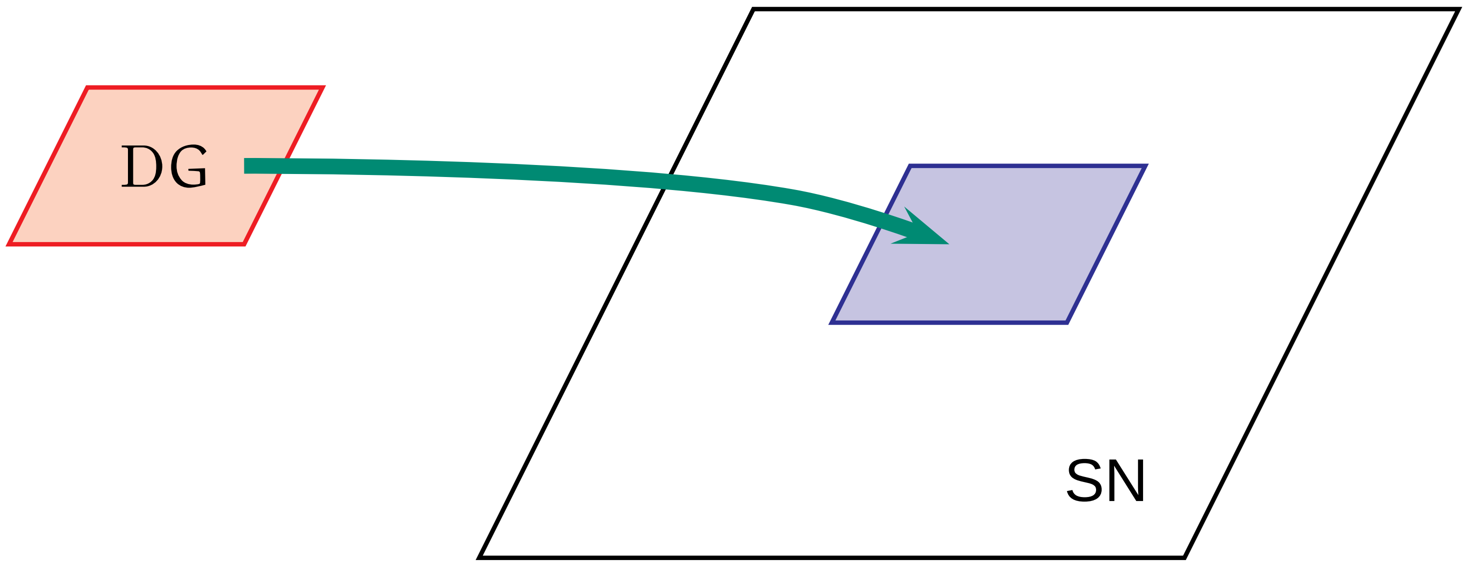
- Quantum double (DG) models:
  - — Good local understanding of excitations.
  - — Everything is representation theory.
  - — Plaquettes and vertices similar: hint of duality.

Topological order



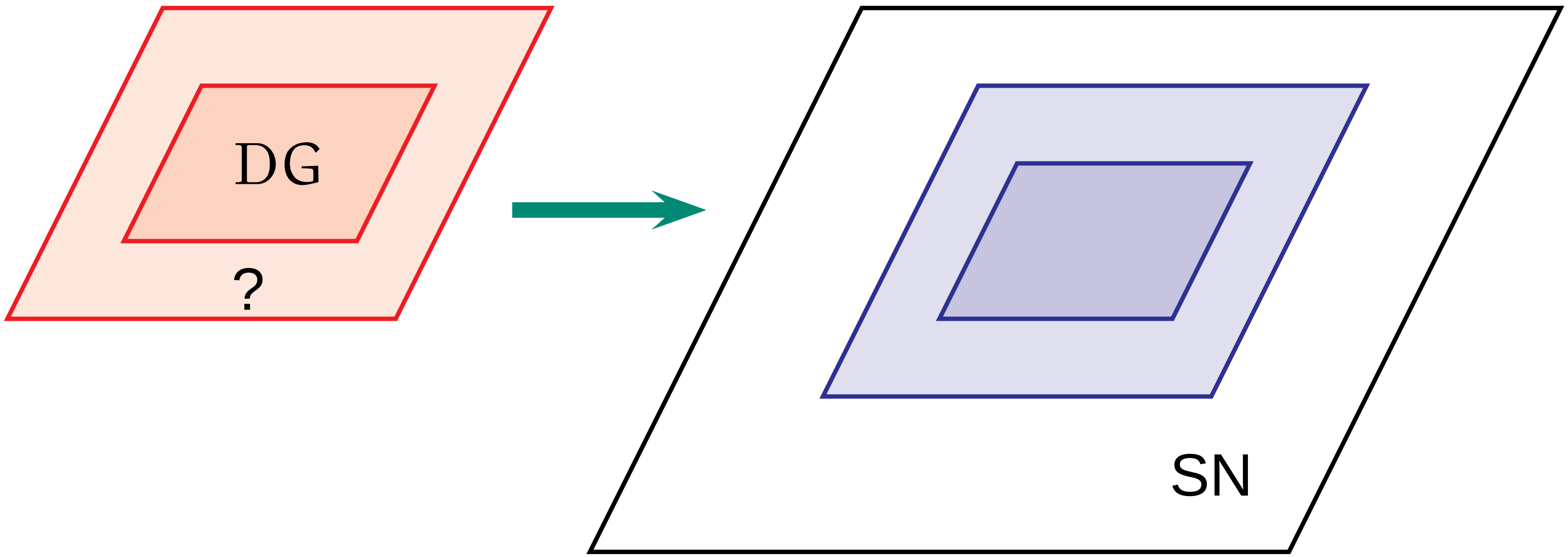
- String-net (SN) models:
  - General class (for P, T-symmetric phases).
  - Excitations not completely understood locally.
  - Asymmetry between plaquettes and vertices.
  - Language: category theory.

Topological order



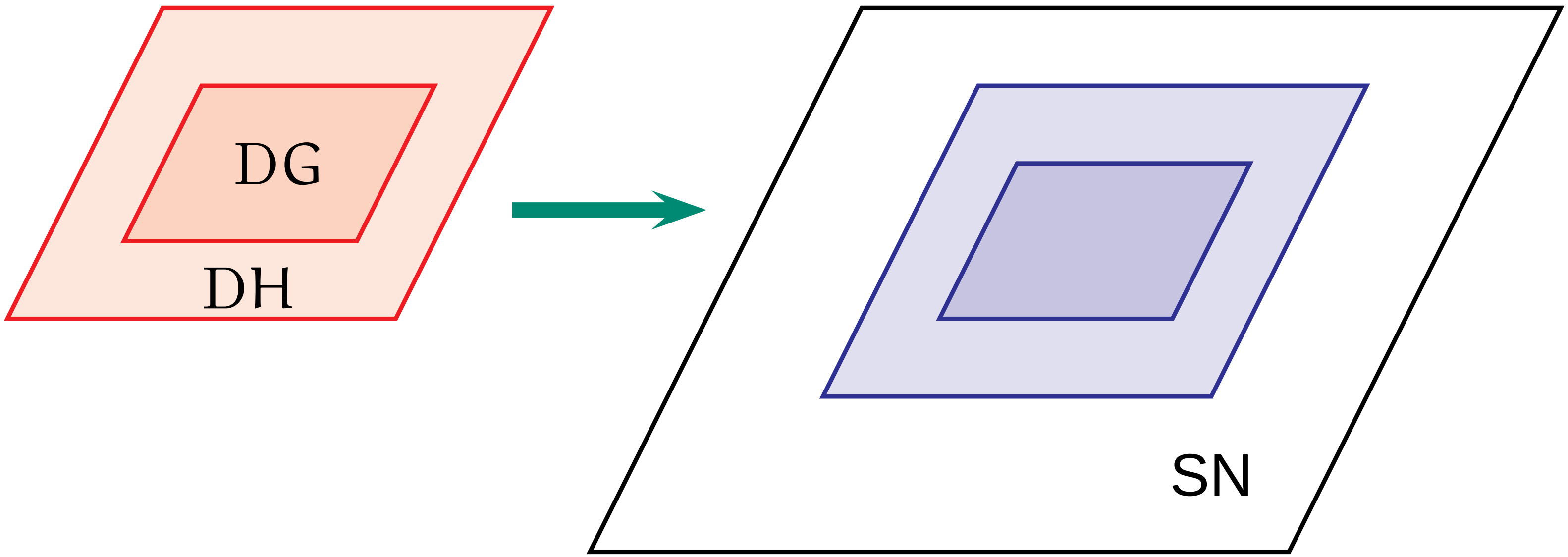
- Know how to map $\mathrm{DG} \mapsto \mathrm{SN}$ (Buerschaper + M. A., PRB '09).
  - QD $\rightarrow$ extended SN (same ground level, extra d.o.f.'s).
  - Complete local picture of excitations (representations).
  - Restore some symmetry between plaquettes and vertices.

