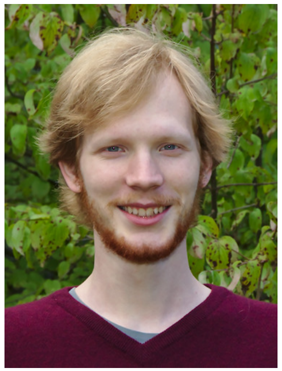
What is topological order?

based on work with Marius Lewerenz, Leander Fiedler, Roman Kossak



The vaguest answer

Start with locality

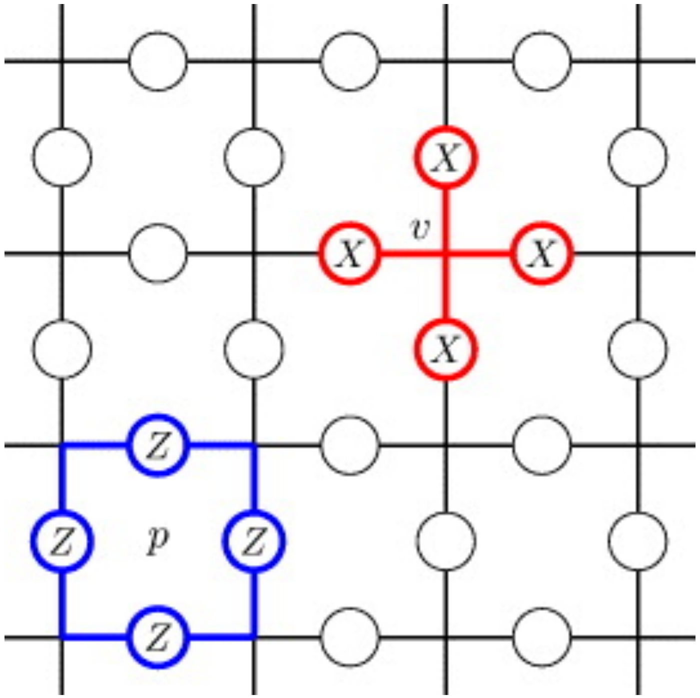
Topology ~ global stuff

Topological order ~ interesting global stuff

Why is it useful?

Global stuff is robust to local deformation

The second vaguest answer

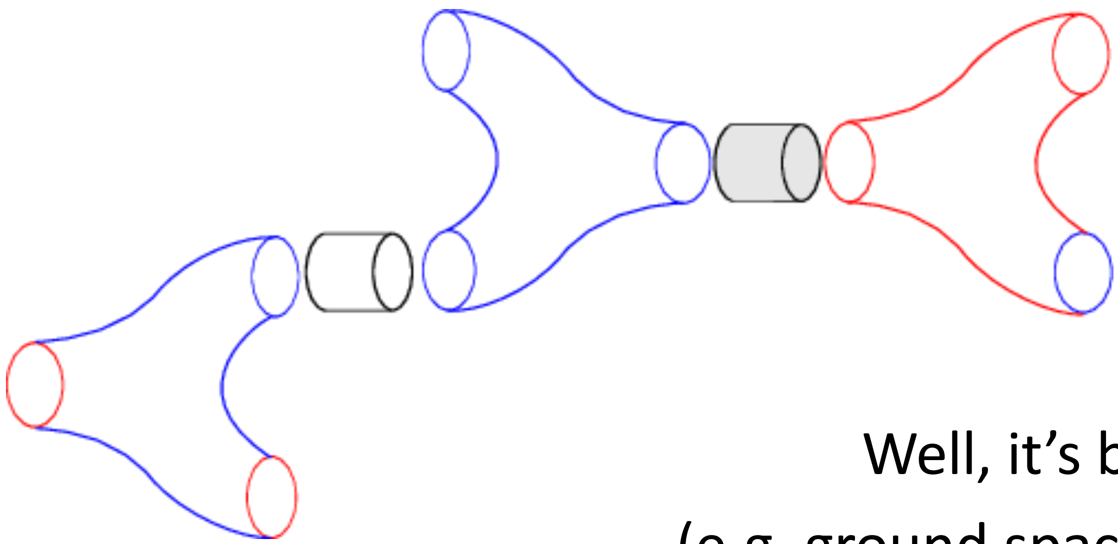


Topological order

=

Things that are like the toric code

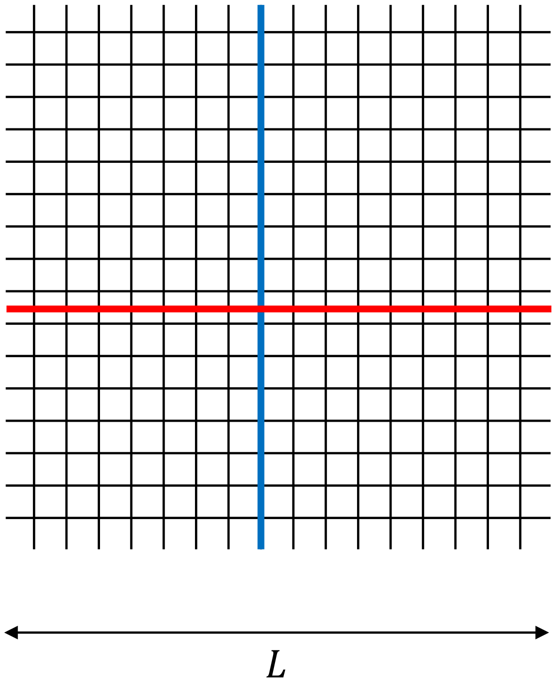
What is topological about the toric code?



