

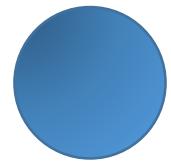
2D SET order in tensor networks

Dominic Williamson

Verstraete Group

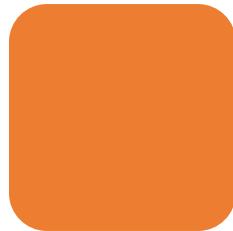
University of Vienna

arXiv:1412.5604
arXiv:1511.08090



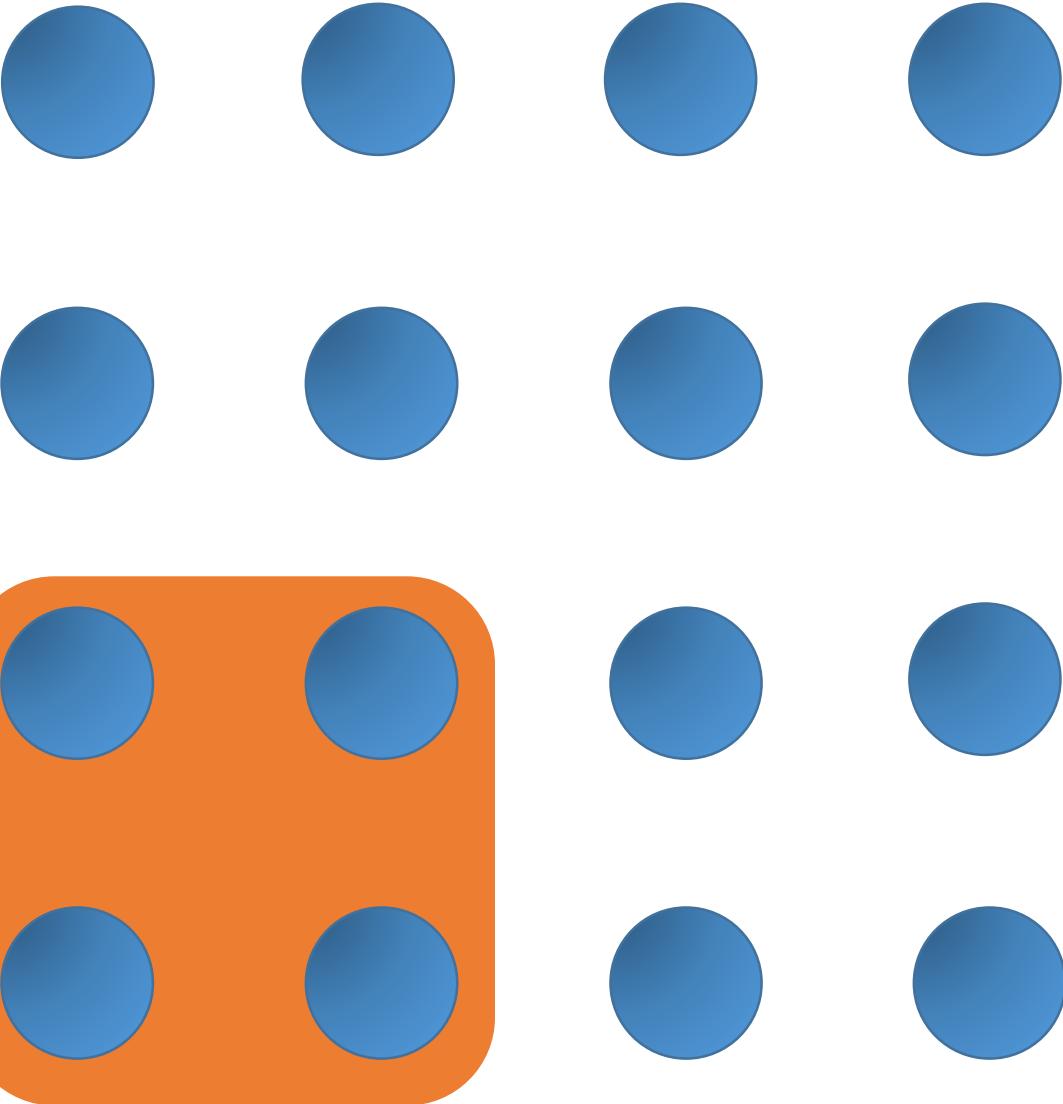
$$\mathbb{C}^d$$

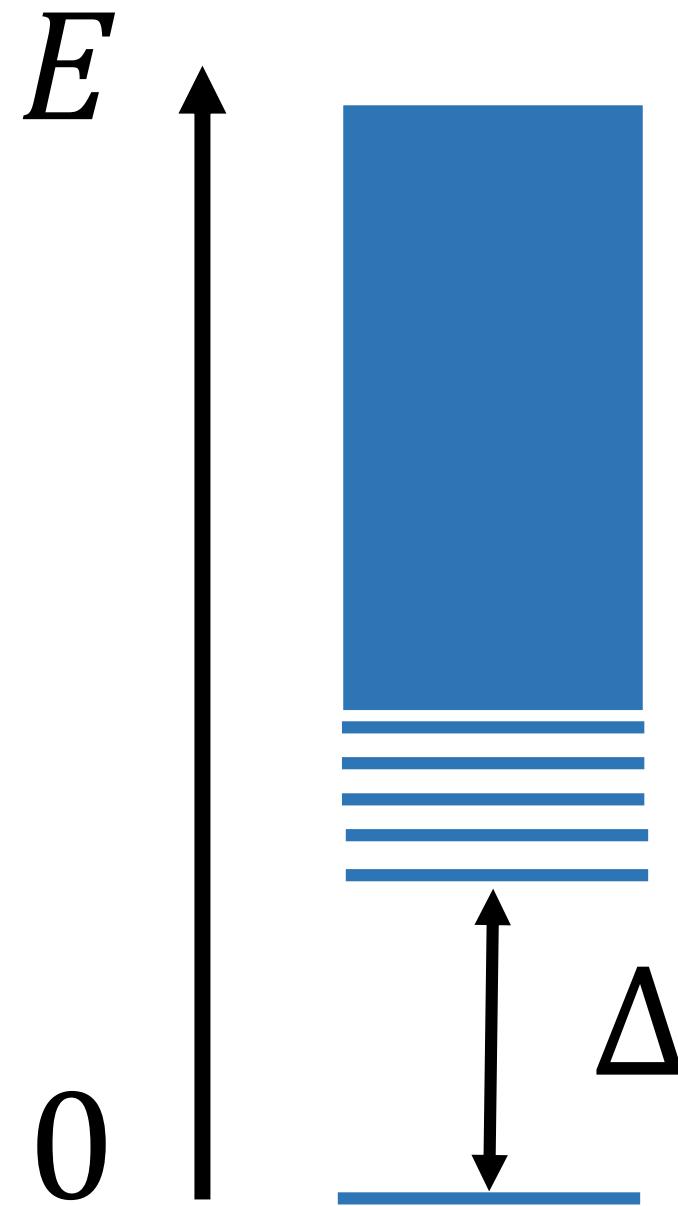
$$\mathbb{H} = \bigotimes_v \mathbb{C}^d$$