Topological order



- Can we understand more string-nets à la Kitaev?
  - Using representation theory rather than categories (!).
  - Cannot use groups, need more general structures.
  - Conjecture: all SN's from weak Hopf C^*-algebras.

Topological order



- As an appetizer, we can use Hopf algebras (Kitaev!).
  - Representation theory very much like groups.
  - Get nontrivial examples.
  - Also many interesting results for free...

$\mathrm{D(G)}$ quantum double models

- Quantum double, or $\mathrm{D(G)}$, models: Kitaev, Ann.Phys. **303** ('03) 2.
  $\sim$ discrete gauge theories in $2\mathrm{d}$.

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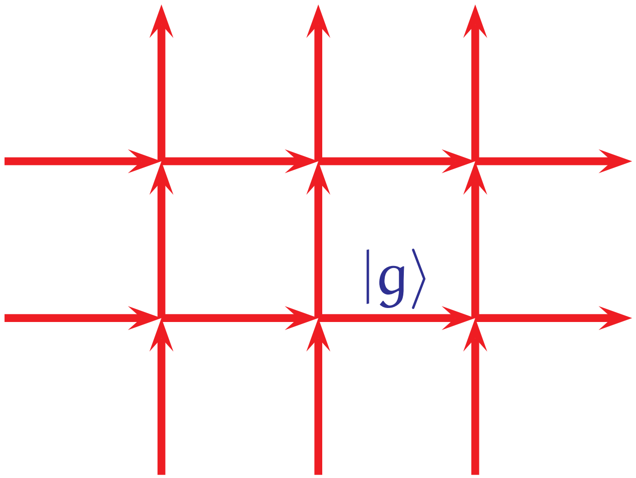
- Quantum double, or $\mathrm{D(G)}$, models: Kitaev, Ann.Phys. **303** ('03) 2.
  Hilbert space at edges: group algebra $\mathbb{C}\mathrm{G}$.
  Edge basis labelled by $\mathrm{g} \in$ finite group G.



$$\mathrm{g} \sim \mathcal{P}\exp \mathrm{i} \int \vec{\mathrm{A}} \cdot \mathrm{d}\vec{\ell}$$

D(G) quantum double models

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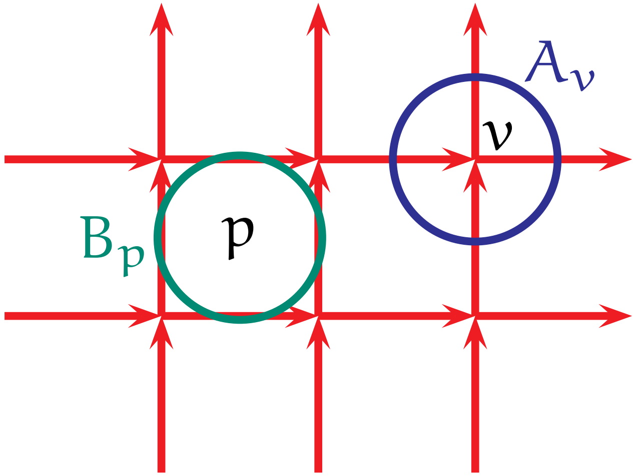
  Hilbert space at edges: group algebra $\mathbb{C}G$.
- Hamiltonian $\mathbb{H} = -\sum_v A_v - \sum_p B_p$.
  All A_v, B_p commuting projectors.



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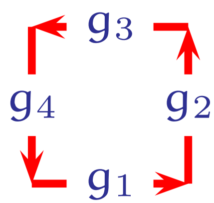
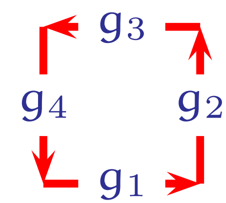
  — Plaquette operators: project onto trivial magnetic flux.

$B_p \Big|$ 

$\Big\rangle = \delta(g_4 g_3 g_2 g_1,\, e) \Big|$ 

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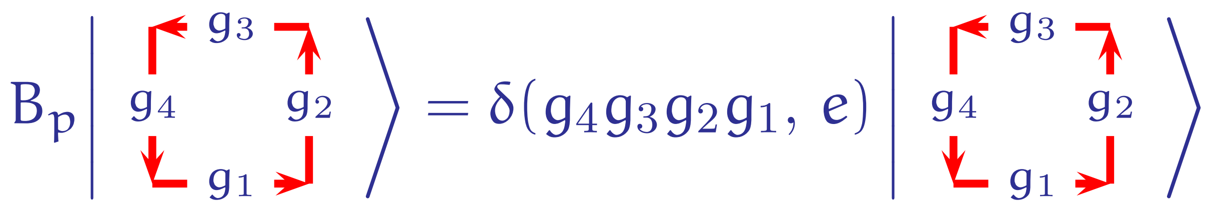
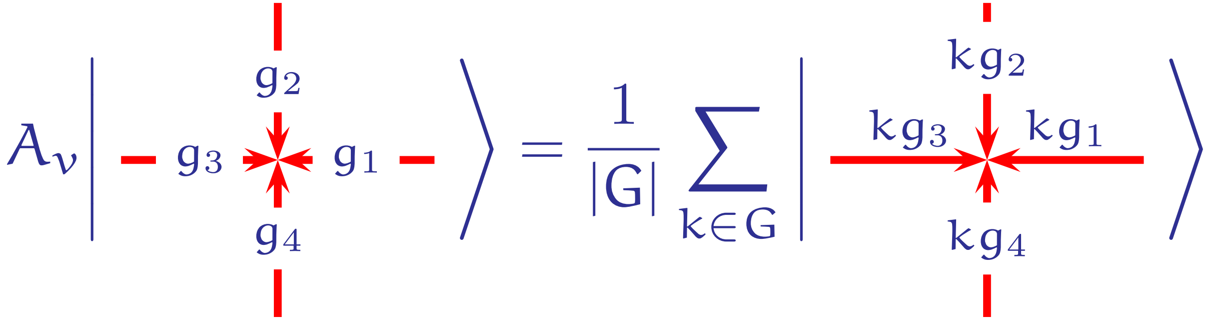


  - — Vertex operators: impose gauge invariance.



