Local topological order inhibits thermal stability in 2D

arXiv:1209.5750

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Self-correcting memory

Self-correcting memory = physical system which encode (quantum) information

- reliably
- for a macroscopic period of time
- letting the memory interact with its environment (thermal noise)
- *without* active error correction

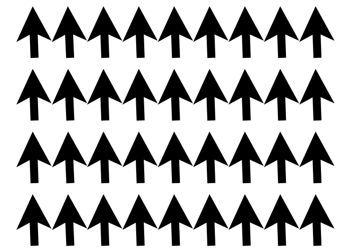
Encoding $|\psi_i\rangle$ —Noise—  $\longrightarrow |\psi_i'\rangle \xrightarrow{\text{Decoding}} |\psi_f\rangle$

Code = degenerate groundspace of a local Hamiltonian of spin particles on a (2D) lattice.

Self-correcting classical memories

2D ferromagnetic Ising model

$$H_{\text{Ising2D}} = -\sum_{\langle i,j \rangle} \sigma_z^i \otimes \sigma_z^j$$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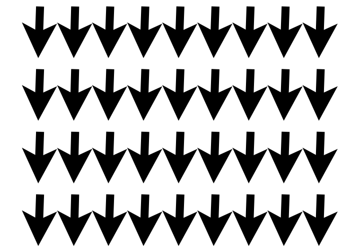
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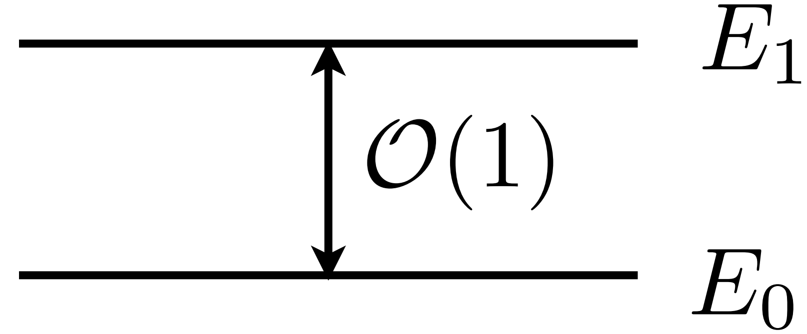
- thermally stable: for T<TCurie, no macroscopic error droplets
- contrasts with 1D case : point-like excitations which diffuse freely

Not stable under perturbation!
➡ (small) magnetic field breaks degeneracy
➡ true for any system with local order parameter

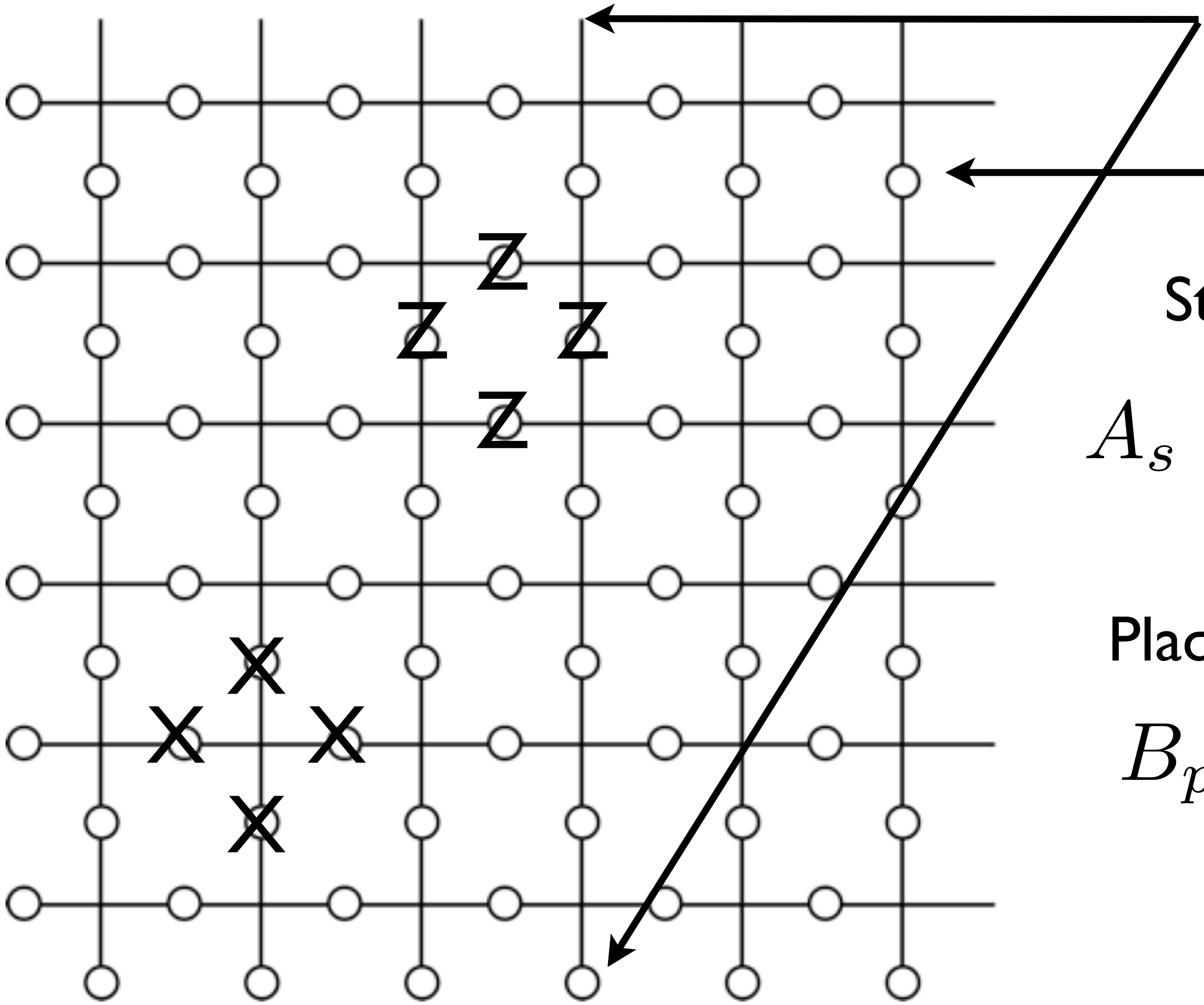
Quantum systems
➡ with no local order parameter ?
➡ stable spectrum ?



Topologically ordered system !