Well, it's based on a TQFT

(e.g. ground space and transformations)

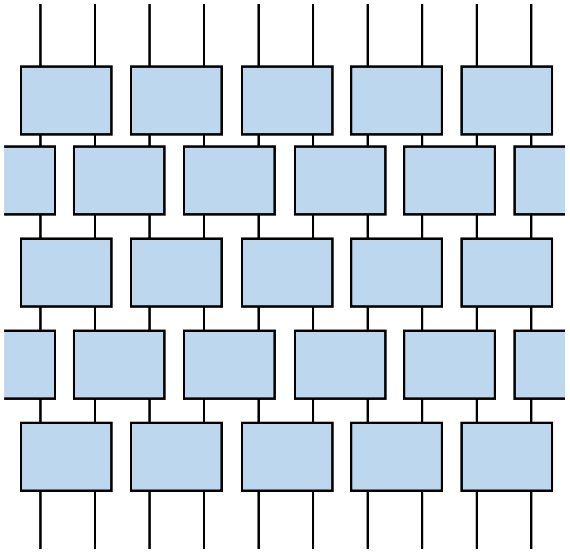
What is topological about the toric code?



It's a macroscopic distance code

$$d = O(L) = O(\sqrt{N})$$

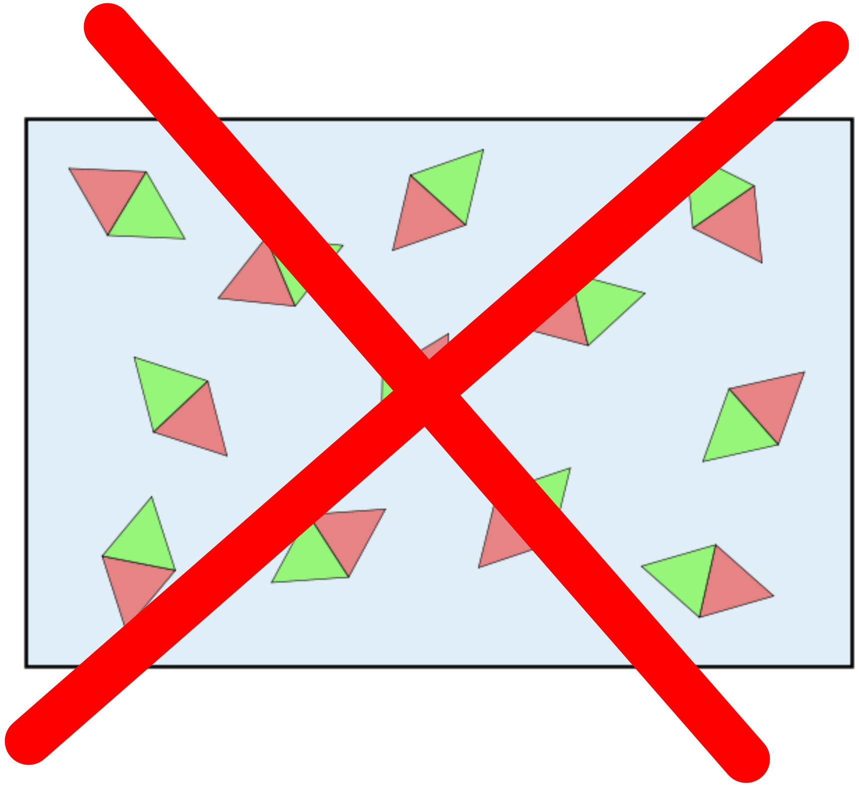
What is topological about the toric code?



It's difficult to prepare

depth $\sim O(L) = O(\sqrt{N})$

What is topological about the toric code?



It’s very different from a paramagnet

$$H = \sum_i \sigma_z^i$$

What is topological about the toric code?

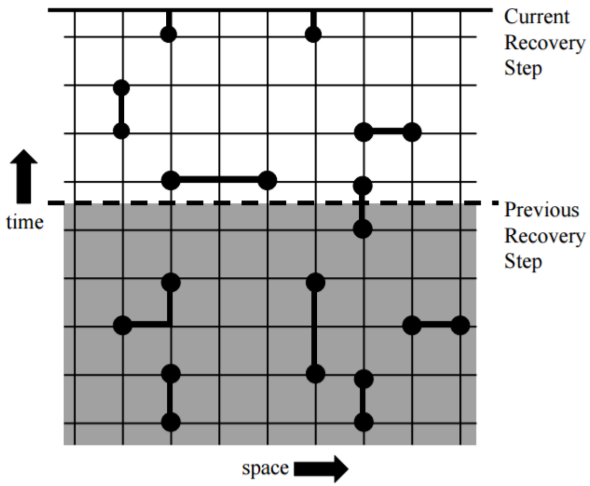
It’s got topological invariants

What is topological about the toric code?