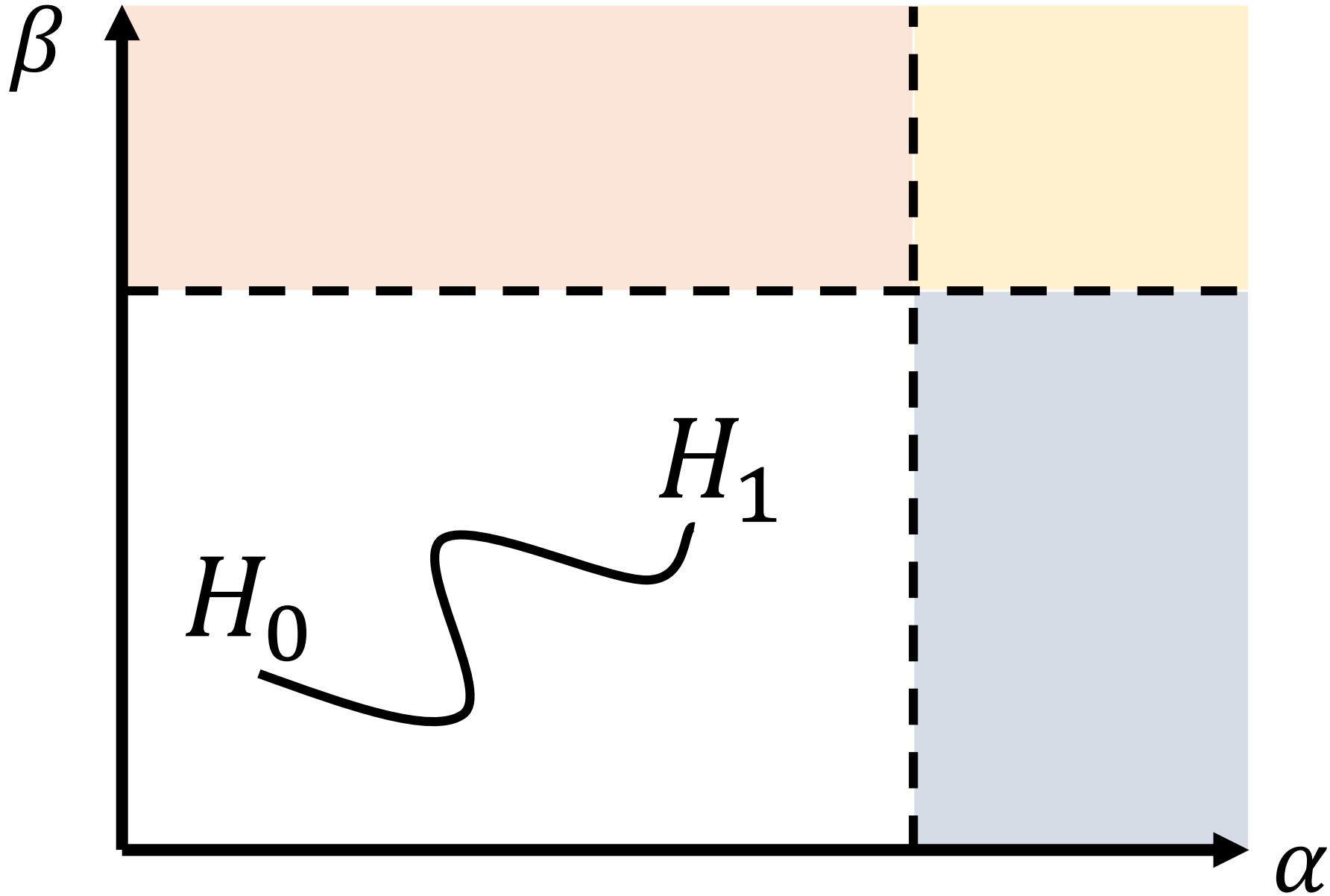


$$h_v$$

$$H = \sum_v h_v$$







$$\text{ПОП} = c(o)\Pi$$

Π -ground space projector
 o -local operator

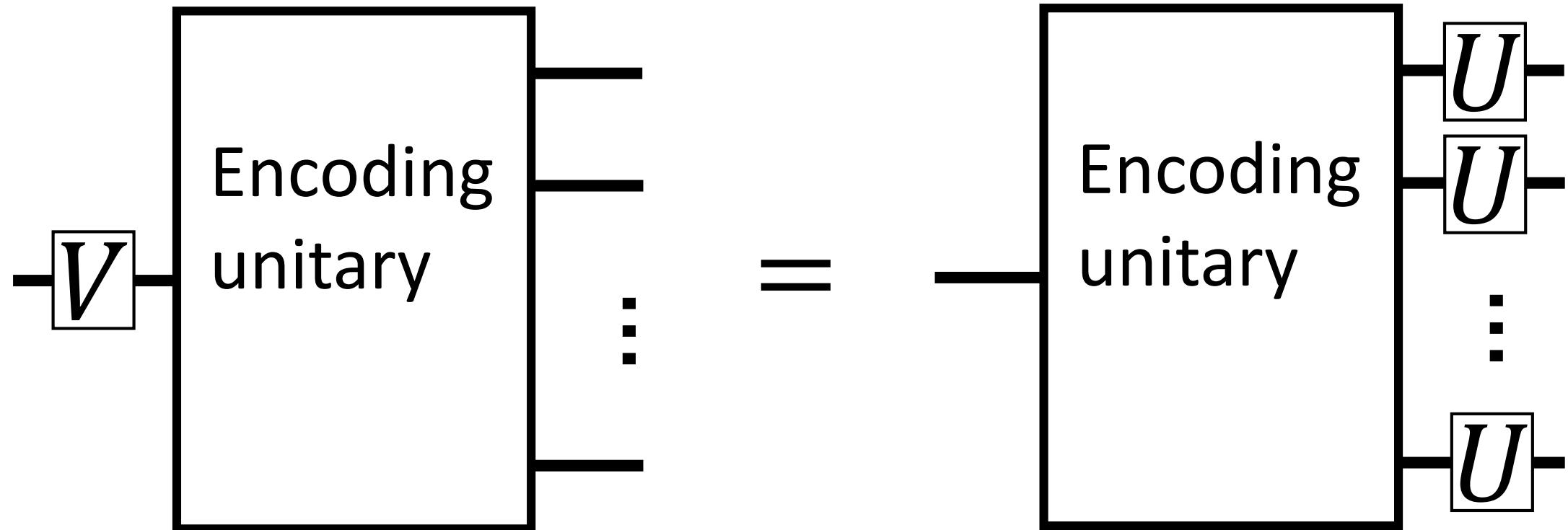
Topological order

Quantum code

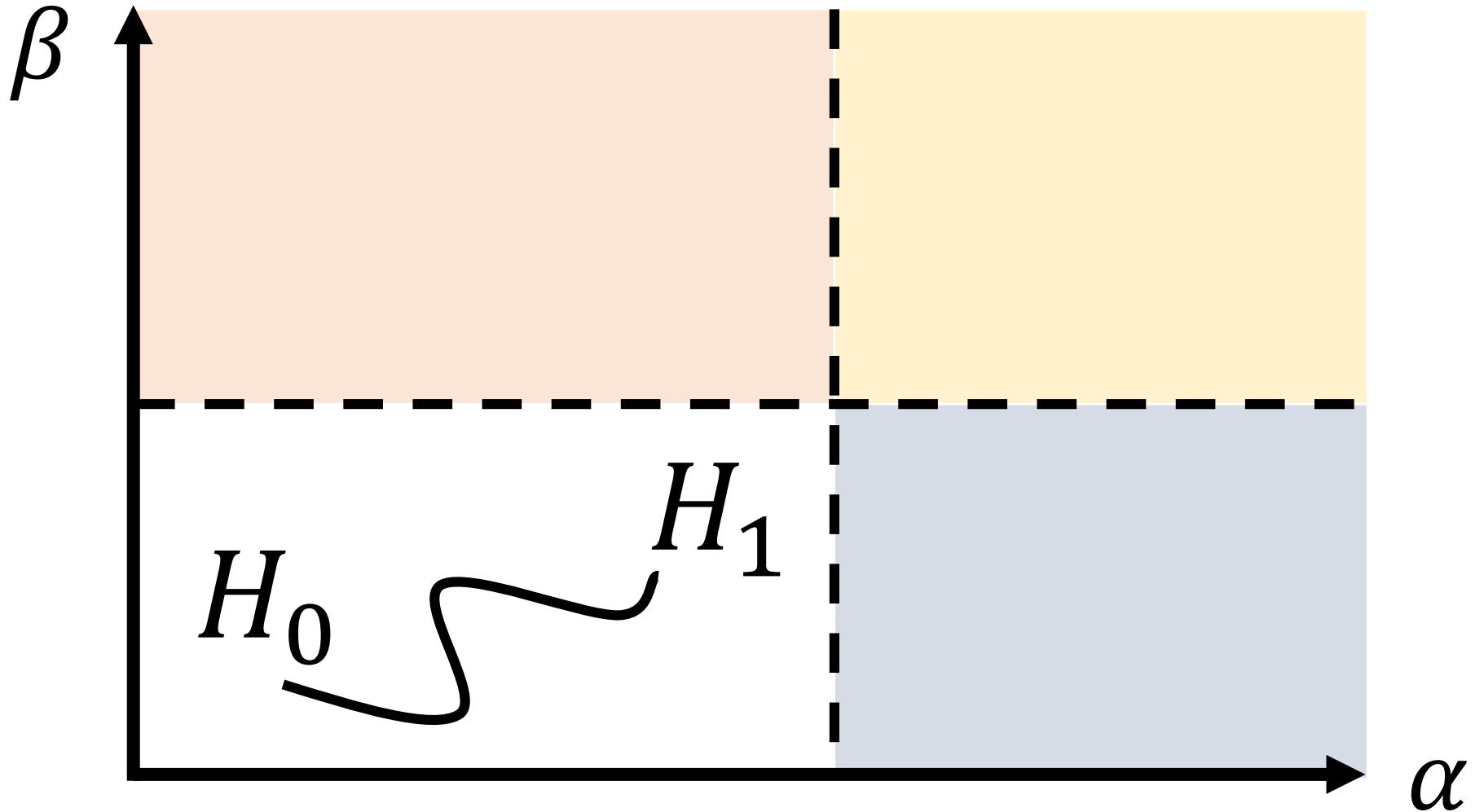
$$\Pi E_b^\dagger E_a \Pi = c_{ab} \Pi$$