(Archetypical) example : Kitaev's toric code (1997)

A Kitaev. Ann. Phys. 303(1), 2–30 (2003)

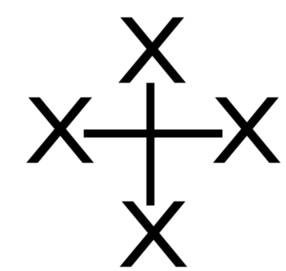


Periodic boundary conditions

Spin-1/2 on the *edges*

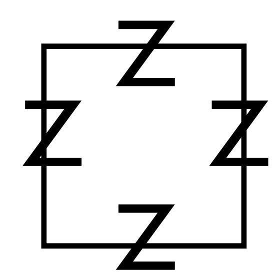
Star operator

$$A_s = \bigotimes_{i \in \mathcal{N}(s)} \sigma_x^i$$



Plaquette operator

$$B_p = \bigotimes_{i \in p} \sigma_z^i$$



$$H = -\sum_s A_s - \sum_p B_p$$

All operators commute pairwise.

Kitaev's toric code is spectrally stable.
Is it thermally stable ?

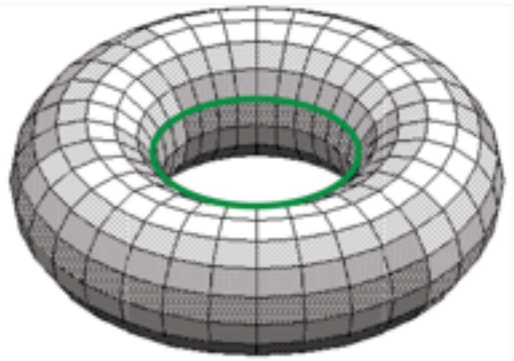
Unstability of Kitaev's toric code

Groundstates

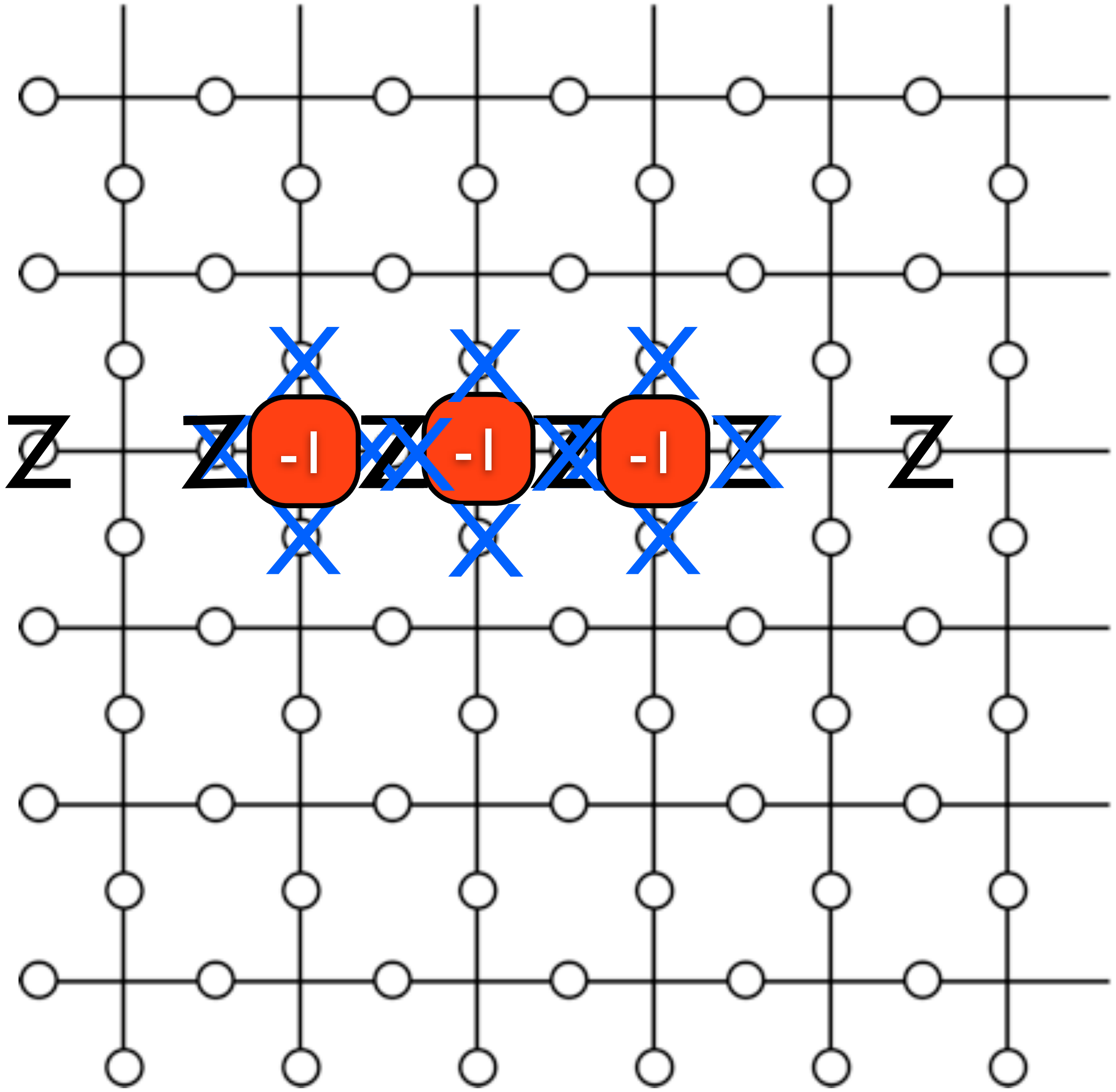
$$\forall s \; A_s|\psi\rangle = +|\psi\rangle$$

$$\forall p \; B_p|\psi\rangle = +|\psi\rangle$$

Logical operator : string of Z



Excitations



No energy for anyon propagation.

Thermal fluctuations can accumulate and corrupt the information.