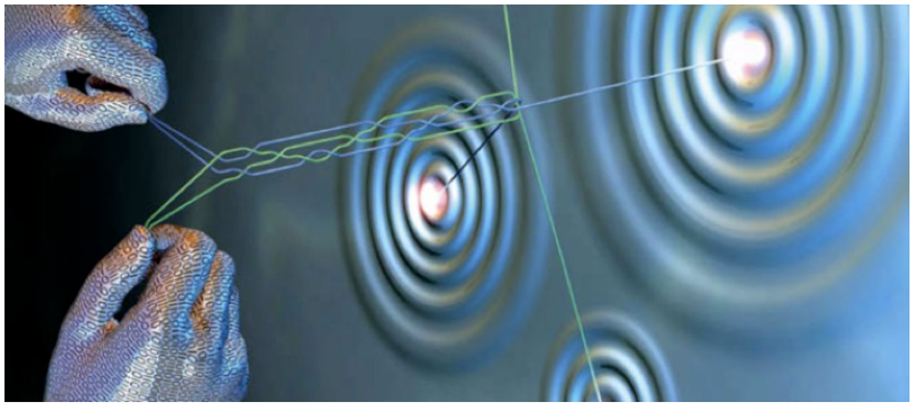
- Lattice TQFT
- Macroscopic distance code
- Hard to prepare
- Not a paramagnet
- Topological invariants

What do we use topological order for?

Topological codes – high fault tolerance thresholds



Topological quantum computation – robust gates



Operational considerations?

Outline

Part I: Zero-temperature

- Classification strategies
- Existing definitions
- Relations
- Subtleties

Part II: Non-zero temperature

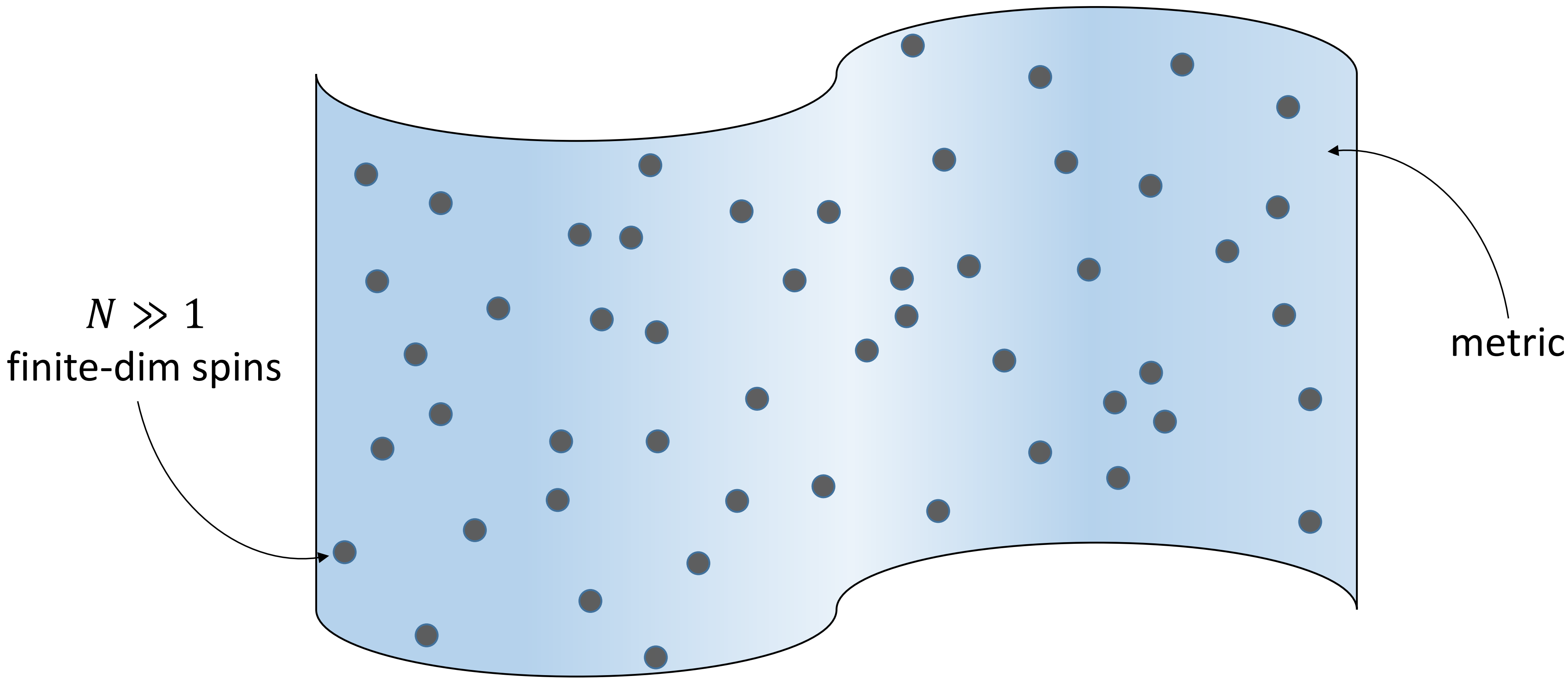
- Existing definitions
- Separating examples

Part I: Zero temperature

aka pure state

aka closed system

General setup



What are we even classifying?