Π -code space projector
 E_a -error operator

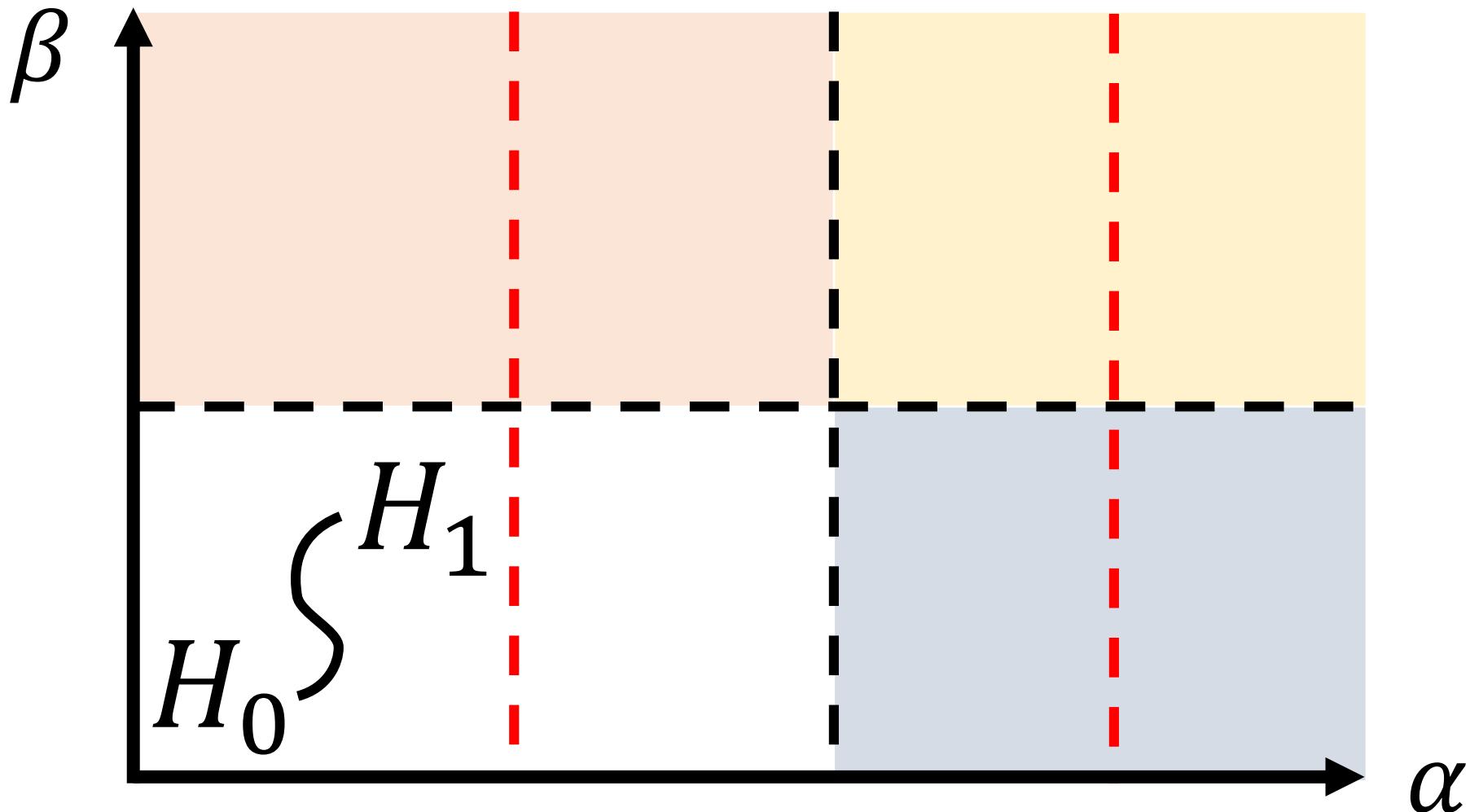
Transversal gates



Transversal \subset Locality preserving



$$H_s = \sum_v h_v^s,$$



$$H_s = \sum_v h_v^s, \quad U_g^{\otimes N}, g \in G, \quad [h_v^s, U_g^{\otimes N}] = 0$$

Transversal Gates on Quantum Codes

Eastin Knill etc.
⇒ Restrictions?