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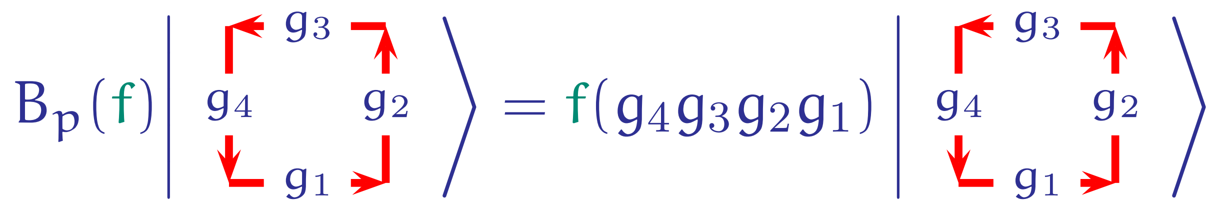
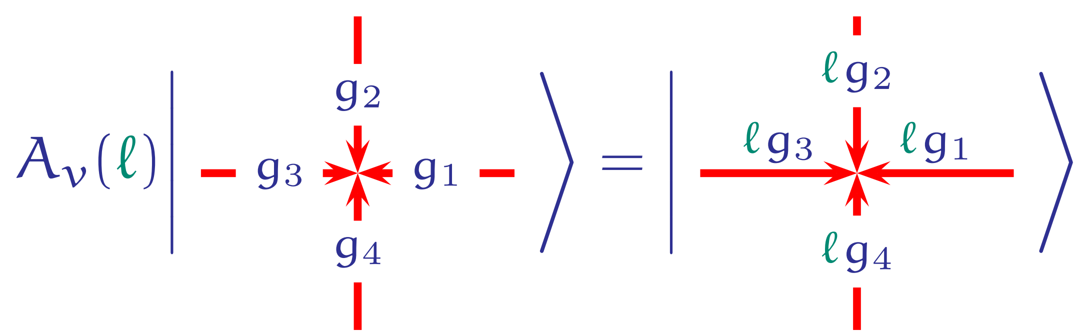
- Classification of charges: algebraic:

  — Plaquette representations: functions $\mathbb{C}G \to \mathbb{C}$.



  — Vertex representations: action of $\mathbb{C}G$.



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- Take $A_v(\ell)$ and $B_p(f)$ together when $v \cap p \neq \emptyset$:

$$A_v(g)\, B_p(\delta_k) = B_p(\delta_{gkg^{-1}})\, A_v(g).$$

  Representation of Drinfel'd's quantum double $D(G)$.

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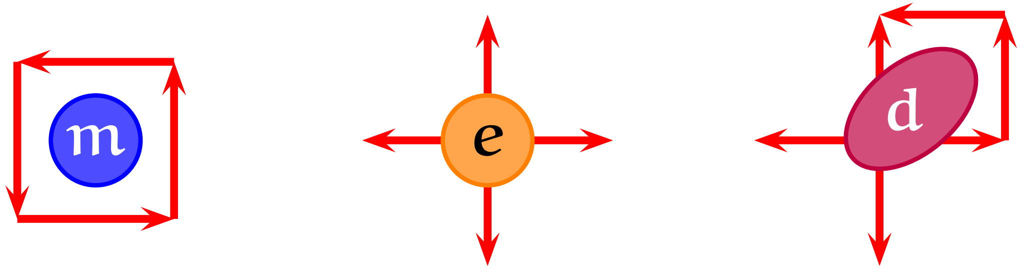

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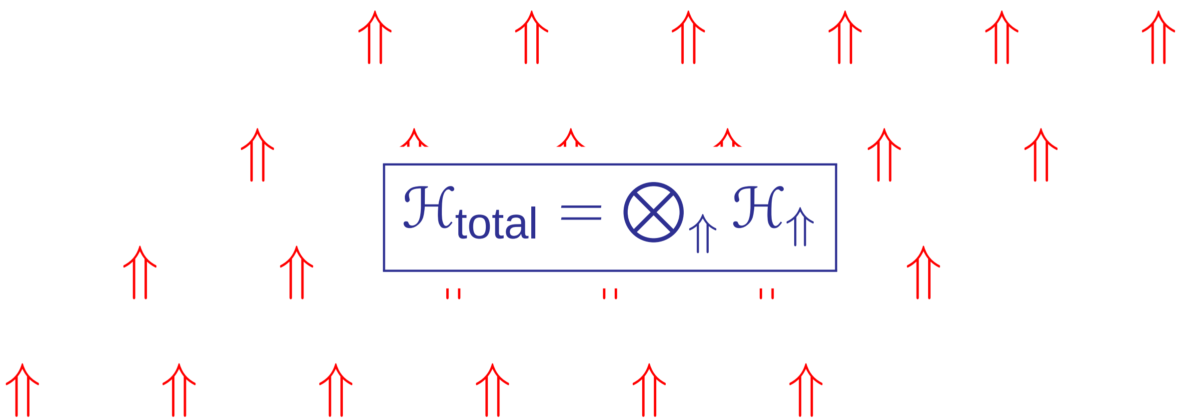
- Already the $D(S_3)$ model universal for QC.

Hopf algebras

- Hopf algebras for condensed matter physicists:

  Language for symmetries in many-body quantum systems.



$$\mathcal{H}_{\text{total}} = \bigotimes_{\Uparrow} \mathcal{H}_{\Uparrow}$$

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  — Identity transformation: unital algebra: $e \cdot g = g$.
  — Representations on tensor products: coalgebra: $g \mapsto g \otimes g$.
  — Trivial representation: counital coalgebra: $\varepsilon(g) = 1$.
  — Conjugate representations: antipode: $S(g) = g^{-1}$.

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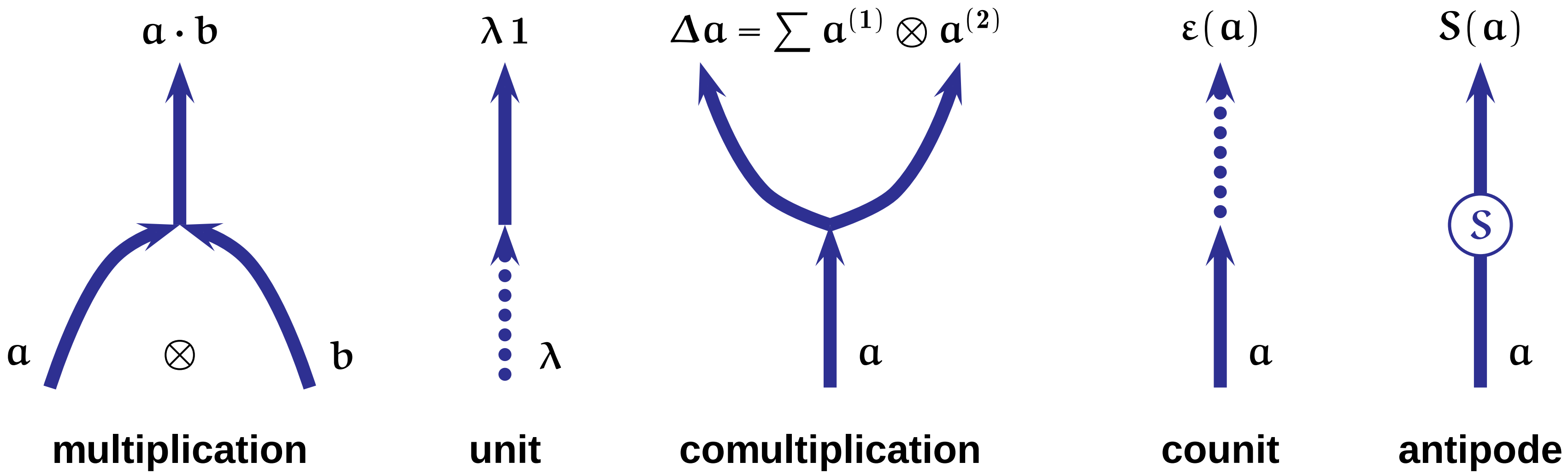