Outline

Introduction

- Thermal stability
- Spectral stability

Setting and known results

- Local commuting projectors code (LCPC)
- Topological order
- Formal definition of self-correction
- Known results for stabilizer codes and LCPCs

Main result and sketch of the proof

- Noise model(s)
- No dead-ends
- Expected number of trials
- Equivalence between models

Broad class of 2D codes: LCPCs

N finite dimensional spins located on the vertices of a 2D lattice (V, E).

$$H = -\sum_{X \subset V} P_X$$

- bounded strentgh $\|P_X\| \leq 1$
- terms commute $[P_X, P_Y] = 0$
- local $\mathrm{diam(X)} \geq w \Rightarrow P_X = 0$
- frustration-free $\forall X \; P_X|\psi\rangle = +|\psi\rangle$

We are interested in the groundspace of H and scaling of the energy gap.
Without loss of generality, P_X = projector

Local commuting projector codes (LCPCs)

$$[P_X, P_Y] = 0$$

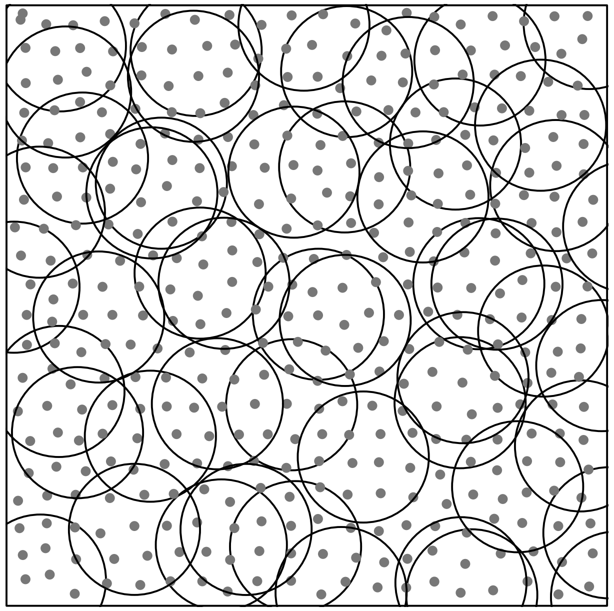
$$P_X|\psi\rangle = +|\psi\rangle$$

$$(P_X)^2 = P_X$$

Code projector $P = \prod P_X$

Stabilizer

$$P_k \rightarrow S_k = \bigotimes_{i_k} \sigma_{i_k}^{[i]}$$



TQO inhibits thermal stability

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Spectrum stability: local topological order

Spectrum of LCPC Hamiltonian is stable if the Hamiltonian has local topological quantum order (LTQO).

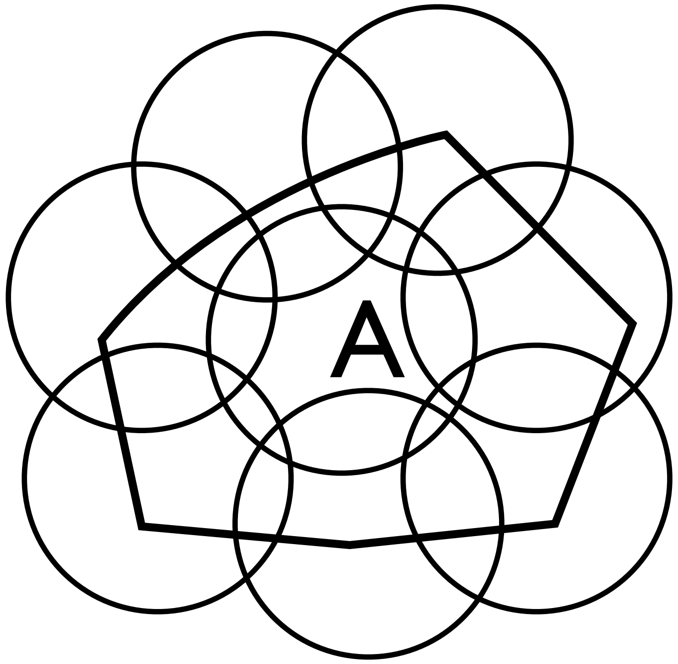
Bravyi, Hastings, Michalakis (2010)

Local topological quantum order (LTQO)

- local indistinguishability: local operators cannot discriminate groundstates.
- local consistency: local groundstate is compatible with global groundspace.

LI $\forall O_A \, \exists c_A \; P O_A P = c_A P$ forbids local order parameter

LC Local projector on the code $P_A = \prod_{X \cap A \neq \emptyset} P_X$



$$\rho_A^{\mathrm{loc}} = \mathrm{Tr}_{\bar{A}} P_A$$
$$\rho_A = \mathrm{Tr}_{\bar{A}} P$$
have same kernel.

Formal definition of self-correction

Thermalization requires detailed knowledge of system dynamics.

Simplified model for thermalization

- penalize high energy states (Boltzmann factor) $\propto e^{-E/k_B T}$
- local moves in noise model

Encoding $|\psi_i\rangle$ —— Noise ——

$\longrightarrow |\psi_i'\rangle$ —— Decoding $\longrightarrow |\psi_f\rangle$

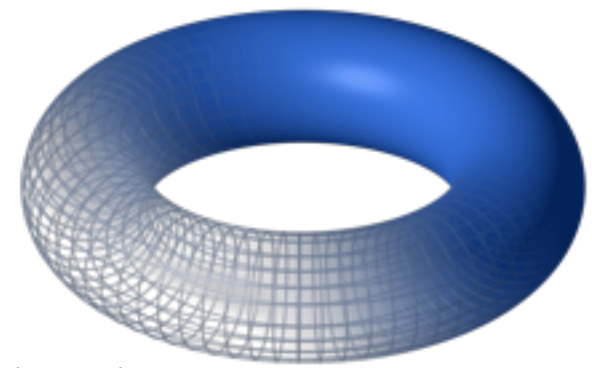
Logical operator : operator that maps groundstate to gs. $[K, P] = 0$

↳ Sequence of local moves (CPTP maps) that implements logical op?

↳ Maximum energy of intermediate states : energy barrier?