- (Pure) states
  - Arbitrary states vs ground states of gapped local Hamiltonians
- Spaces
  - Arbitrary spaces vs ground spaces of gapped local Hamiltonians
- (Closed) systems
  - Gapped local Hamiltonians

Basic strategies

Direct definition

1) Define topological order
2) Finished

Equivalence relation

1) Define an equivalence relation
2) Define a "topologically trivial" representative
3) Anything that is not topologically trivial is topologically ordered

Definitions

- Lattice TQFTs
- Macroscopic distance code
  - Local indistinguishability
  - LTQO
- Hard to prepare
  - Local unitary circuit equivalence
- Not a paramagnet
  - Gapped path equivalence
- Topological invariants
  - Topological entanglement entropy
  - Haah invariants

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Pros:

- Well developed theory of TQFT
- Easy to motivate
- Lots of interesting examples

Cons:

- Isn't an easily checkable criterion
- Very restrictive definition
  - excludes e.g. Haah code

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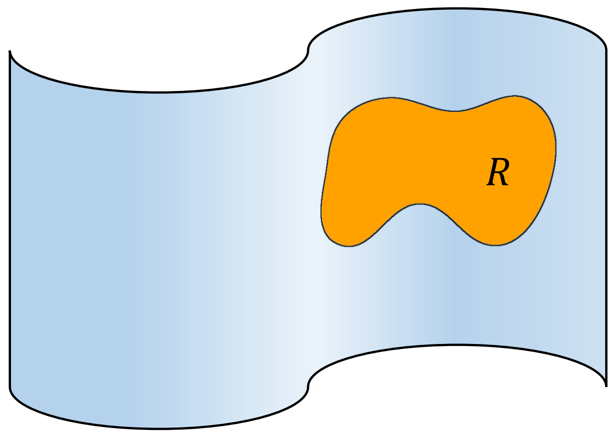
Direct definition on spaces

Local indistinguishability:

$$\langle\psi|\hat{O}_R|\psi\rangle \approx \text{const.}$$



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Pros:

- Operational interpretation: good code against local errors
- Physical consequences: with other conditions, implies perturbative stability

Cons:

- All 1-dim spaces are locally indistinguishable (paramagnet)

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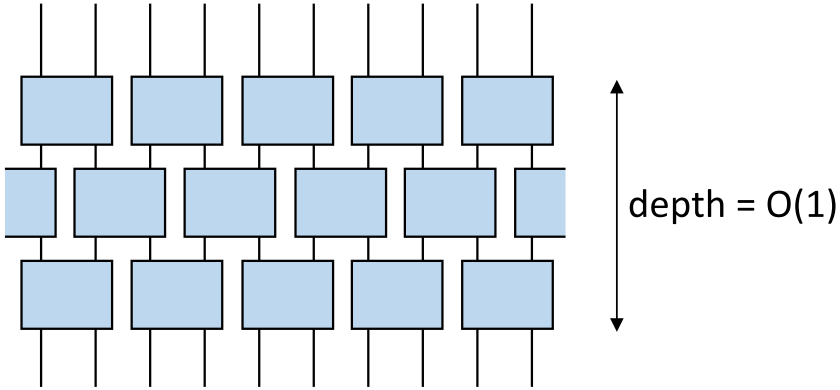
Equivalence relation on states

$$|\psi\rangle \sim |\phi\rangle$$

iff there exists local unitary circuit U

$$|\psi\rangle = U|\phi\rangle$$



$|0\rangle^{\otimes N}$ is trivial

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Pros:

- Defines topological order for all states
- Easy to work with
- Local operators mapped to local operators, global operators mapped to global operators

Cons:

- A little imprecise

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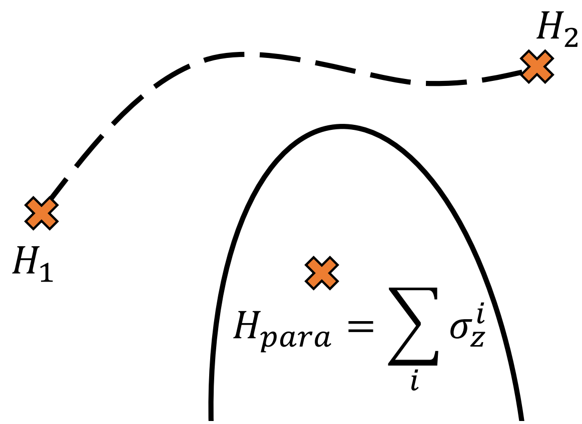
Equivalence relation on Hamiltonians

$$H_1 \sim H_2$$

iff there exists a uniformly gapped smooth path $H(s)$ with

$H_1 = H(0)$ and $H_2 = H(1)$



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Pros:

- Physically motivated: ground state energy is analytic if gap stays open
- Implies existence of quasi-adiabatic continuation between ground spaces ⇒ stability

Cons:

- Only works for gapped Hamiltonians

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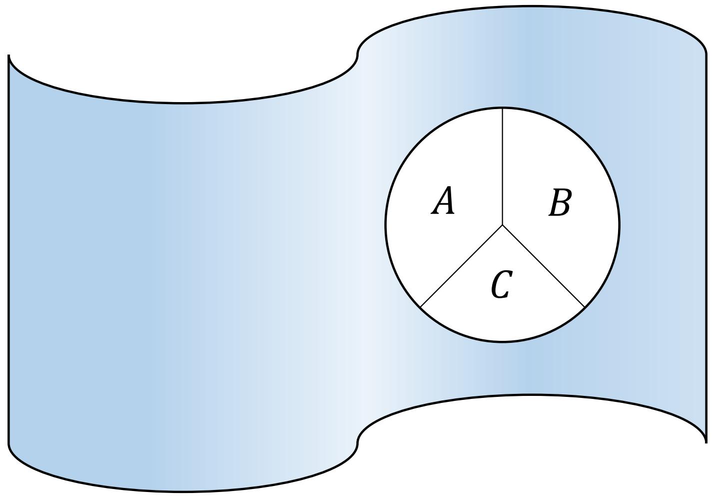
Direct definition on states