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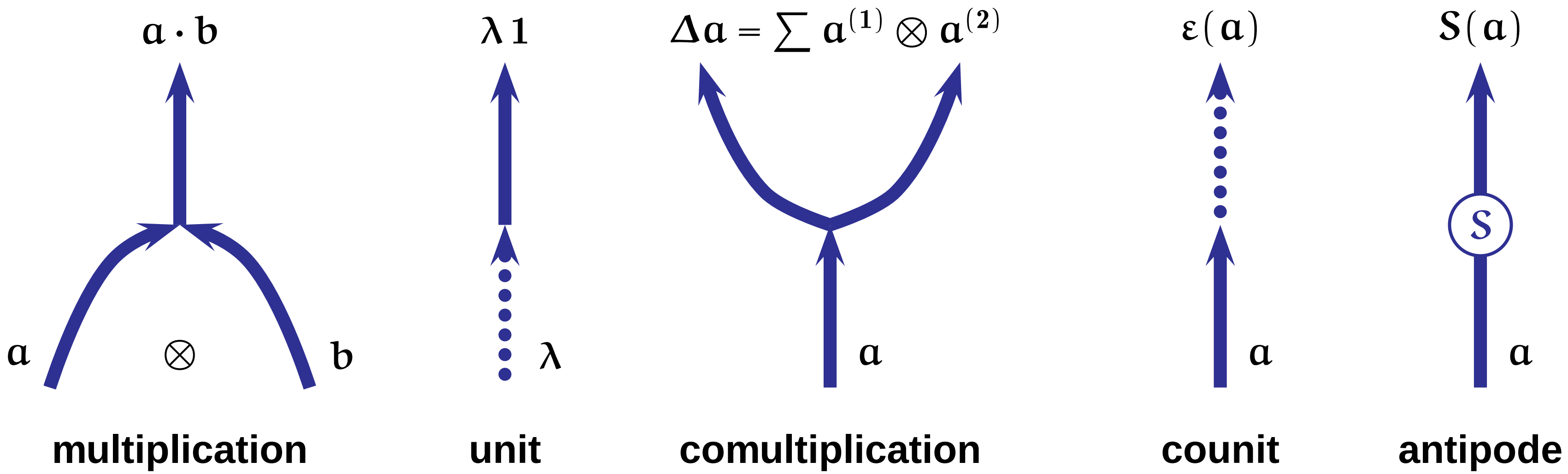

- Hilbert spaces: use C^* structure.

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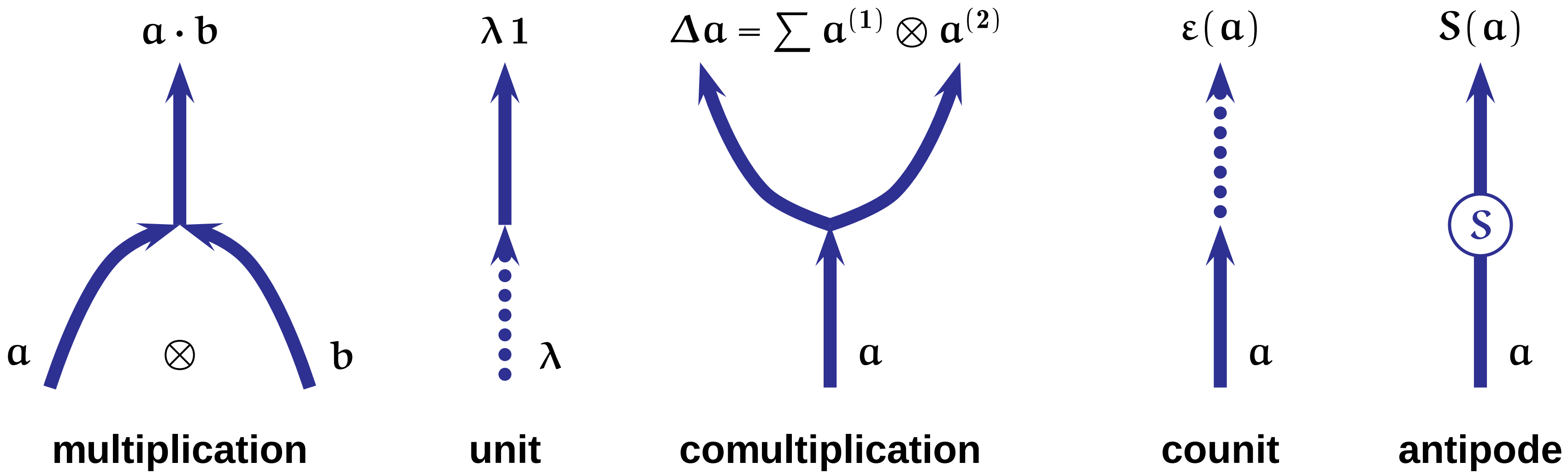
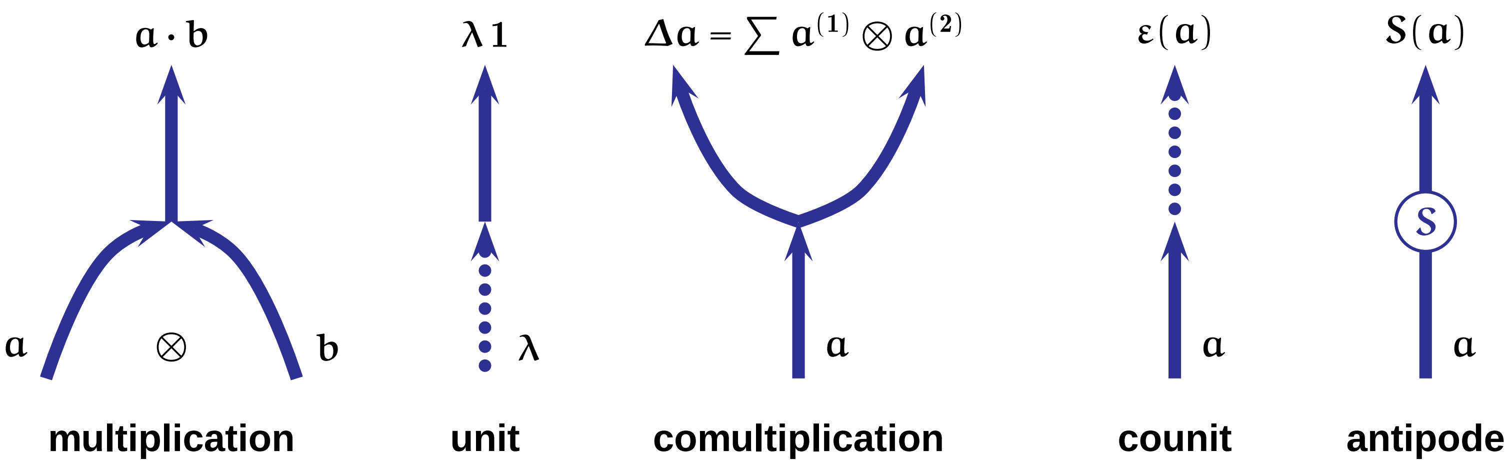


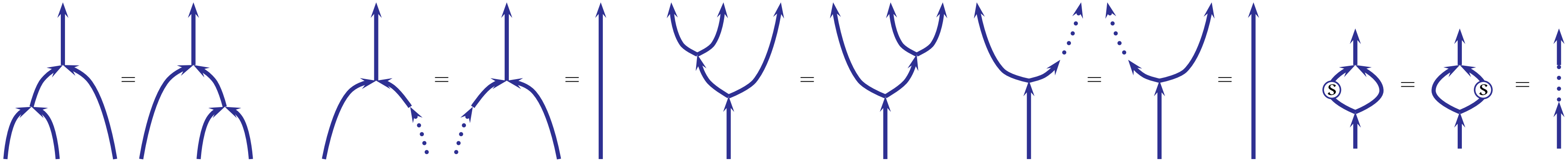
- Hilbert spaces: use C^* structure.
- Spin systems (lazy physicists): everything finite dimensional.