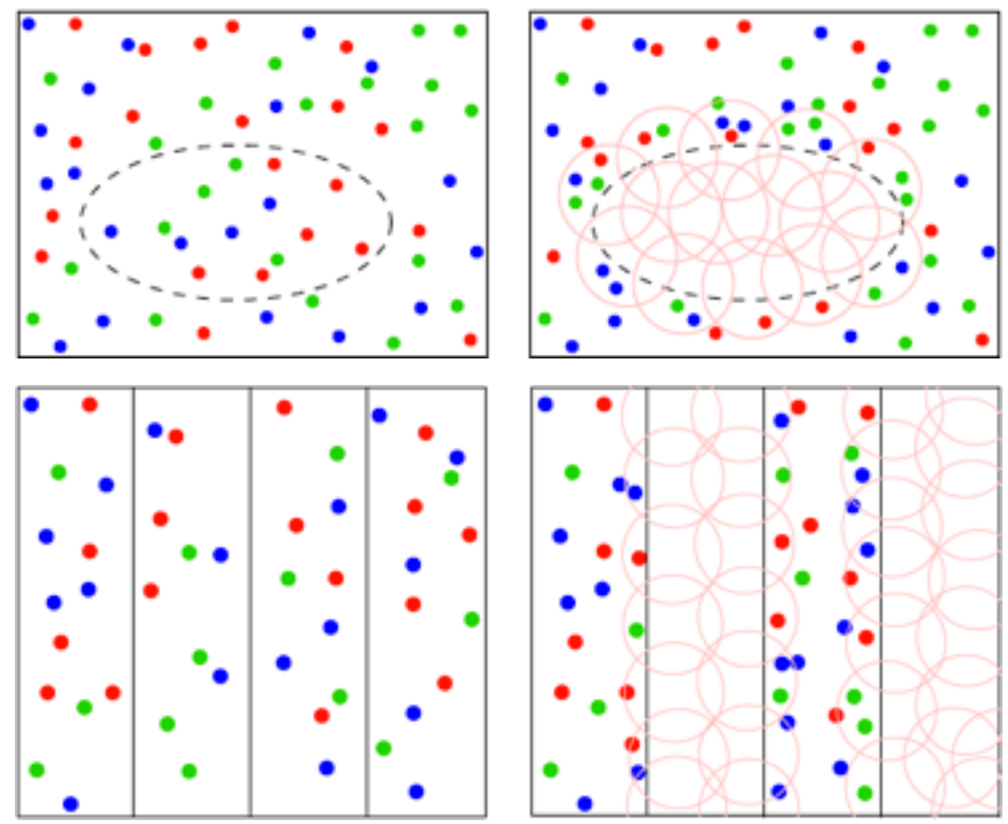
Known results: stabilizer codes & LCPCs

Instability in Kitaev's toric code



Key features

- logical operator is supported on a string of particles
- ~~logical operator is a tensor product of single-body unitaries~~



General result for stabilizer codes
➡ cleaning lemma (Bravyi & Terhal '09)

Generalization to LCPCs
➡ disentangling lemma
Bravyi, Poulin & Terhal '10
➡ Haah & Preskill '12

LCPCs : logical operator is supported on a strip, but not a tensor product.

How to apply it

- through a sequence of local moves?
- without creating too much energy?

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Main result and sketch of the proof

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Main result

Main result (arXiv:1209.5750)
For any 2D *local topologically ordered* LCP code, we exhibit a physically realistic error model corrupting the information.

BHM '10

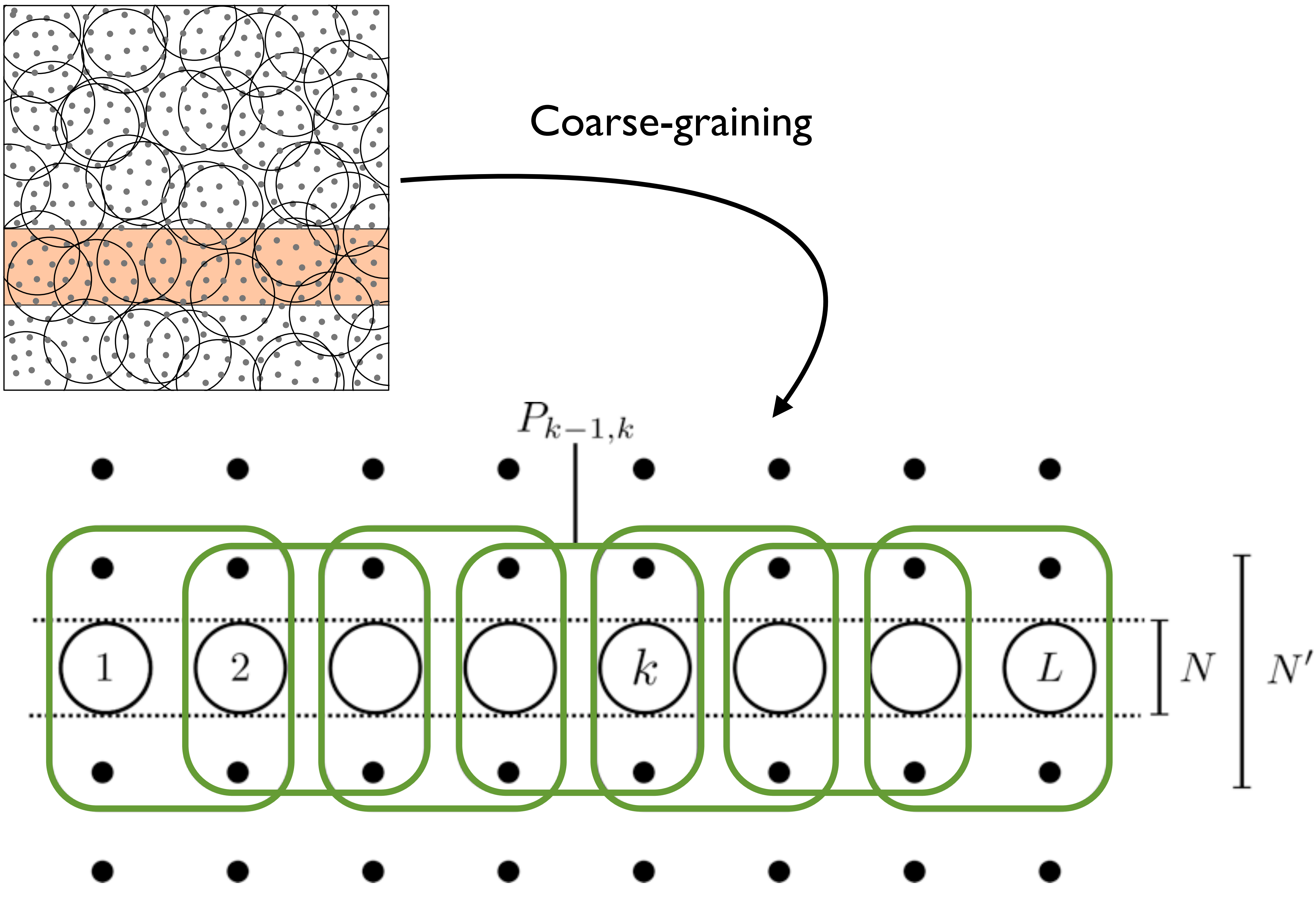
Local topological order → Spectral stability

↓ this work

Thermal unstability

Tradeoff between spectral and thermal stability.