Topological entanglement entropy:

$$\gamma = S_A + S_B + S_C - S_{AB} - S_{BC} - S_{CA} + S_{ABC}$$



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Pros:

- Direct motivation from TQFT
- Simple
- Easily verified
- Can distinguish some phases

Cons

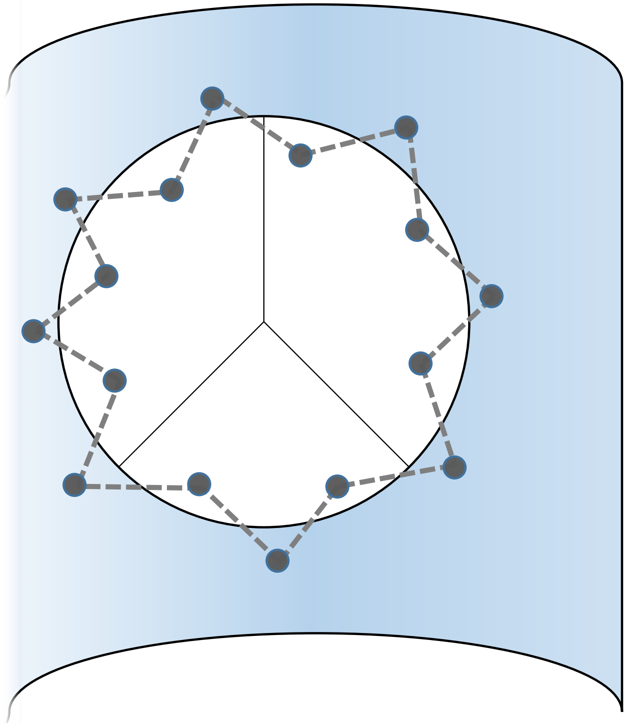
- Not always well-defined
- Can be tricked

Aside: fixing up the TEE

Bravyi counterexample strings give anomalous contribution to TEE.

If we average over the whole phase, this should wash out.

But what is a sensible measure over a phase?



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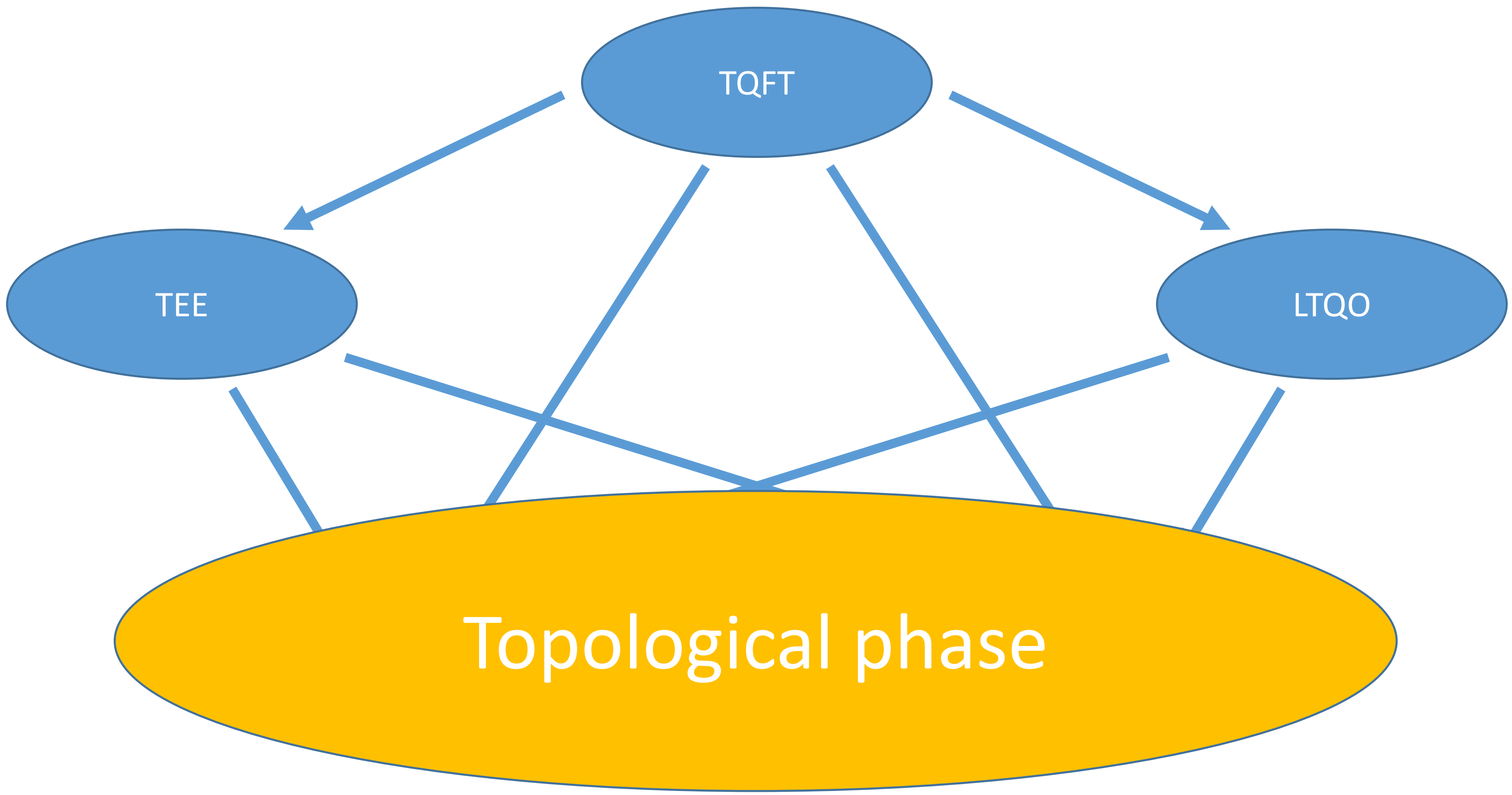
Relations between definitions