All gates based
on anyon
symmetries can
be made
transversal

Symmetry Enriched Topological order

A Motivating Example

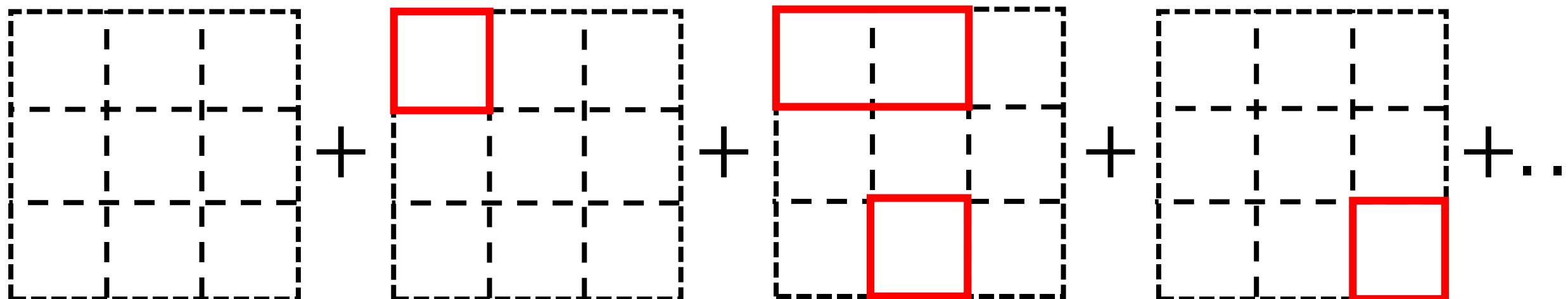
Toric code

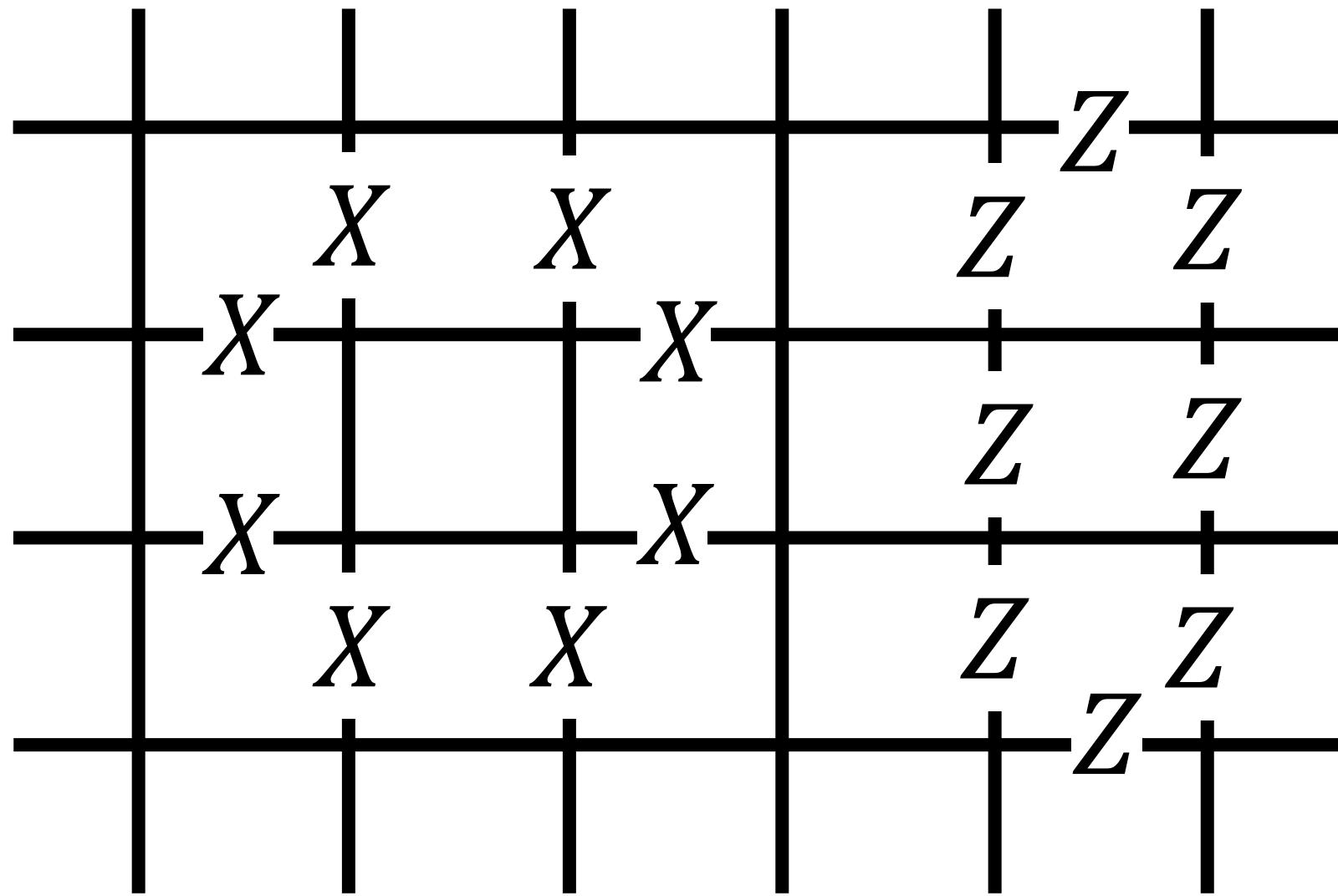
$$\cdots = |+\rangle$$

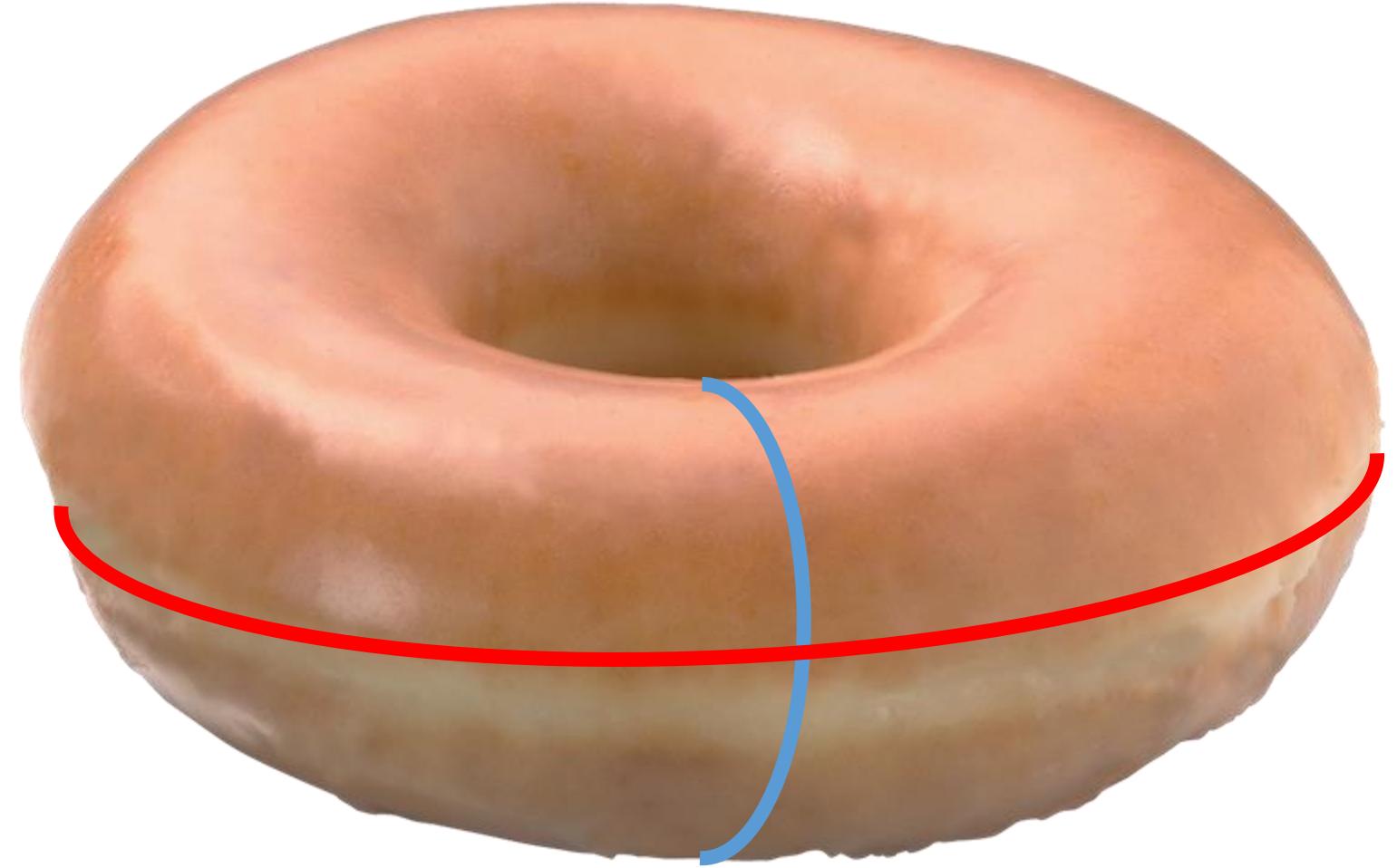
$$\text{---} = |-\rangle$$

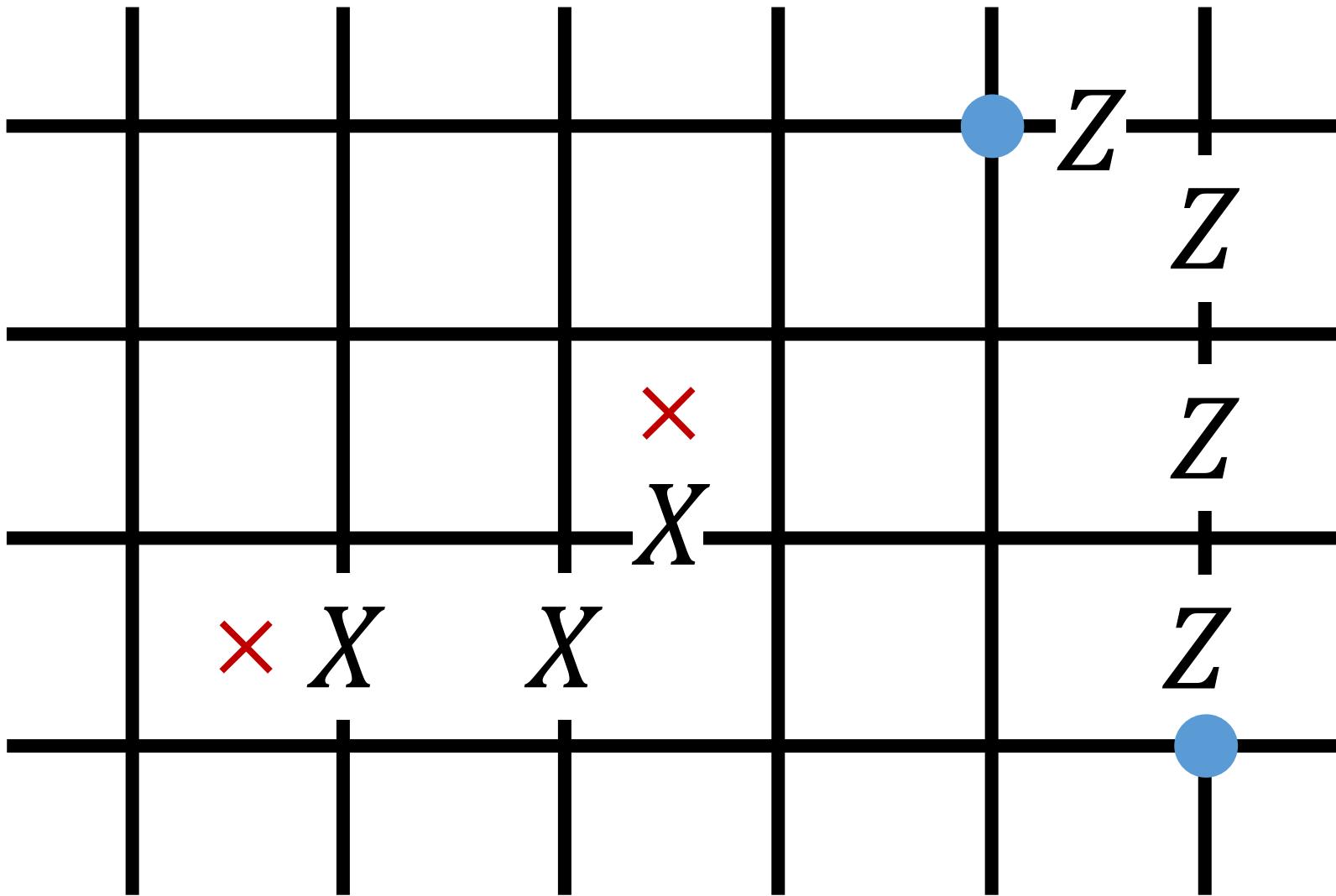
$$H_{TC} = -\sum_p \prod_Z Z^p - \sum_v \prod_X X^v$$

$|\psi_0\rangle =$







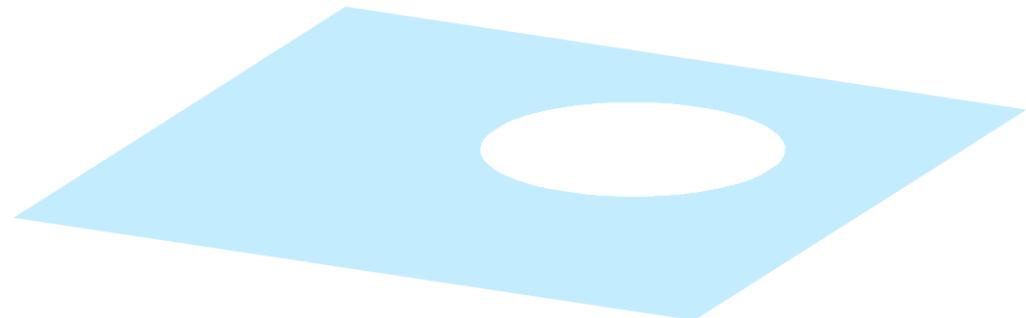
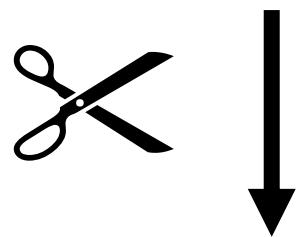
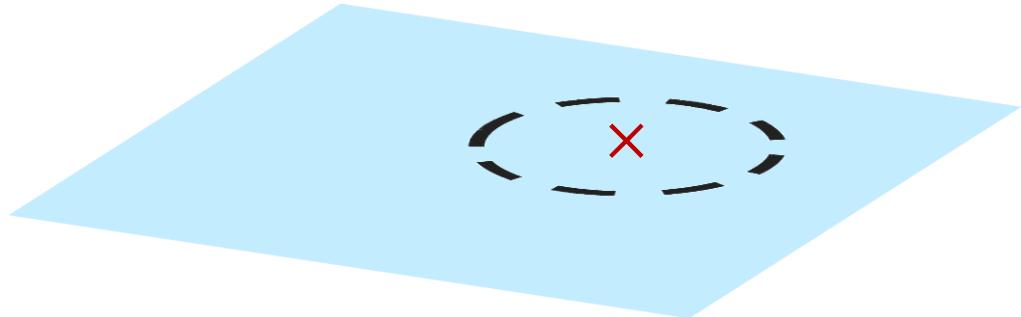
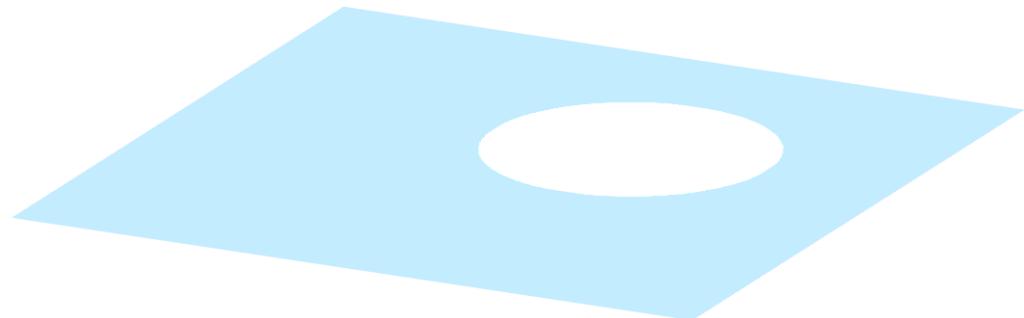
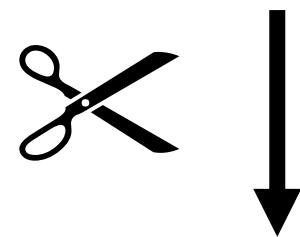
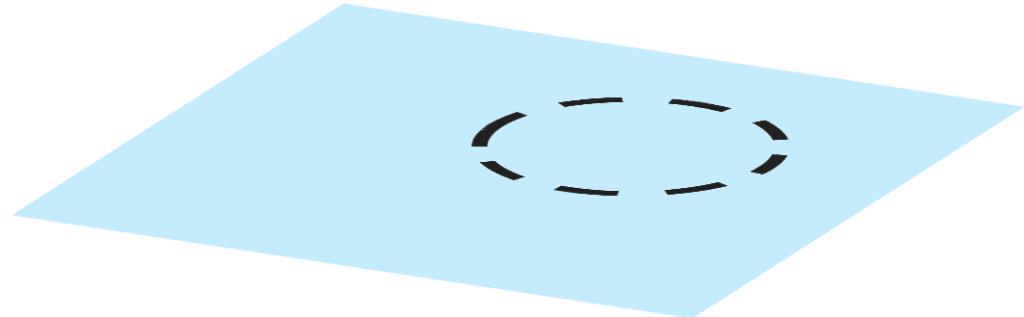