Hopf algebras

- Hopf algebra structure:



Compatibility conditions:



Duality

- Hopf algebras are closed under algebraic dualisation.

  If H finite-dimensional Hopf algebra, $\mathrm{H}^* = \mathrm{Hom}(\mathrm{H}, \mathbb{C})$ also.

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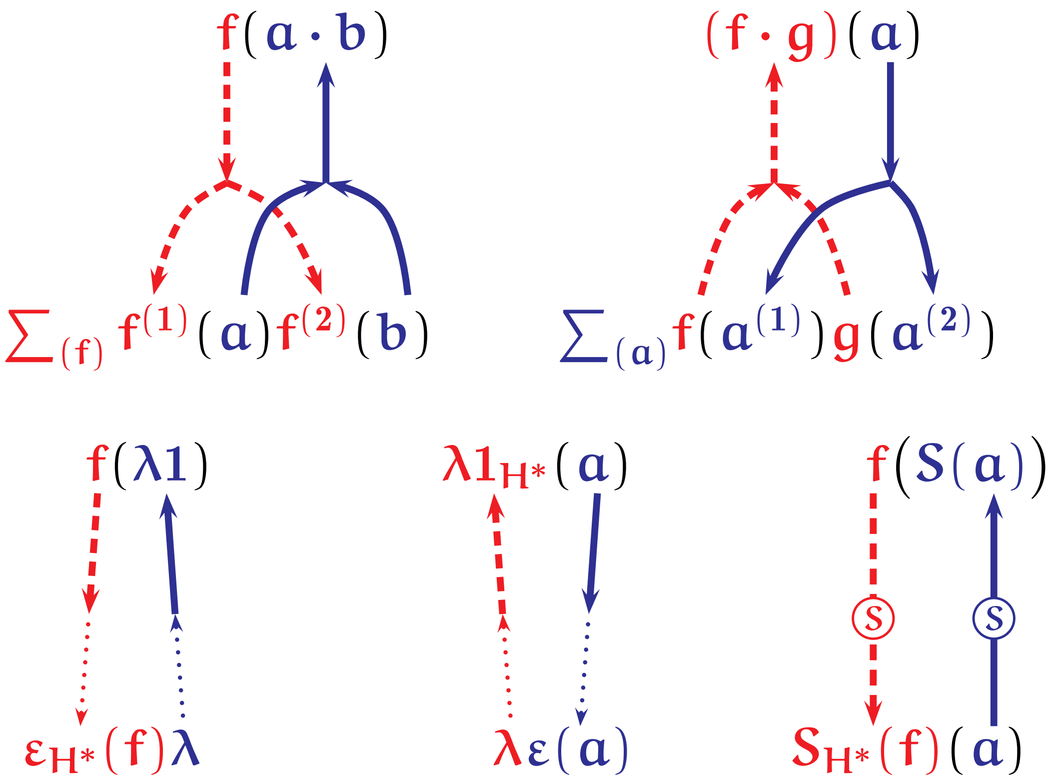
  And the structures are determined from each other:



C^* structure

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  Finite-dimensional Hopf C^*-algebras H are semisimple:

  $\exists$ Haar integrals $h \in H$, $\phi \in H^*$:

  $$a \cdot h = \varepsilon(a)\, h, \quad f \cdot \phi = f(1)\, \phi,$$

  Highly symmetric: $h^2 = h^* = S(h) = h$, $\varepsilon(h) = 1 \ldots$

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In $\mathbb{C}G$, $\quad g^* = g^{-1}, \quad (g, k) = \delta_{g,k}, \quad h = |G|^{-1} \sum_{g \in G} g.$

In $\mathbb{C}^G$, $\quad (\delta_g)^* = \delta_g, \quad (\delta_g, \delta_k) = |G|^{-1} \delta_{g,k}, \quad h = \delta_e.$

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- Hamiltonian: still $\mathbb{H} = -\sum_v A_v - \sum_p B_p$.
  — From Hopf maps and Haar integrals.

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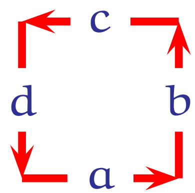
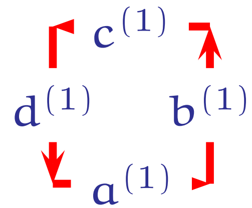
$B_p \Big|$ 

$\Big\rangle = \sum_{(a,b,c,d)} \phi\left(d^{(2)} c^{(2)} b^{(2)} a^{(2)}\right) \Big|$ 

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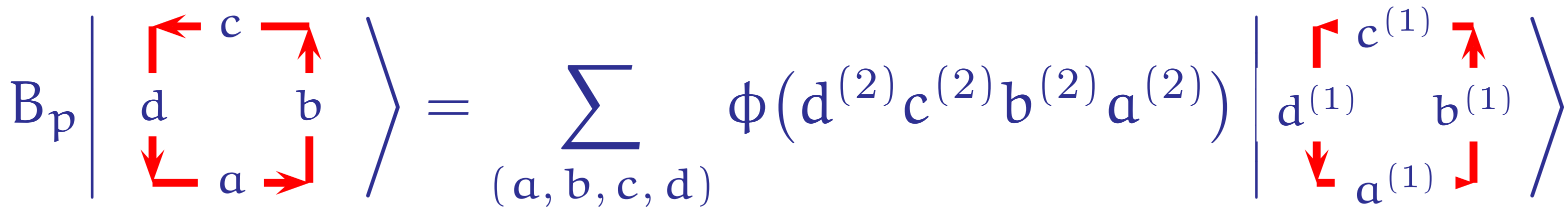
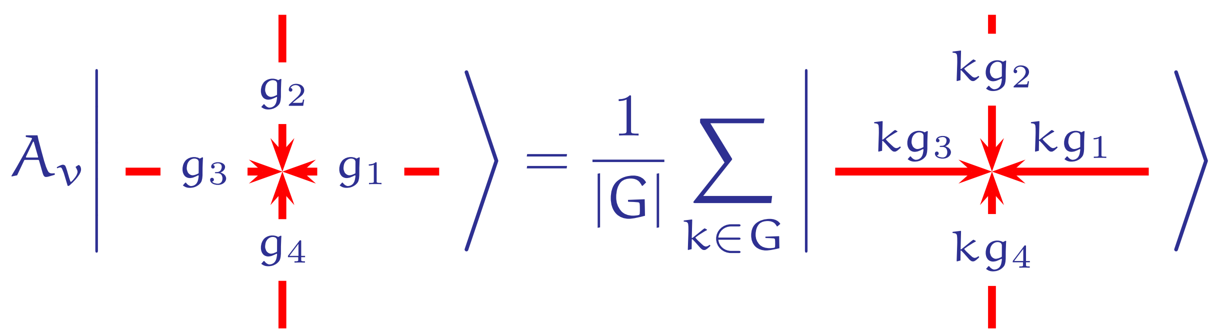