Sketch of the proof (1): coarse-graining

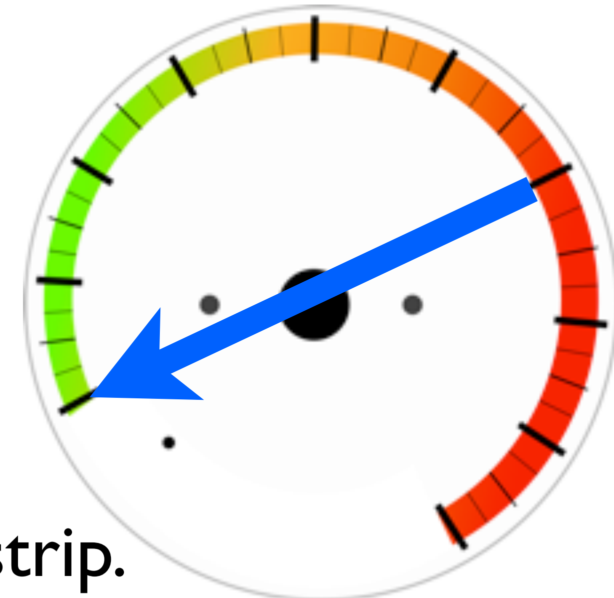


- *Sites* on the strip
- Local constraints

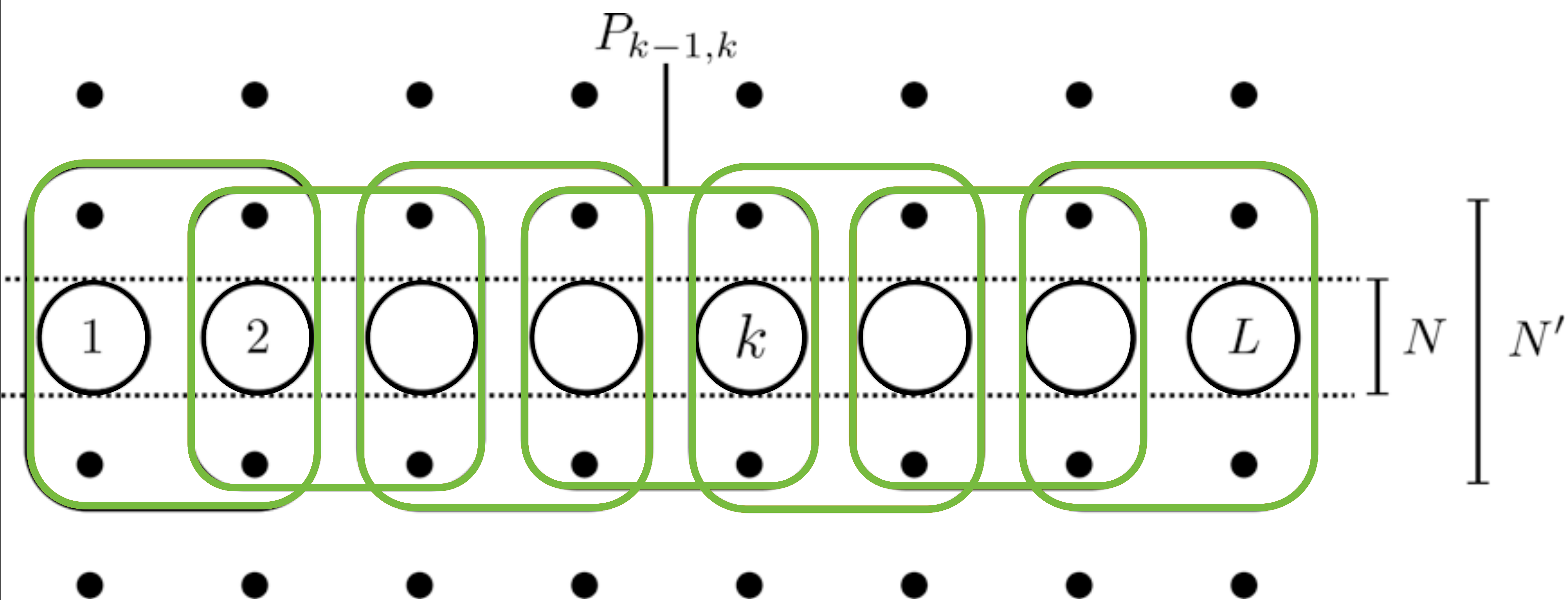
Sketch of the proof (II): fortuitous model

Fortuitous model

- depolarize every site on the strip N
- project back onto the code



Clearly wipes out the information supported on the strip.



Unphysical model: too much energy and projection very unlikely.

Idea : interleave the depolarization step and projection at each site.

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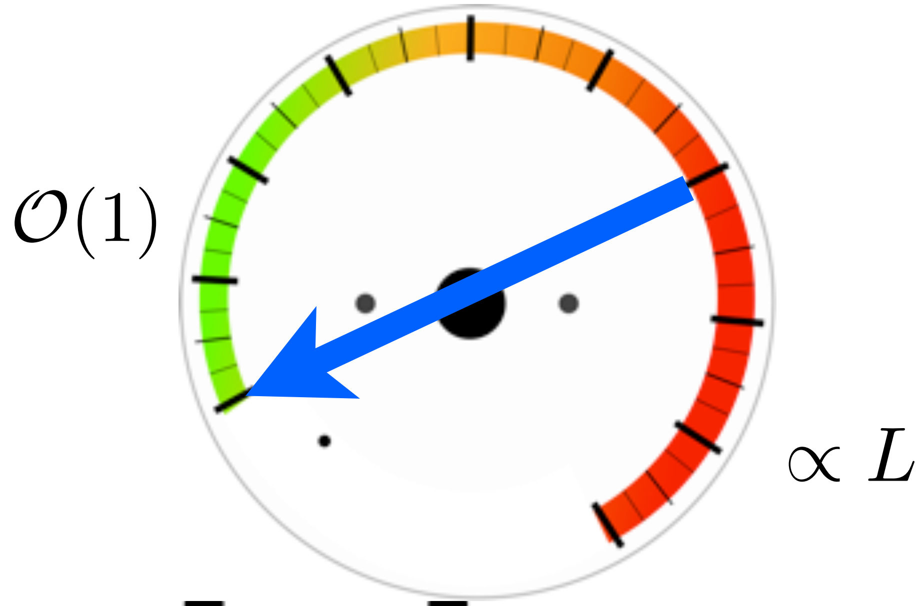
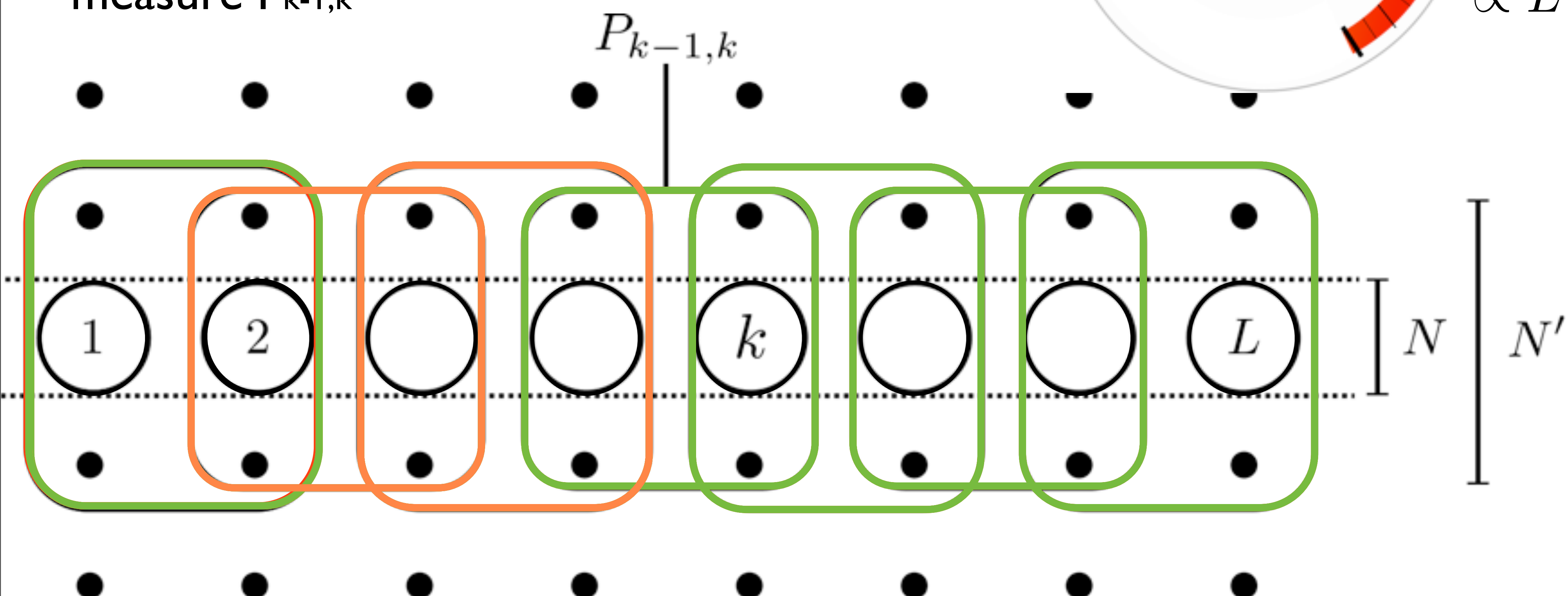
Sketch of the proof (III): iterative randomization model