Anyons / Superselection sectors

(Modular Tensor Categories)

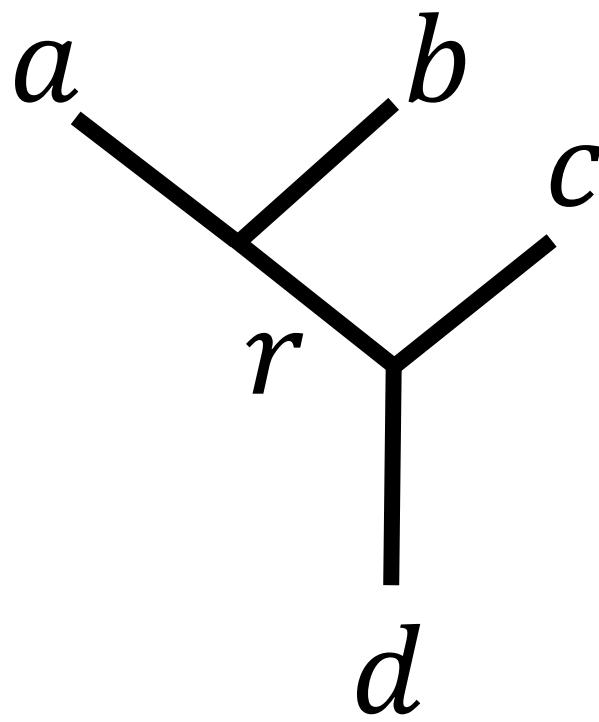
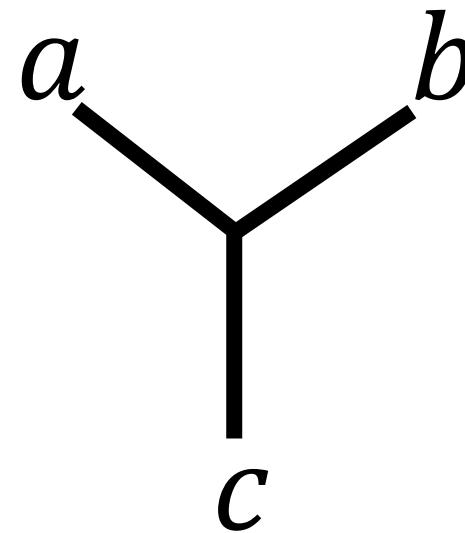
arXiv:cond-mat/0506438 Kitaev

$$[\rho] := \{\rho' \mid \rho' = U\rho U^{-1}, \exists U\}$$

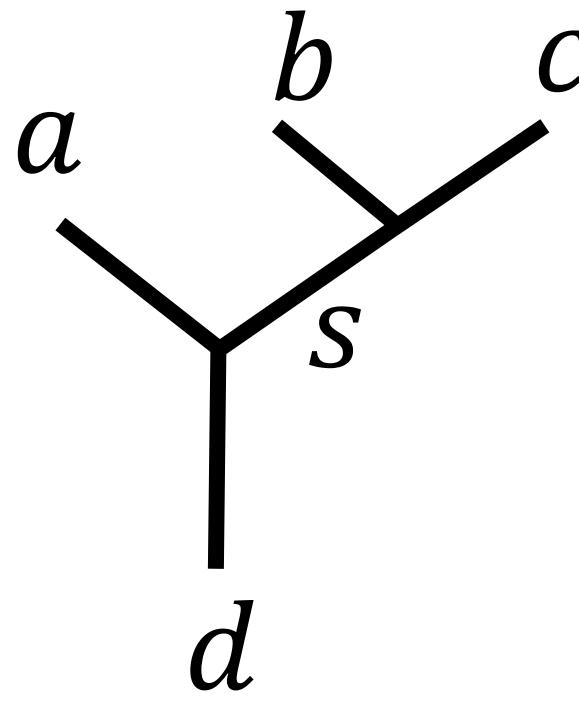
 \neq 

$$a \times b = c + d + \dots$$

$$= \sum N_{ab}^c \ c$$



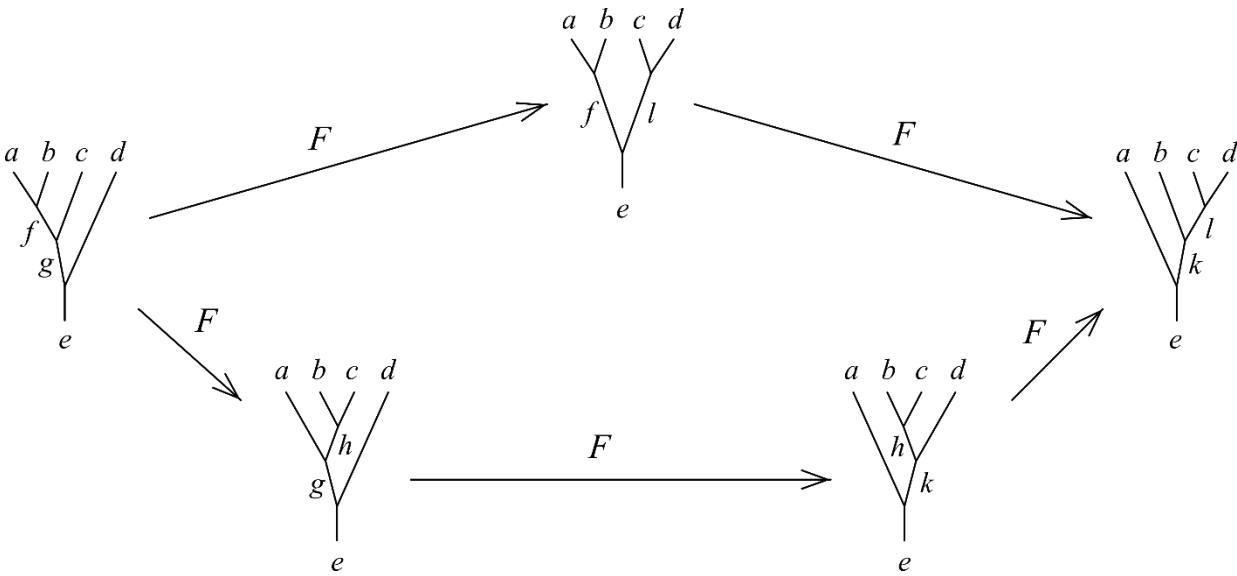
$$= F_{d;rs}^{abc}$$



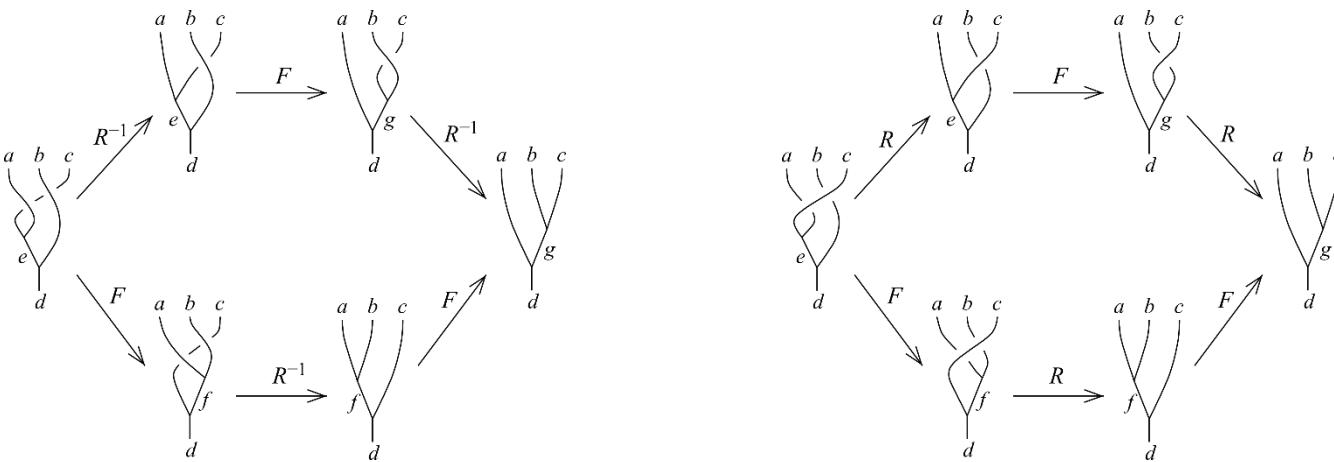
$$R^{ab} = \begin{array}{c} a \\ \diagup \quad \diagdown \\ b \end{array}$$

$$\begin{array}{c} a \\ \diagup \quad \diagdown \\ b \end{array} = R_c^{ab} \begin{array}{c} a \\ \diagup \quad \diagdown \\ b \\ c \end{array}$$