  - Vertex operators:



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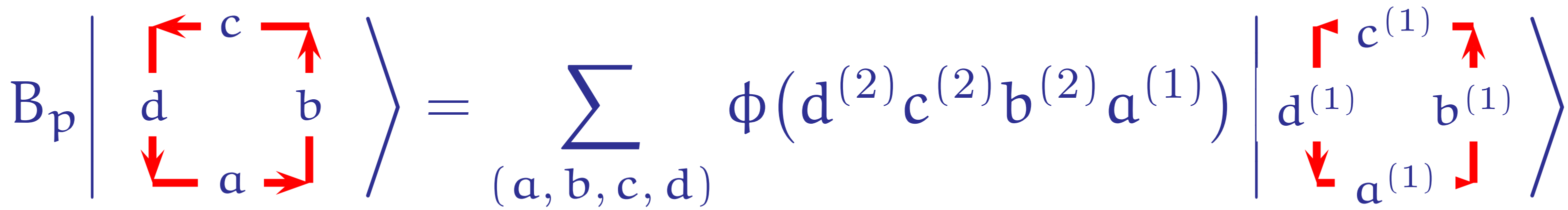

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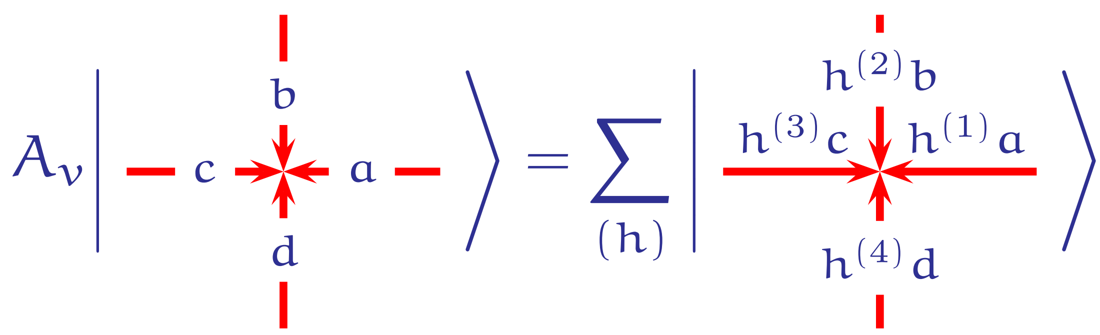

D(H) quantum double models

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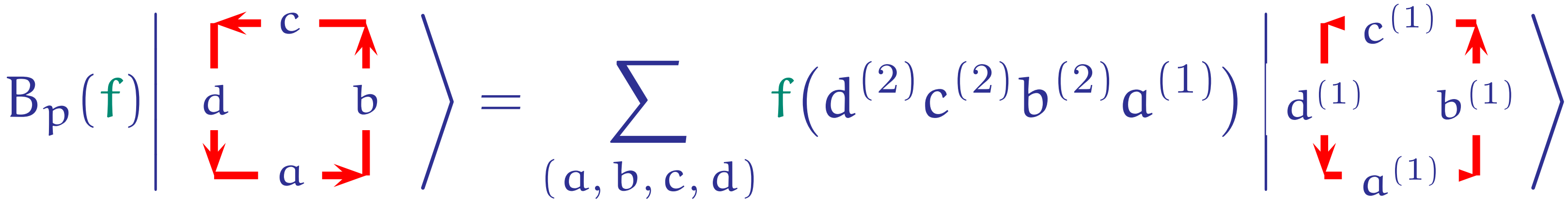
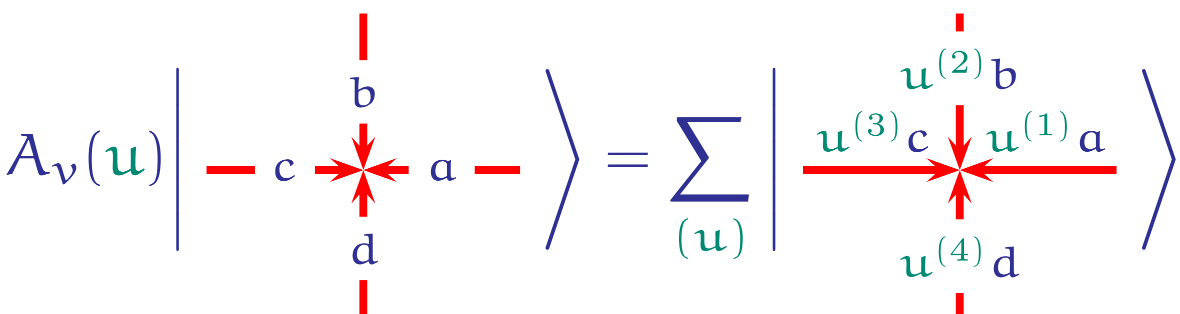
- Hamiltonian: still $\mathbb{H} = -\sum_v A_v - \sum_p B_p$.
  - Plaquettes: representations of $(H^*)^{cop}$:



  - Vertices: representations of H:



  - Build representations of DH, Drinfel'd double of H.
  - Irreps of DH classify excitations.

D(H) quantum double models

- Quantum many-body: Need efficient descriptions of states.
  — $\dim \mathcal{H}$ scales exponentially with system size,

$$|\Psi\rangle = \sum_{i_1,\ldots,i_N} c_{i_1,\ldots,i_N} |i_1,\ldots,i_N\rangle,$$

but Hamiltonian only polynomial # parameters.

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- Quantum many-body: Need efficient descriptions of states.
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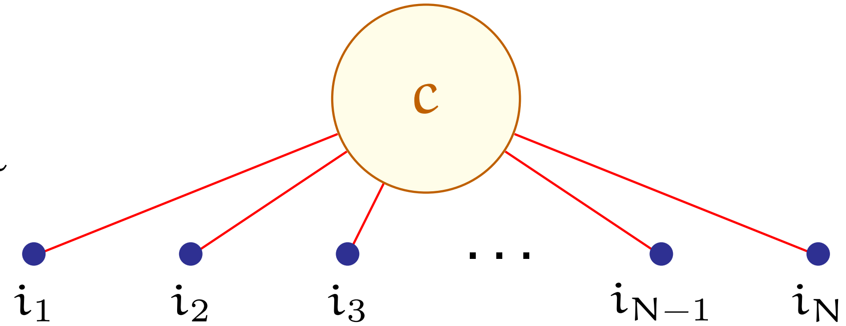
  but Hamiltonian only polynomial # parameters.
- So instead of generic coefficients. . .

$c_{i_1,\ldots,i_N} \sim$



$D(H)$ quantum double models

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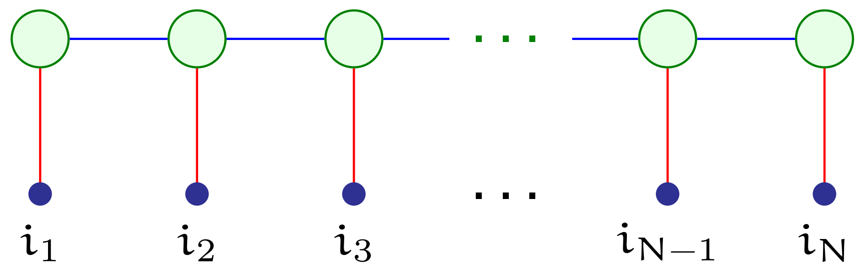

  - Useful for numerics, also exact applications.