Iterative randomization model

For every site k (iteration),
- apply random trial unitary
- measure $P_{k-1,k}$





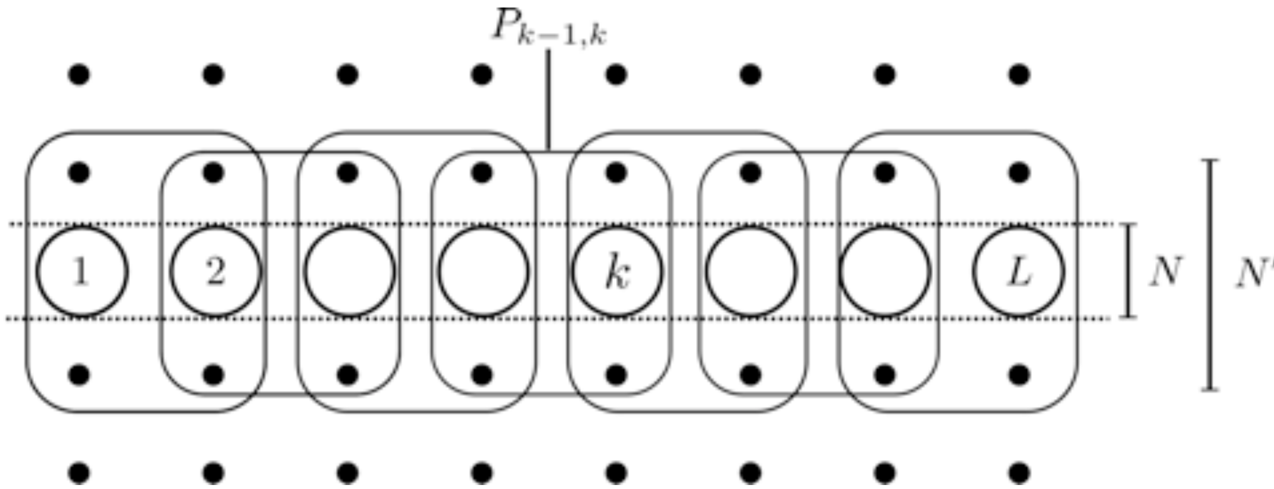
Immediate properties
- at any step, the energy is constant above the gs energy
- no need to backtrack

To show
- no dead-end and expected number of trials at each iteration is constant
- effect of iterative randomization model = effect of fortuitous model

Sketch of the proof (IV): no dead-end

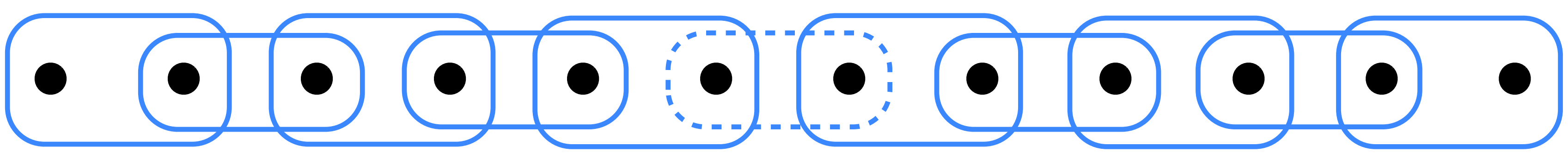
Iterative randomization model

For every site k,
- apply random trial unitary
- measure $P_{k-1,k}$



Dead-end = impossible to find eligible unitary at a given iteration.
State of the strip, yet consistent with previous constraints, can't be extended.

Simple example: chain of qutrits



$$P_{i,i+1} = |00\rangle\langle 00| + |11\rangle\langle 11| + |22\rangle\langle 22|$$

$$P^*_{i,i+1} = |00\rangle\langle 00| + |11\rangle\langle 11| \qquad P = |0\ldots 0\rangle\langle 0\ldots 0| + |1\ldots 1\rangle\langle 1\ldots 1|$$

Dead-end: start preparing all 2 state...