Pentagon



Hexagons



Toric code anyons: $D(\mathbb{Z}_2)$

$$\mathcal{C} = \{1, e, m, em\}$$

$$e \times m = em$$

$$e \times e = m \times m = 1$$

F symbols trivial

$$\begin{aligned} R_{em}^{e,m} &= -1 \\ &= R_m^{e,em} \\ &= R_e^{em,m} \end{aligned}$$

$$U: \begin{array}{c} e \leftrightarrow m \\ em \rightarrow em \end{array}$$

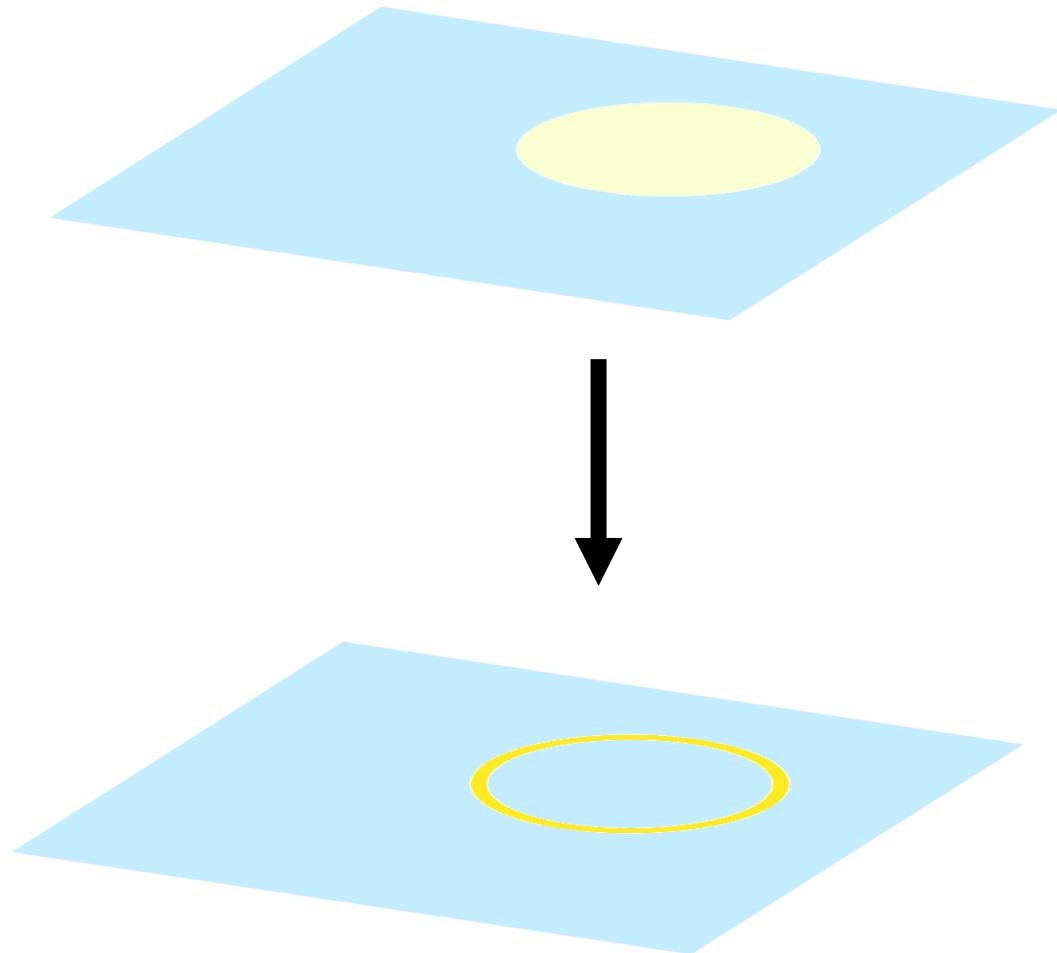
$$U:\begin{array}{c}e\leftrightarrow m\\em\rightarrow em\end{array}$$

$$UH_{TC}U^\dagger=H_{TC},$$

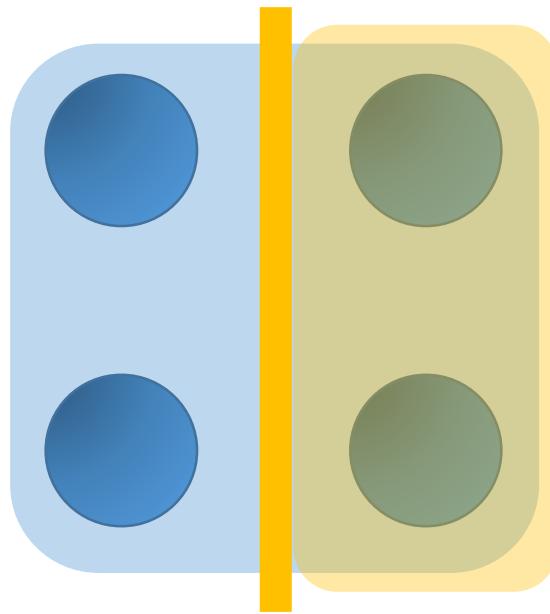
$$U:\begin{array}{c}\lceil^Z \rceil \\ Z \\ \lfloor_Z \rfloor\end{array}\rightarrow\begin{array}{c}\lceil^X \rceil \\ X \\ \lfloor_X \rfloor\end{array}\rightarrow\begin{array}{c}\lceil^X \\ X+X- \\ X \\ \lfloor\end{array}$$

$$U_\mathrm{Torus}=\mathsf{H}\otimes\mathsf{H}\,SWAP$$

Symmetry Domain walls

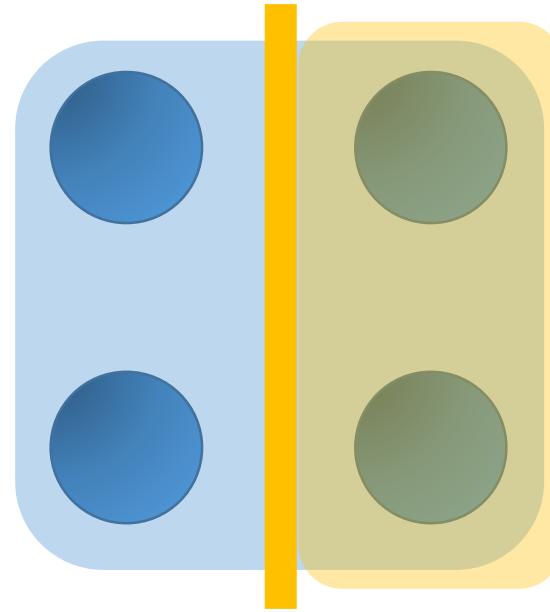
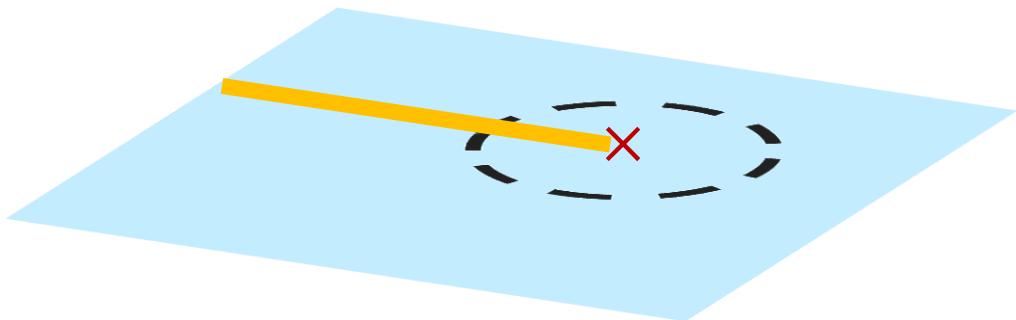