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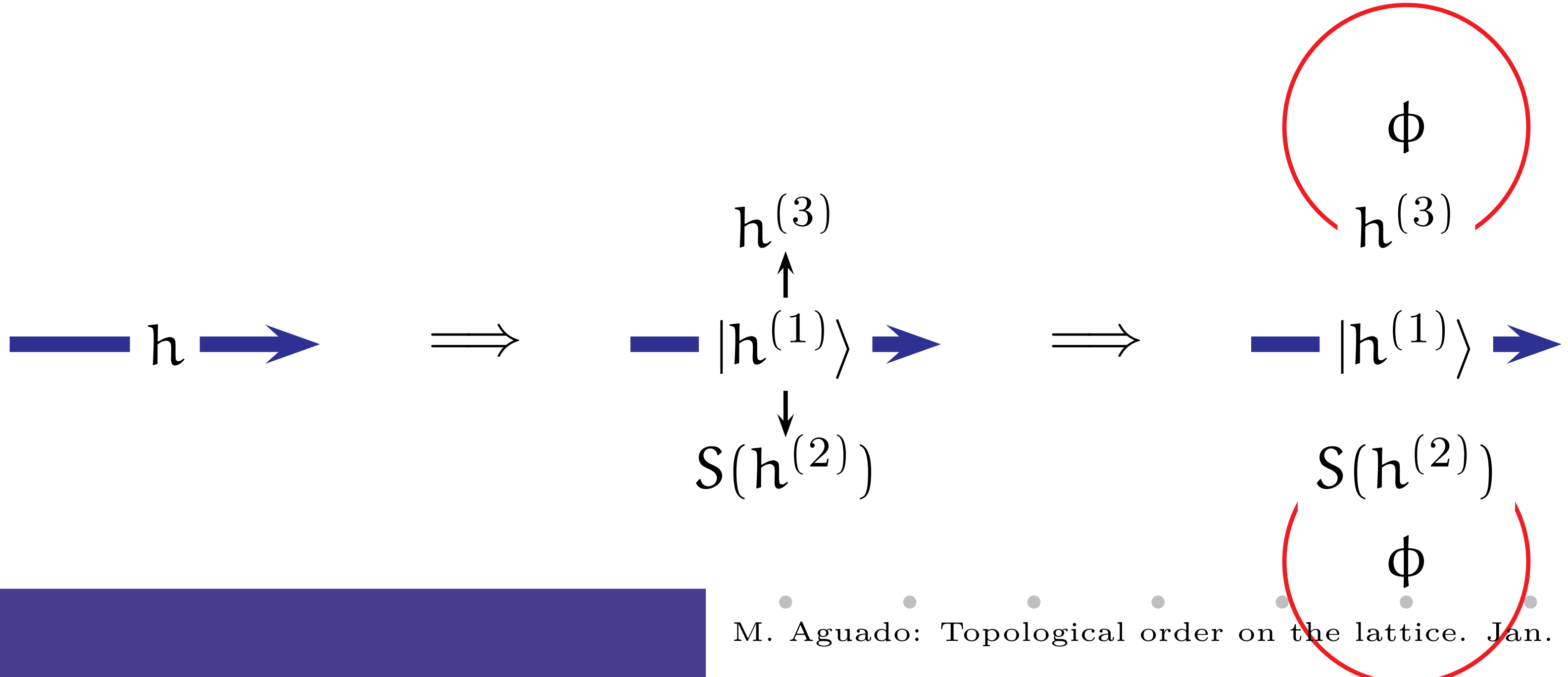

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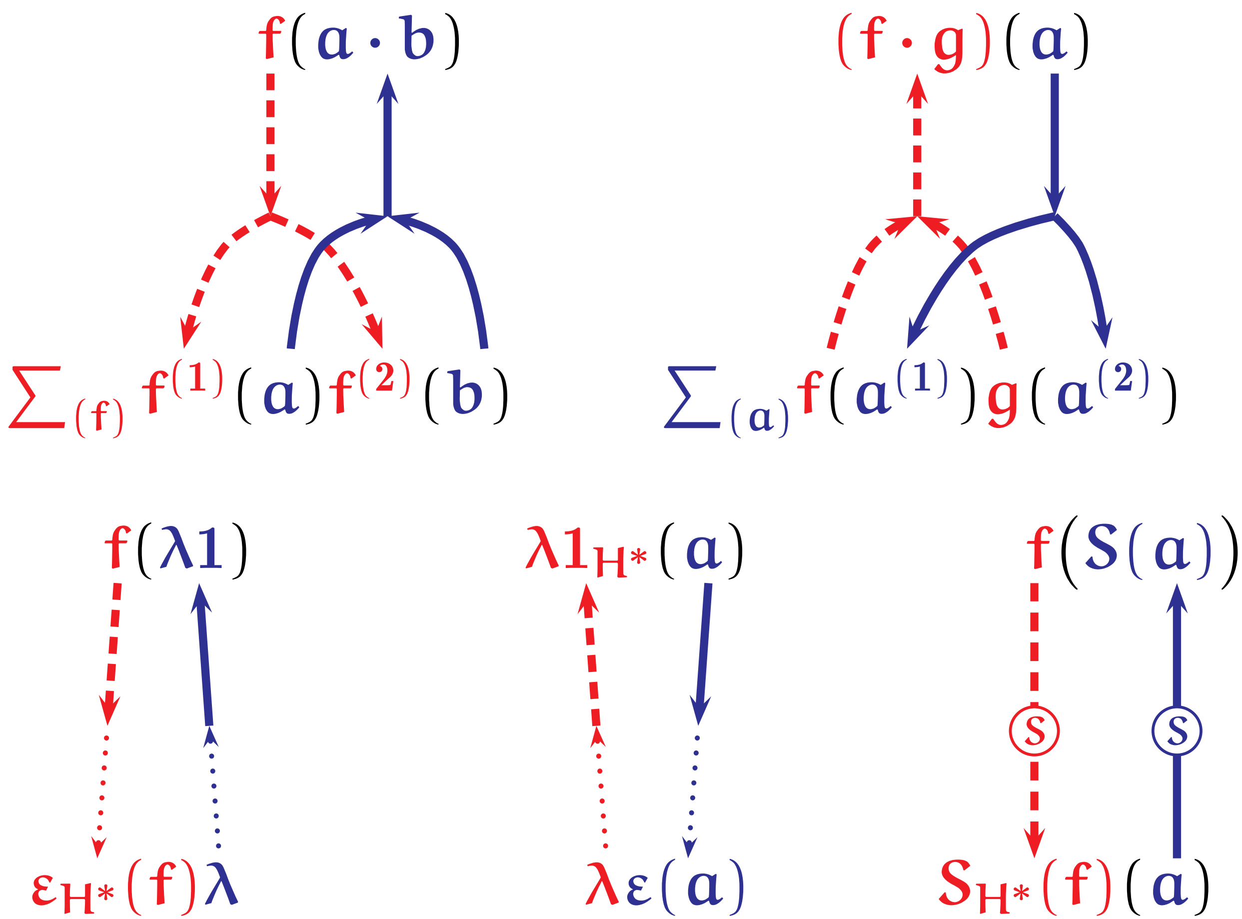
- DH: First in a hierarchy of topological TN's
  using Hopf subalgebras of H, H^*
  (related to condensation of topological charge).