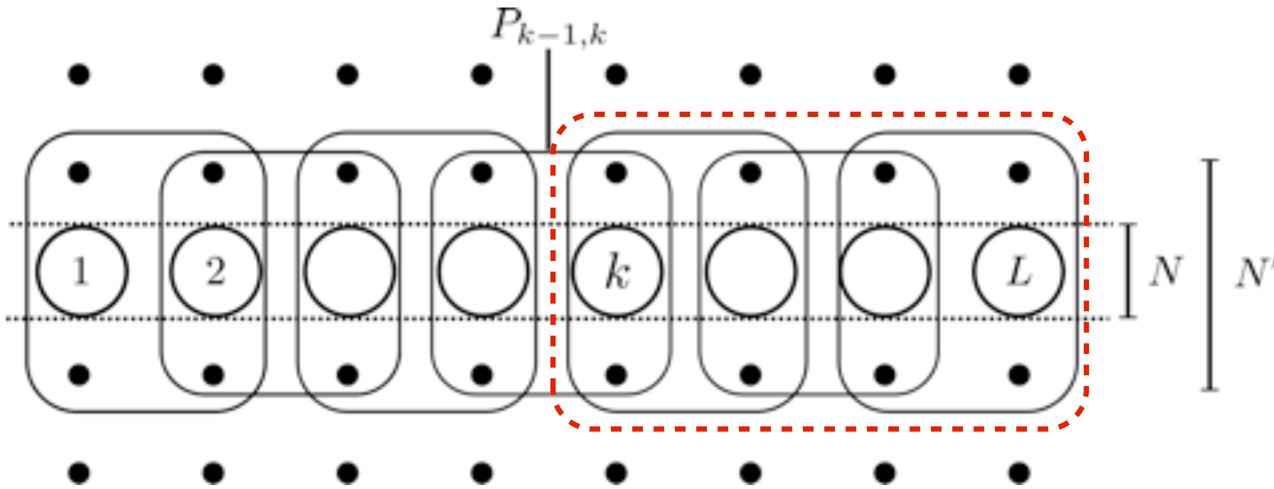
Violates local consistency: look at any site k far from defect

$$\rho_k \equiv \mathrm{Tr}_k P = |0\rangle\langle 0| + |1\rangle\langle 1|$$
$$\rho_k^{\mathrm{loc}} \equiv \mathrm{Tr}_k P_{k-1,k} P_{k,k+1} = |0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2|$$

different kernels

Sketch of the proof (IV): no dead-end

Iterative randomization model



For every site k,
- apply random trial unitary
- measure $P_{k-1,k}$

Proposition Local topological order implies that, at any iteration k, there exists an eligible unitary.

Proof (contrapositive).

Dead end at step k $\forall U_k \; P_{k-1,k} U_k |\psi\rangle = 0$

Average over Haar measure $P_{k-1,k} \left(\mathrm{Tr}_k [\psi] \otimes I_k / D \right) = 0$

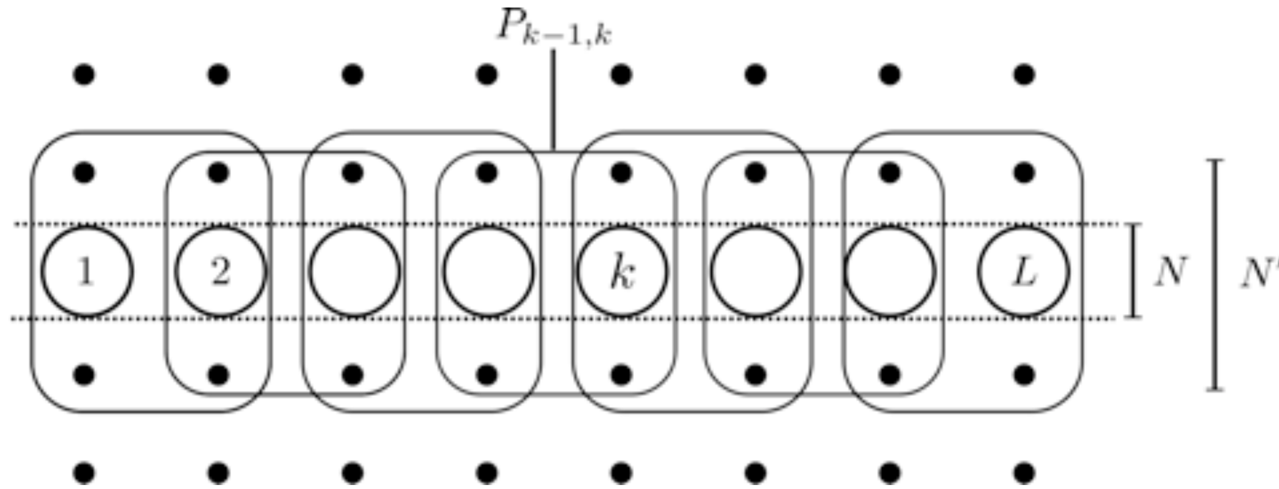
Trace out region at the right of site k $\mathrm{Tr}_k [P_{k-1,k}] \, \mathrm{Tr}_{R_k} [\psi] = 0$

Exists state in image of $P_{i-1,i}$ for i<k and in kernel of $\mathrm{Tr}_k P_{k-1,k}$
Violation of local consistency for site k-2.

□

Sketch of the proof (IV): expected number of trials

Iterative randomization model



For every site k,

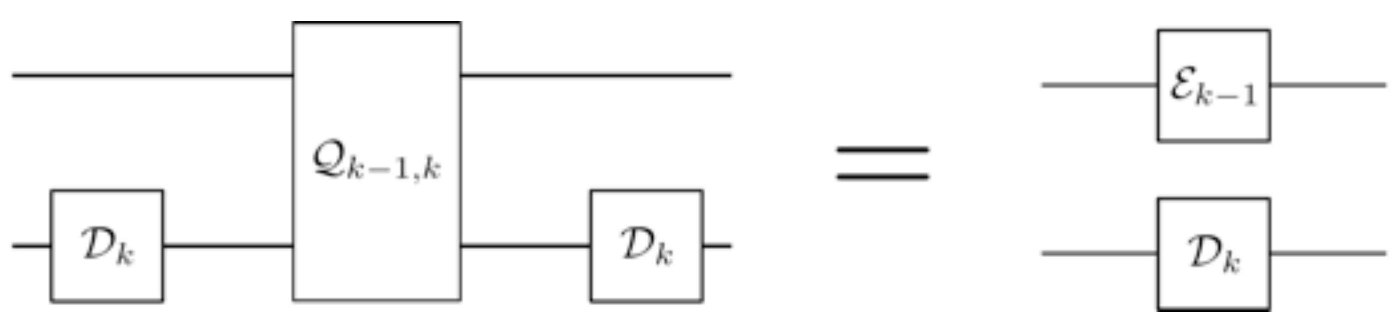
- apply random trial unitary
- measure $P_{k-1,k}$

Proposition Local topological order implies that, the expected # of trials at iteration k is a constant.