$$h_{|} \rightarrow \pi_g h_{|} \pi_g^{\dagger}$$

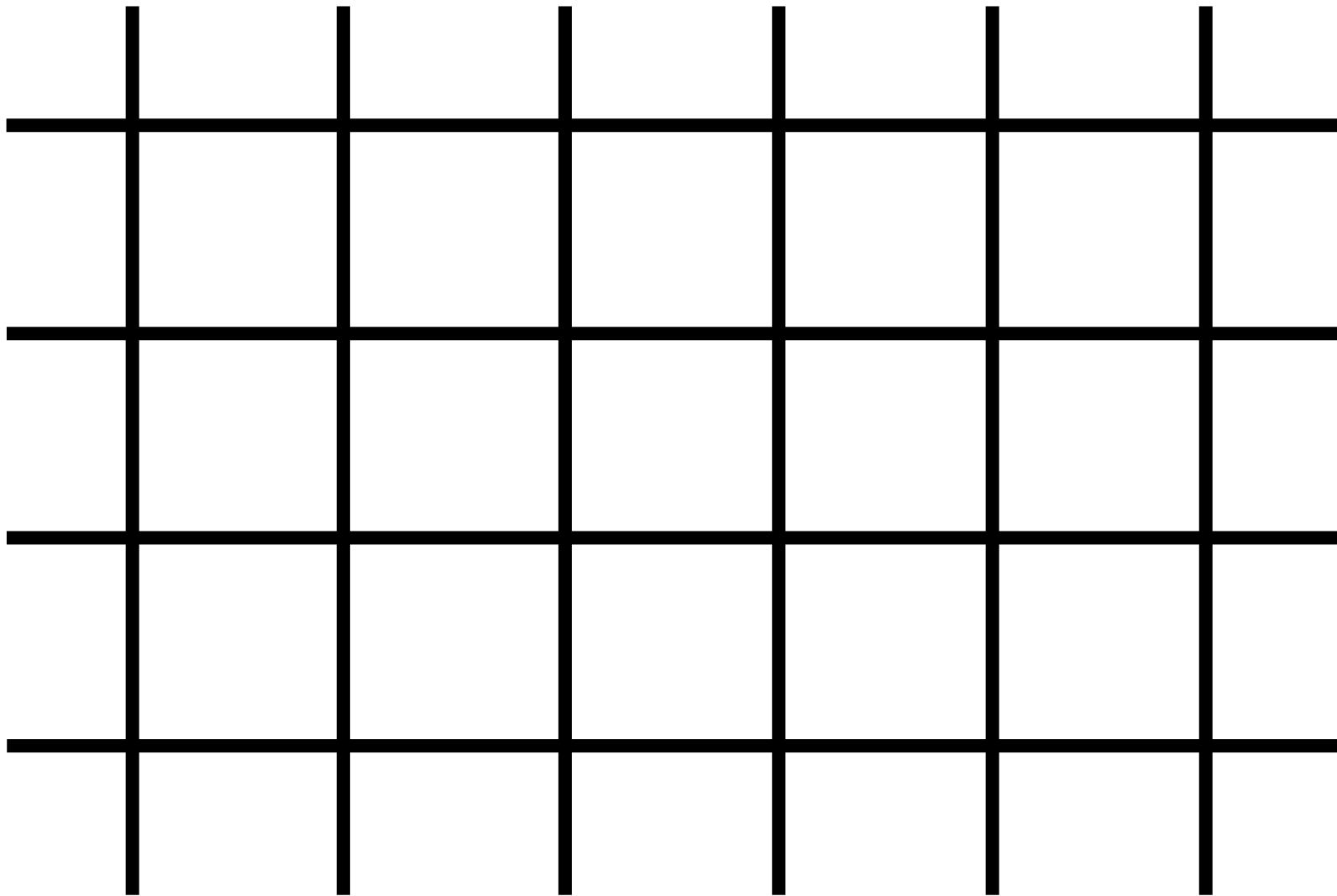


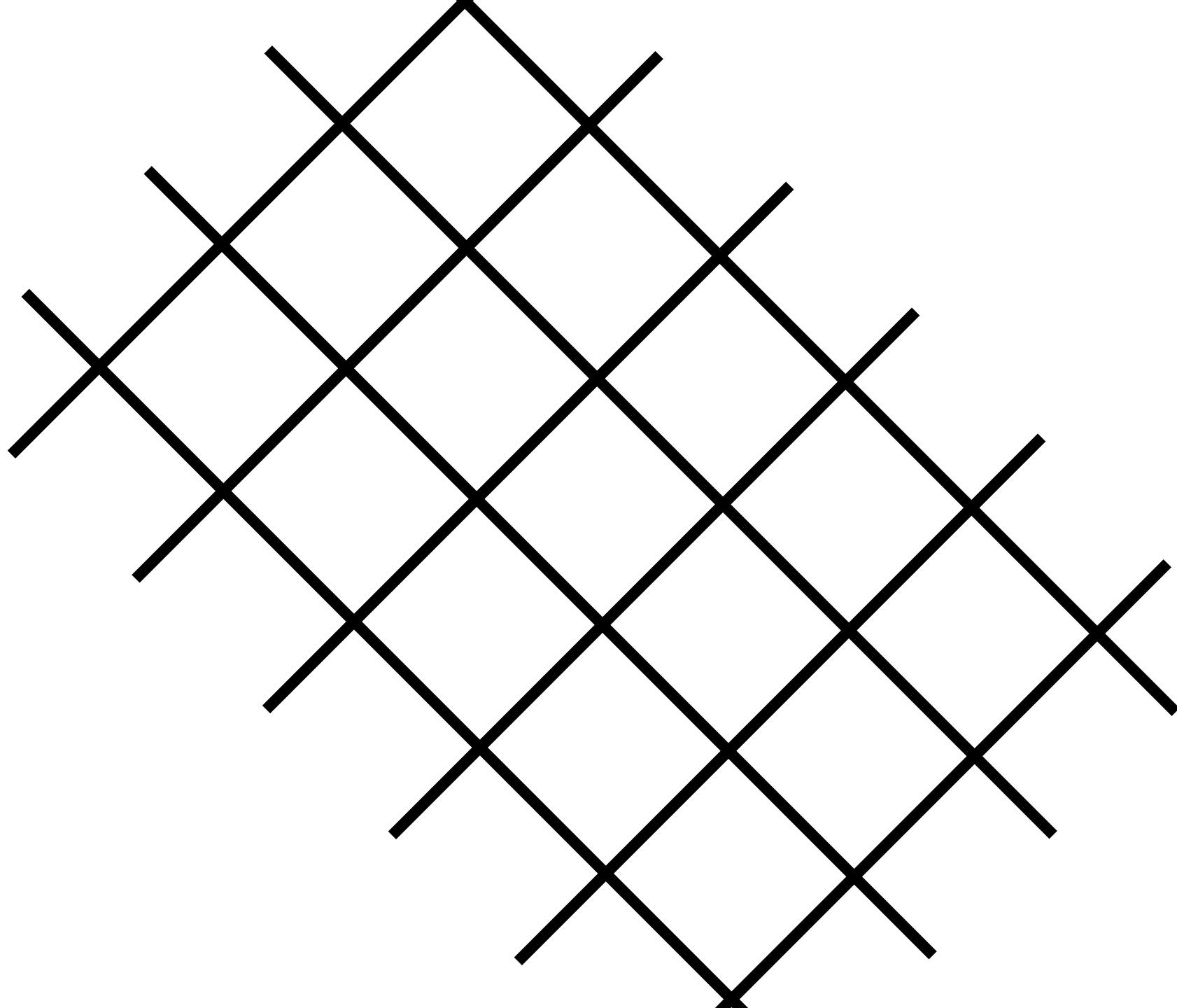
Symmetry Defects

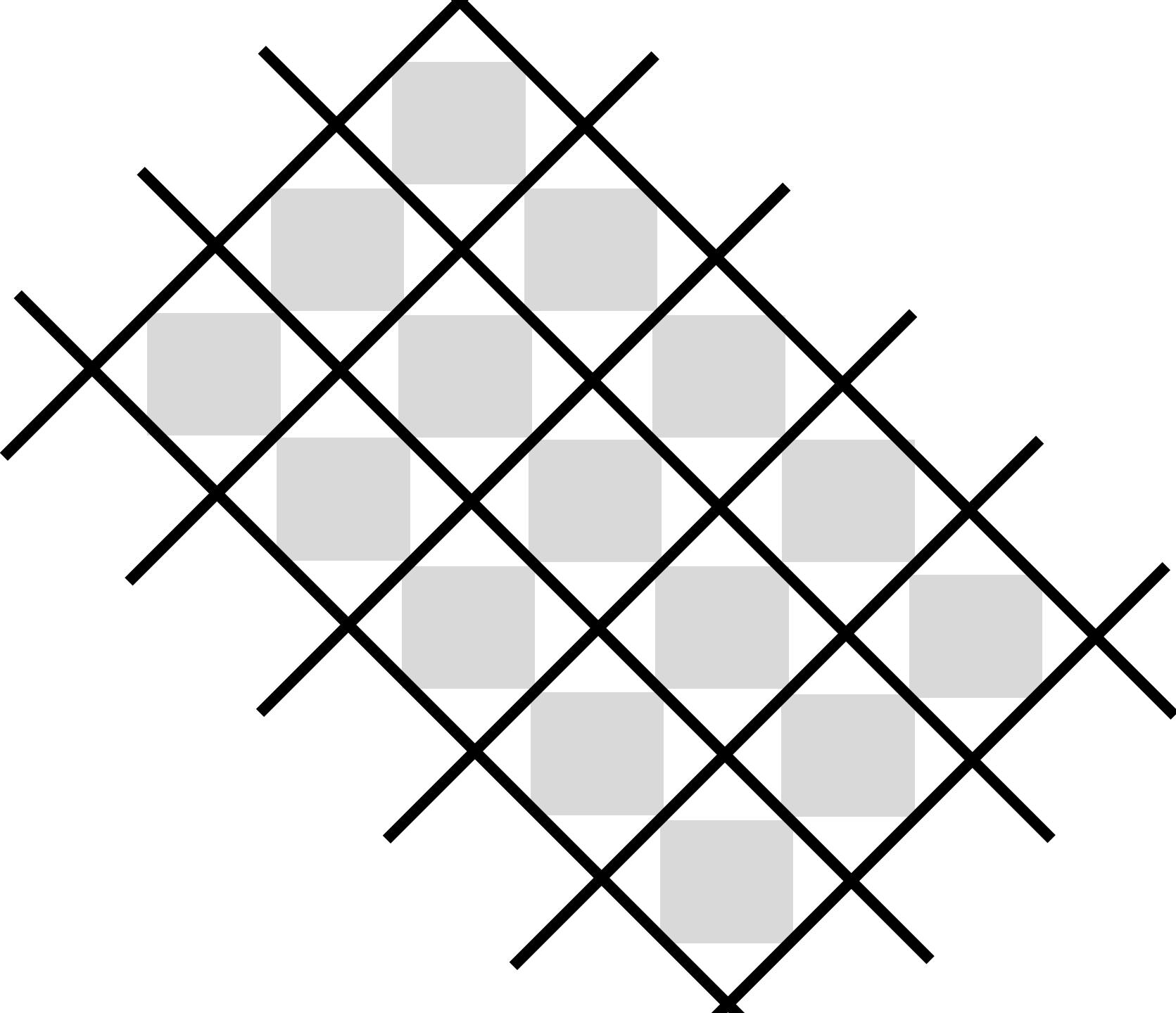
$$h_{|} \rightarrow \pi_g h_{|} \pi_g^{\dagger}$$