D(H) quantum double models

- Hopf algebras closed under algebraic dualisation:



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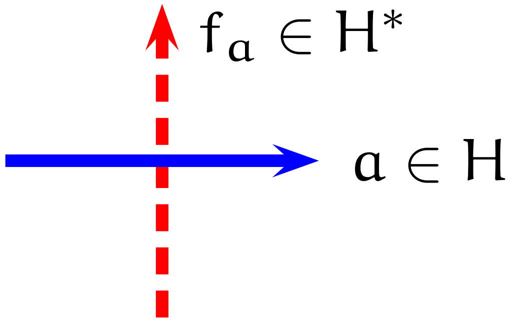
- . . . leading to electric-magnetic duality for DH models.
  - Go over to the dual lattice and the dual of H:

$$a \in H \longmapsto f_a \in H^* , \quad f_a(b) = \sqrt{|H|}\, \phi(a \cdot b)$$



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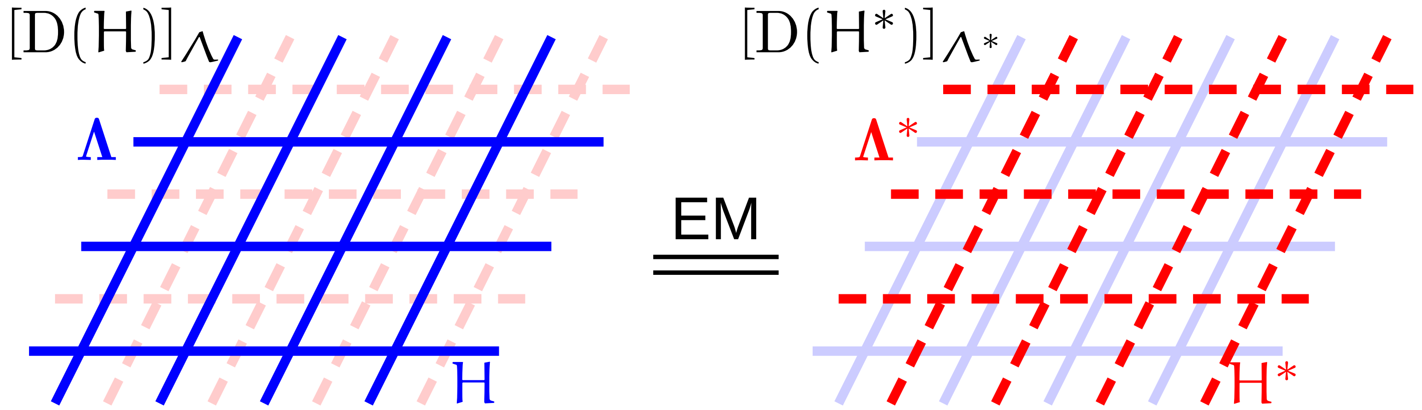

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  - $D($Abelian $G)$: EM-selfdual.

    $D($non-Abelian $G)$: not closed under EM duality.

    DH: smallest class containing all DG's

    and closed under EM duality and tensor products.

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  - Plaquette operators: related to F-symbols $(6j)$.
- Argued to represent fixed points of renormalisation group.
  - All topological lattice phases with P, T symmetry.
- Language: category theory.
- Excitations not so well understood locally.

String-net models

- Can map DH models to string-nets.
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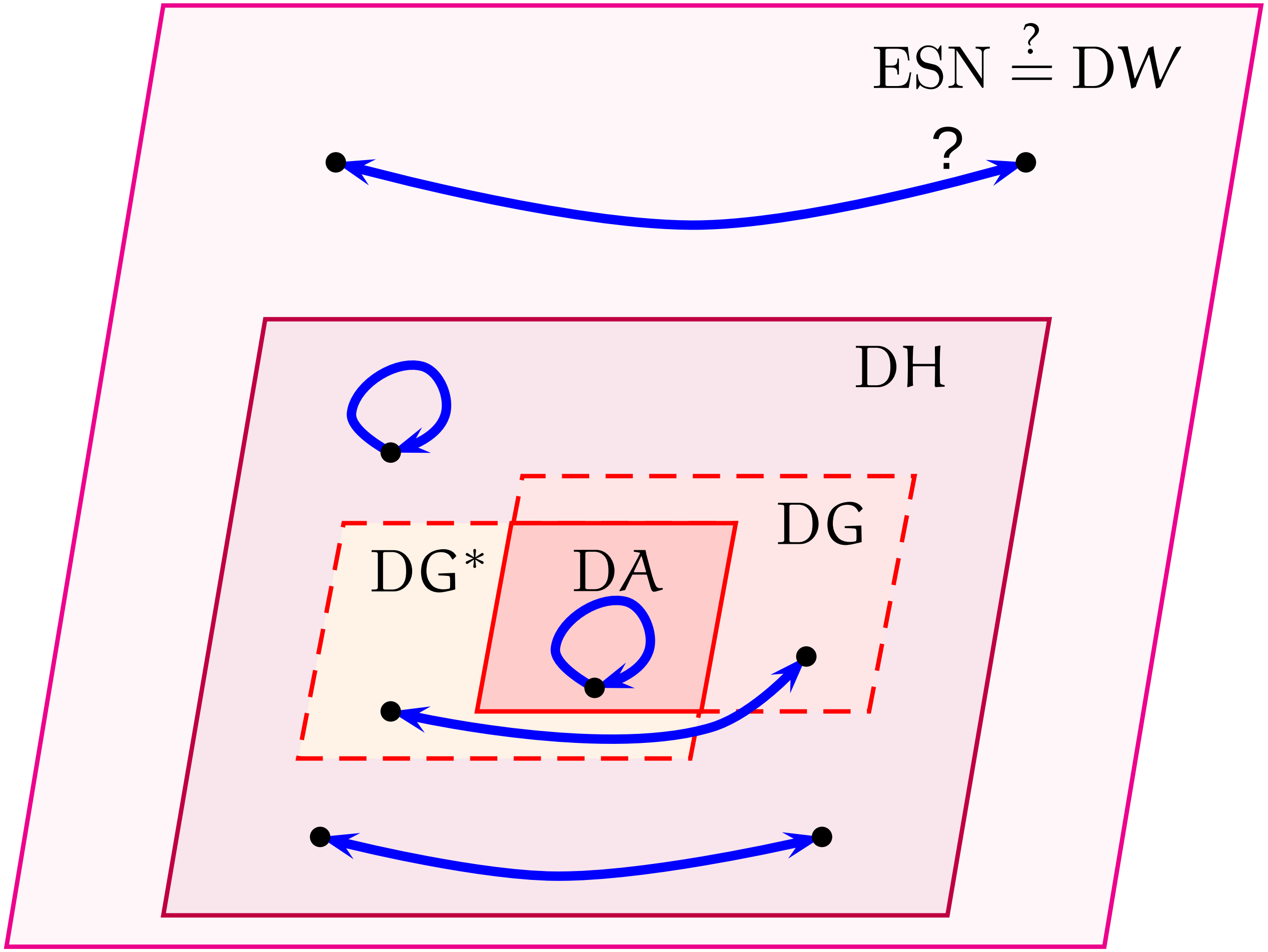
  Ground levels of DH and SN identical.

  — From EM duality: two such maps $\mathrm{DH} \to \mathrm{SN}$.
  SNs: "electric" and "magnetic" projections of DH.

Conclusions



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— A. Yu. Kitaev, *Anyons in an exactly solved model and beyond*, Annals Phys. **321**, 2 (2006), `cond-mat/0506438`.

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