- A TQFT has nonzero TEE
- A TQFT satisfies LTQO
- TEE, if properly defined, should be invariant under local unitaries and gapped paths
- A TQFT is local-unitary non-trivial
- A TQFT is gapped-path non-trivial
- Local indistinguishability is preserved under local unitaries and gapped paths
- Local unitaries and gapped paths, properly defined, should be equivalent (on ground states of gapped Hamiltonians)

(Idealized) relations between definitions



From gapped paths to local circuits

Definition of Quasi-Adiabatic Continuation

We introduce the unitary operator

$$\tilde{V}(s) \tag{17}$$
$$= \mathcal{S}' \exp\left\{-\int_0^s \mathrm{d}s' \int_0^\infty \mathrm{d}\tau e^{-(\tau/t_q)^2/2}[\tilde{u}^+_{s'}(i\tau) - h.c.]\right\},$$

where the symbol $\mathcal{S}'$ denotes that the exponential is S'-ordered, in analogy to the usual time ordered or path ordered exponentials. We define $u_s = \partial_s \mathcal{H}_s = \sum_i u^i_s$, and define $\tilde{u}^+_s(i\tau)$ following Ref. [27]: for any operator A

$$\tilde{A}(t) \equiv A(t) \exp[-(t/t_q)^2/2], \tag{18}$$

$$\tilde{A}^{\pm}(\pm i\tau) = \frac{1}{2\pi} \int \mathrm{d}t\, \tilde{A}(t) \frac{1}{\pm it + \tau}. \tag{19}$$

From gapped paths to local circuits

- This is basically evolution under a (quasi)-local Hamiltonian
- We can just Trotterize!

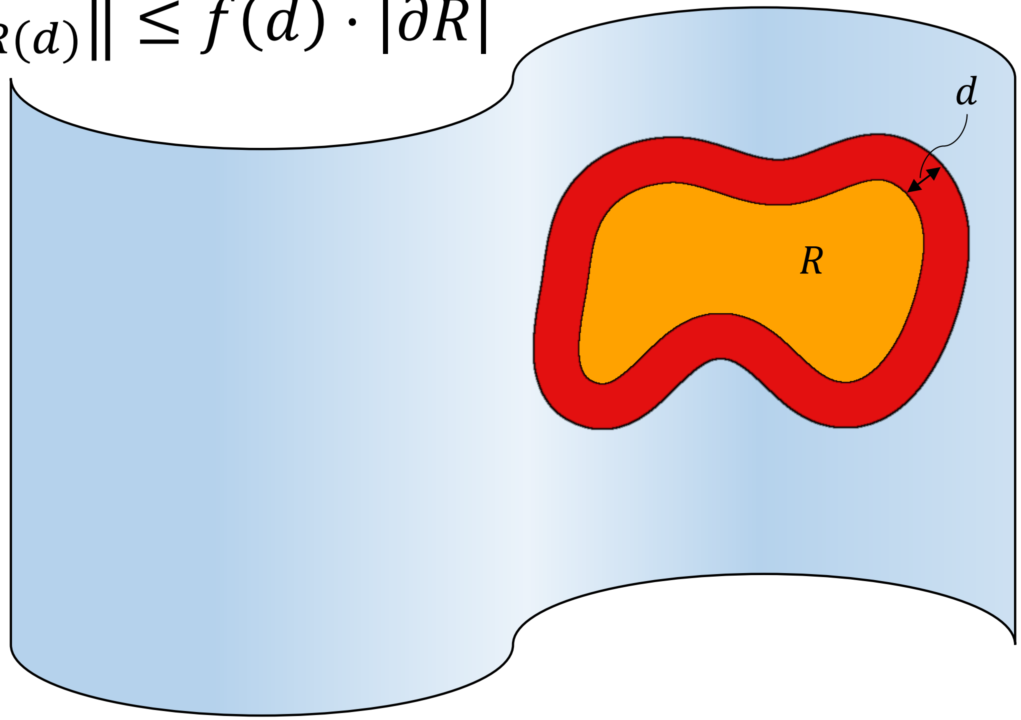
… up to errors

… which don’t play nicely

Controlling errors

We can approximate a quasi-adiabatic continuation by breaking it into strictly local pieces:

$$\left\| V - (V_R \otimes V_{R^c}) \cdot V_{\partial R(d)} \right\| \leq f(d) \cdot |\partial R|$$