Proof. Introduce maps

- successful measurement of $P_{k-1,k}$ $\mathcal{P}_{k-1,k}$
- failed measurement of $P_{k-1,k}$ $\mathcal{Q}_{k-1,k}$
- depolarizing of site k $\mathcal{D}_k$

Biasing map



Succes after m failed trials $\mathcal{P}_{k-1,k}\mathcal{D}_k\left(\mathcal{Q}_{k-1,k}\mathcal{D}_k\right)^m = \mathcal{P}_{k-1,k}\left(\mathcal{E}_{k-1}^m \otimes \mathcal{D}_k\right)$

Expected # of trials

$$A_k(\psi) = \sum_{m=1}^{\infty}(m+1)\mathrm{Tr}\left[\mathcal{P}_{k-1,k}\left(\mathcal{E}_{k-1}^m \otimes \mathcal{D}_k\right)[\psi]\right]$$
$$= \mathrm{Tr}\left[\mathcal{P}_{k-1,k}\left(\left(\mathcal{I}_{k-1} - \mathcal{E}_{k-1}\right)^{-2} \otimes \mathcal{D}_k\right)[\psi]\right] \quad \square$$

Sketch of the proof (V): equivalence between models

Fortuitous model

- depolarize every site on the strip N
- apply arbitrary transformation
- project back onto the code

Iterative randomization model

For every site k,

- apply random trial unitary
- measure $P_{k-1,k}$

Proposition Both models have same average effect.

Proof. Average effect of iterative randomization model.

Average effect of iteration k

$$\mathcal{K}_{k-1,k} = \sum_{m=0}^{\infty} \mathcal{P}_{k-1,k}\left(\mathcal{E}_{k-1}^{m} \otimes \mathcal{D}_k\right)$$
$$= \mathcal{P}_{k-1,k}\left(\left(\mathcal{I} - \mathcal{E}_{k-1}\right)^{-1} \otimes \mathcal{D}_k\right)$$

Average total effect

$$\mathcal{K} = \prod_{k=2}^{L} \mathcal{P}_{k-1,k}\left(\left(\mathcal{I} - \mathcal{E}_{k-1}\right)^{-1} \otimes \mathcal{D}_k\right)\mathcal{D}_1$$

Reorder terms

$$\mathcal{K} = \boxed{\prod_{k=2}^{L} \mathcal{P}_{k-1,k}} \boxed{\prod_{k=2}^{L+1} \left(\mathcal{I} - \mathcal{E}_{k-1}\right)^{-1}} \boxed{\prod_{k=1}^{L} \mathcal{D}_k}$$

projection onto the code | bias | depolarize

□

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Discussion

- Towards a better definition of self-correction
- Topologically ordered 2D Hamiltonian -> Anyons?

Towards a better definition of self-correction

I) Entropy plays a critical role...

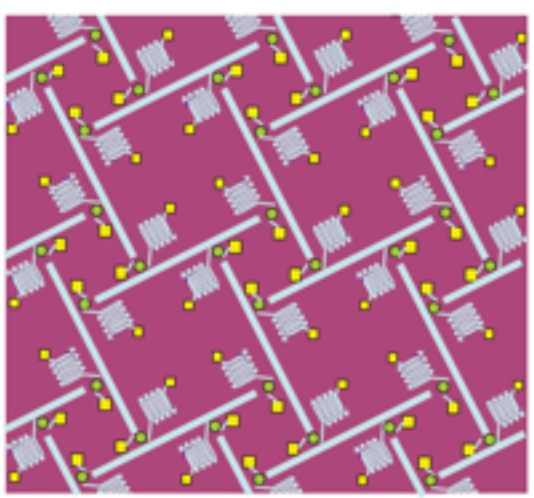
ex: 2D ferromagnetic Ising model

Energy barrier: $\mathcal{O}(L)$

Available energy, assuming constant density of defects: $\mathcal{O}(L^2)$

Non-zero temperature: minimization of free energy E-TS

II) Distinction between self-correction and active QEC?



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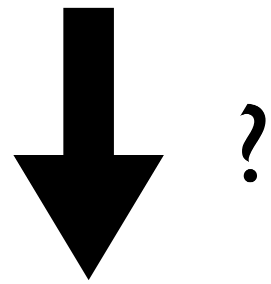
= self-correcting memory?

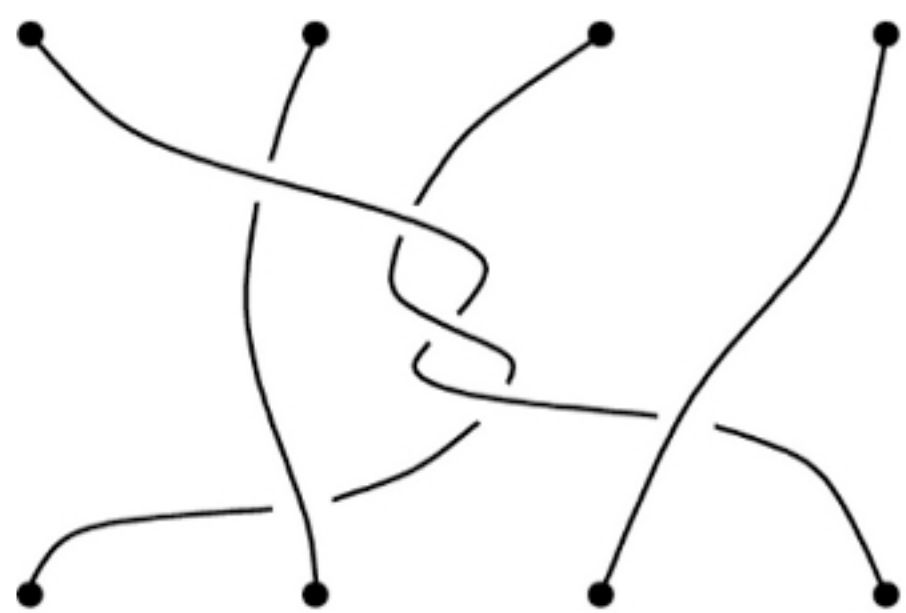
Topologically-ordered 2D Hamiltonian implies anyons?

2D Local commuting projectors code + TQO

$$H = - \sum_{X \subset V} P_X$$



Anyons model



L. Cincio and G. Vidal. arXiv:1208.2623 .

Conclusion

Main result (arXiv:1209.5750)
For any 2D *local topologically ordered* LCP code, we exhibit an physically realistic error model which corrupts the information.

Hope for self-correcting quantum memories

2D Entropy-protected memory
Non-zero temperature: minimization of free energy E-TS
Entropy barrier: few local noise sequences corrupting info.

3D Codes with scalable energy barrier
➡ Haah's cubic code   Haah, PRA, **83** (2011)   Bravyi & Haah, PRL, **107** (2011)
➡ Welded codes   K. Michnicki, arXiv:1208.3496.

Conclusion

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Thank you for your attention.