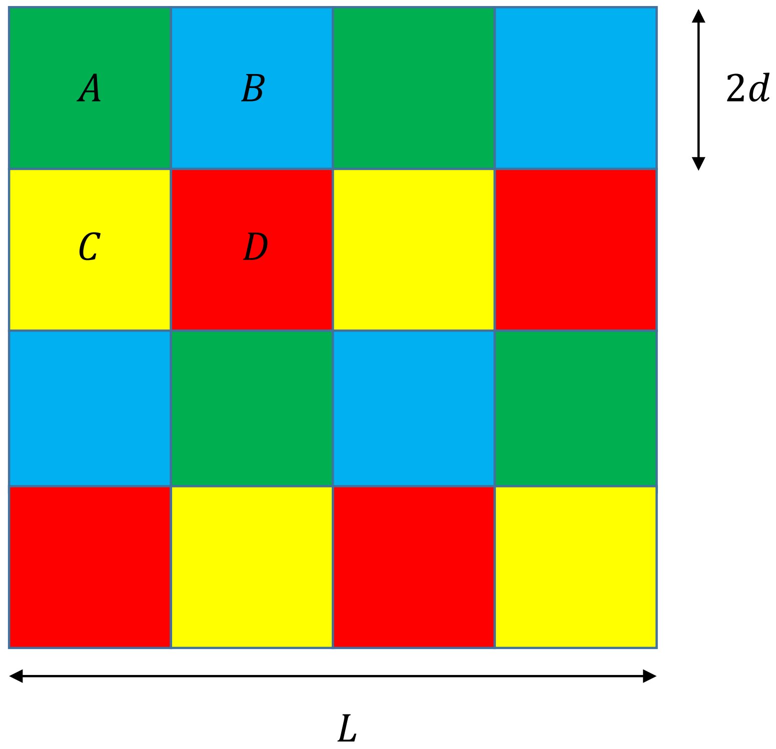
Controlling errors

Break up the (2d) lattice into 4 pieces

Recursively split up the QAC

Now everything is local

Error is under control

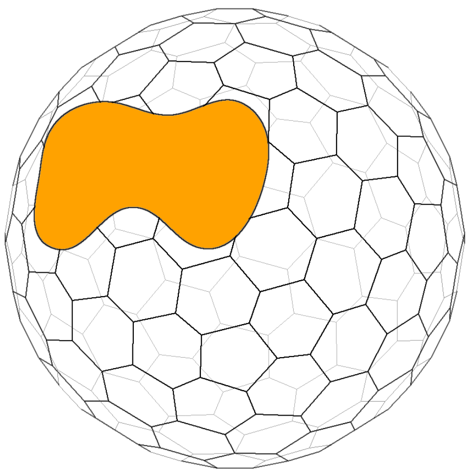


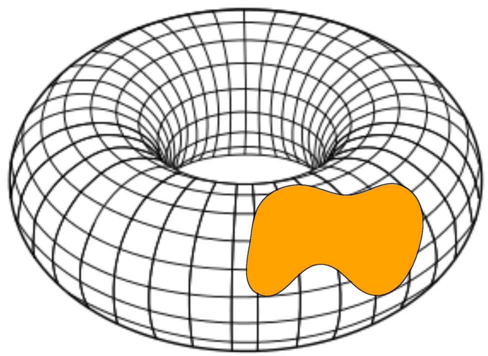
Aside: an application

- The procedure only uses Lieb-Robinson bounds
- Equally well applies to simulation of local Hamiltonian dynamics
- Previously known in 1D, now in any finite dim Euclidean space

Topology sensitivity

- Should the toric code on the sphere be in the same phase as the toric code on the torus?





- Equivalence up to local indistinguishability = equivalence of local patches

Notable generalizations

- Add symmetries to phase classifications
  - SPT / SET
- Quasi-topological order

Part II: Finite temperature

aka mixed state

aka open system

What are we even classifying?

- Mixed states
  - Arbitrary states vs thermal states of gapped local Hamiltonians
- Open systems
  - Gapped local Hamiltonians under thermalizing dynamics
  - General Liouvillians

What should be topologically ordered?

Topologically ordered pure states

Low temperature thermal states of 4D toric code (self-correcting)

Low temperature thermal states of 3D toric code (non-zero TMI)

Thermal states of Haah code

Thermal states of 2D toric code

Definitions

- Not a classical state (Hastings)
- Local purification (Osborne)
- Local CPTP circuits equivalence
- Not a mixture of trivial states (Osborne)
- Topological invariants

Definitions

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ρ is trivial if a spin K_i can be added at each site i, and a local unitary circuit U found such that

$$\rho = \mathrm{tr}_K(U\rho_{cl}U^\dagger)$$

for some “classical” state:

$$\rho_{cl} \propto \exp(-\beta H_{cl})$$

Definitions

- Not a classical state (Hastings)
- Local purification (Osborne)
- Local CPTP circuits equivalence
- Not a mixture of trivial states (Osborne)
- Topological invariants

Pros:

- Tractable criterion
- Often gives “right” answer
- Most of the way to an equivalence relation

Cons:

- Not so easily motivated
- Tailored to thermal states
- Does it always give “right” answer?

Definitions

- Not a classical state (Hastings)
- Local purification (Osborne)
- Local CPTP circuits equivalence
- Not a mixture of trivial states (Osborne)
- Topological invariants

ρ is trivial if there exists a local purification of it that is pure-state trivial

Definitions

- Not a classical state (Hastings)
- Local purification (Osborne)
- Local CPTP circuits equivalence
- Not a mixture of trivial states (Osborne)
- Topological invariants

Pros:

- Easily connects to pure state definitions

Cons:

- Difficult to check
- Does it give “right” answers?

Definitions

- Not a classical state (Hastings)
- Local purification (Osborne)
- Local CPTP circuits equivalence
- Not a mixture of trivial states (Osborne)
- Topological invariants

ρ is trivial if there exists local CPTP circuit $\mathcal{C}$ such that

$$\rho = \mathcal{C}(|0\rangle\langle 0|^{\otimes N})$$

Definitions

- Not a classical state (Hastings)
- Local purification (Osborne)
- Local CPTP circuits equivalence
- Not a mixture of trivial states (Osborne)
- Topological invariants

Pros:

- Connects intuitively with pure-state case
- Half-way to an equivalence relation

Cons:

- Difficult to check
- Does it give “right” answers?

Definitions

- Not a classical state (Hastings)
- Local purification (Osborne)
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- Topological invariants

ρ is trivial if it can be written as a convex combination of topologically trivial pure states

Definitions

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Pros:

- Connects easily to pure-state case
- Operationally motivated

Cons:

- Difficult to check
- Does it give “right” answer?

Definitions

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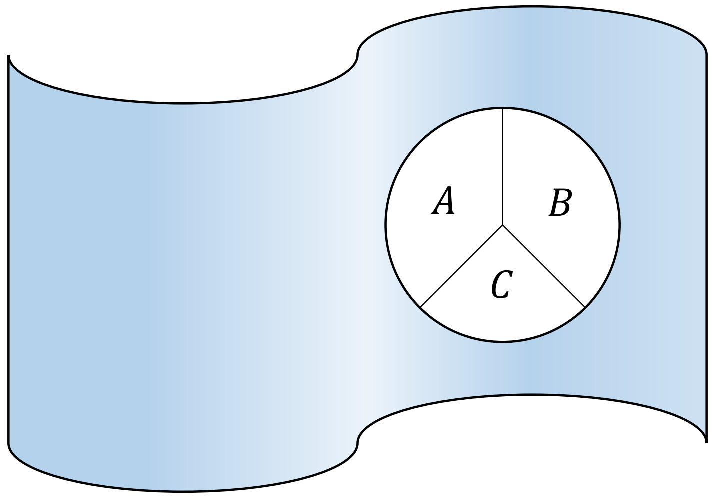
Topological mutual information:

$$\gamma = I_A + I_B + I_C - I_{AB} - I_{BC} - I_{CA} + I_{ABC}$$



Definitions

- Not a classical state (Hastings)
- Local purification (Osborne)
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- Not a mixture of trivial states (Osborne)
- Topological invariants

Pros:

- Easy to calculate
- Close to pure-state condition

Cons:

- Not always well-defined

How can we test these definitions?

- ITO states
- Critical classical states

Invertible topological order

$|\psi\rangle$ is invertibly topologically ordered if it has an "anti-state"

i.e. $|\psi\rangle$ is local-unitary topologically ordered

and

$\exists\ |\phi\rangle$ s.t. $|\psi\rangle \otimes |\phi\rangle$ is local-unitary trivial

Is an ITO state topologically ordered?

- ✓ Pure state definitions

- ✗ Not a classical state (Hastings)
- ✗ Local purification
- ✓ Local CPTP circuits equivalence
- ✓ Not a mixture of trivial states
- ✓ Topological invariants

Classical critical states

e.g. - thermal states of q-clock model at intermediate temperature
- critical thermal states of Ising model

- These states have algebraically decaying correlations
- States with algebraically decaying correlations are difficult to prepare

Is a classical critical state topologically ordered?

~~Pure state definitions~~

- ✗ Not a classical state (Hastings)
- ✓ Local purification
- ✓ Local CPTP circuits equivalence
- ✗ Not a mixture of trivial states
- ✗ Topological invariants

Operational motivations?

	Zero temperature TEE	Finite temperature TMI
2D toric code		
3D toric code		
4D toric code		

Operational motivations?

	Zero temperature TEE	Finite temperature TMI
2D toric code	2	0
3D toric code		
4D toric code		

Operational motivations?

	Zero temperature TEE	Finite temperature TMI
2D toric code	2	0
3D toric code		
4D toric code	2	2

Operational motivations?

	Zero temperature TEE	Finite temperature TMI
2D toric code	2	0
3D toric code	2	1
4D toric code	2	2

Big questions to leave you with

- What do we want topological order to mean?
- How should we characterise topological phases?
- Is there an operational definition for topological order?

Advertisement: Hopf algebras in quantum double models

Mathematical connections from gauge theory to topological quantum computation and categorical quantum mechanics

Co-organizers: Lucy Zhang & Prince Osei

July 31 – August 4