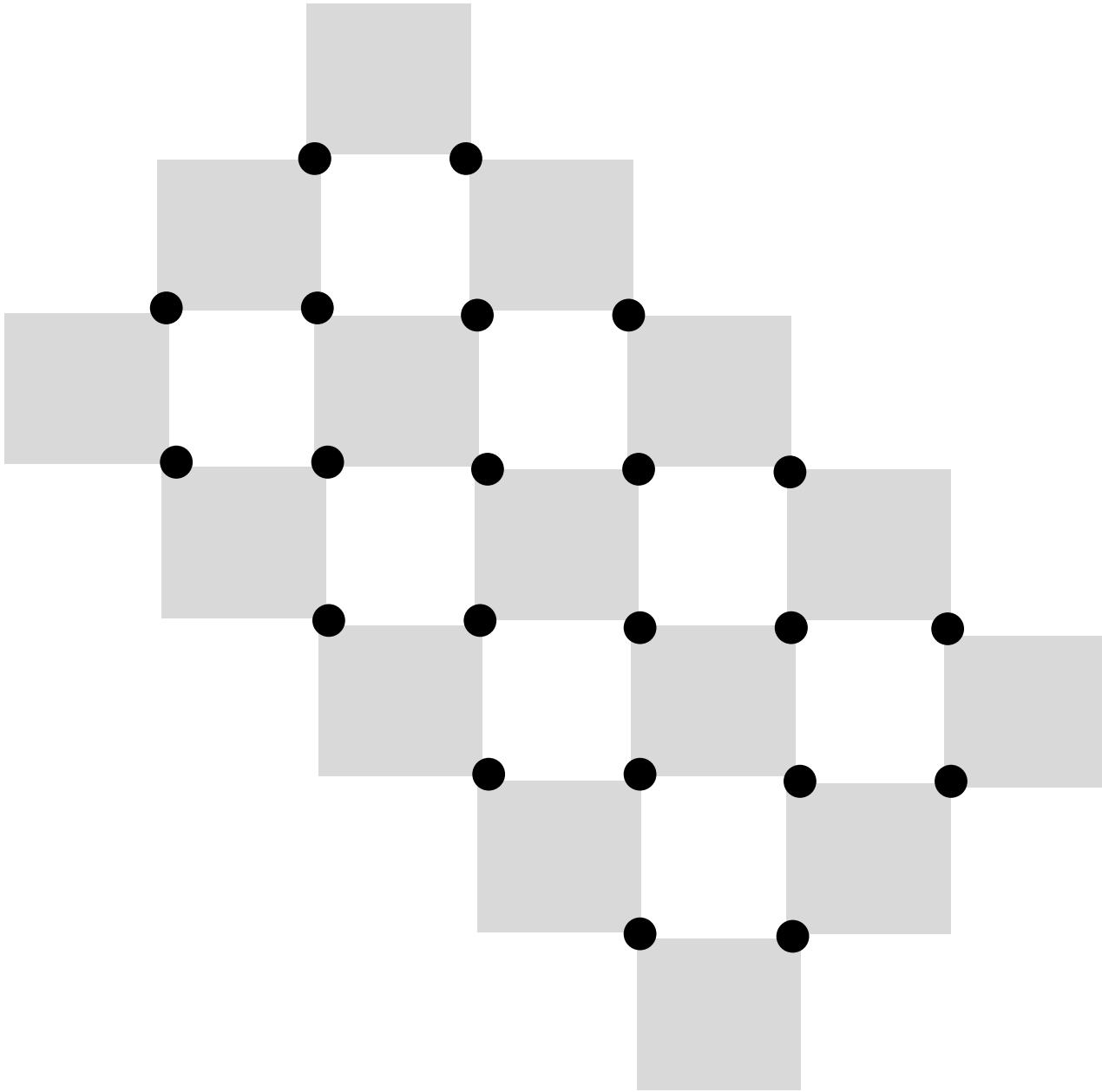


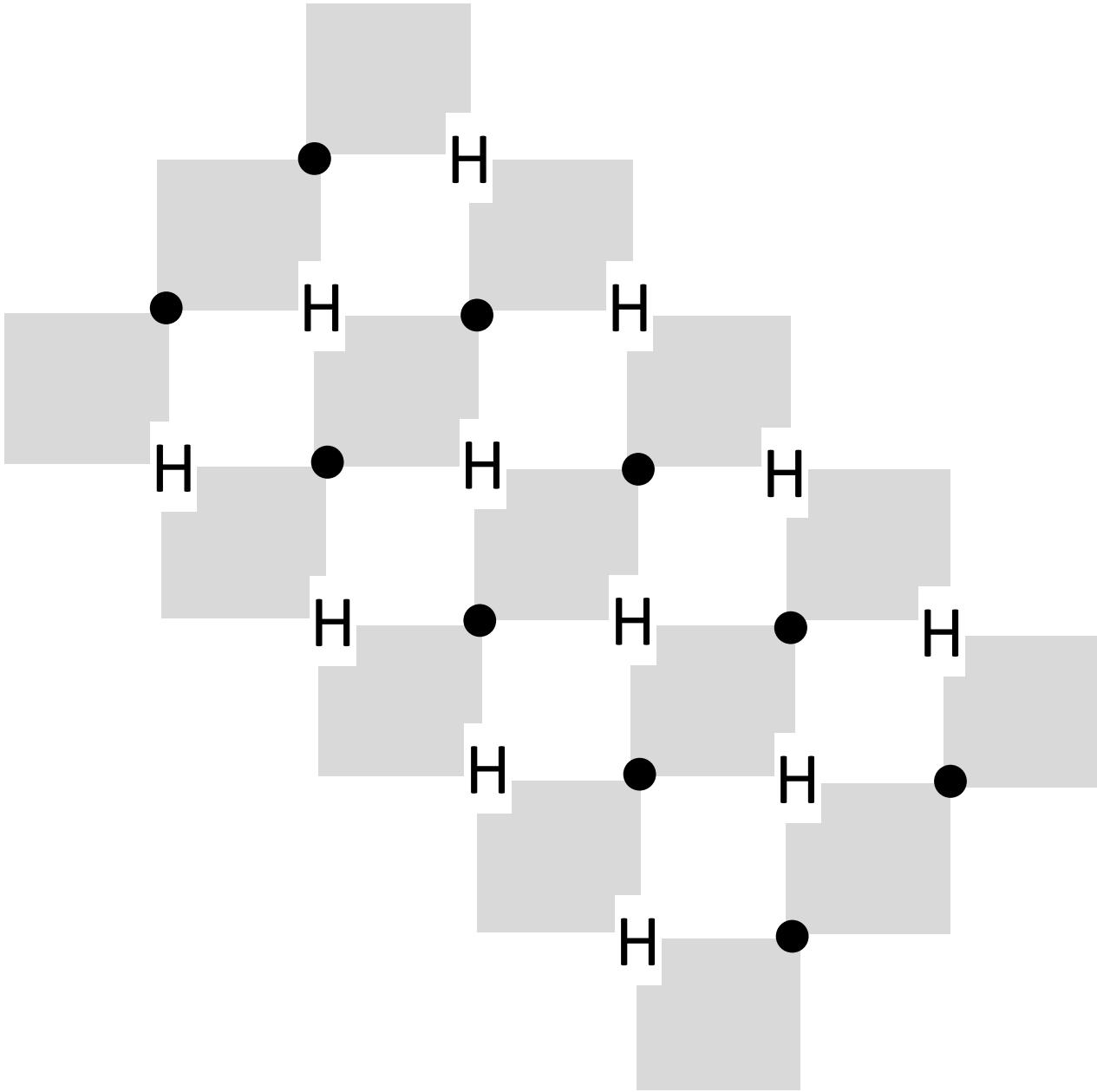
arXiv:cond-mat/0506438 Kitaev
arXiv:1004.1838 Bombin

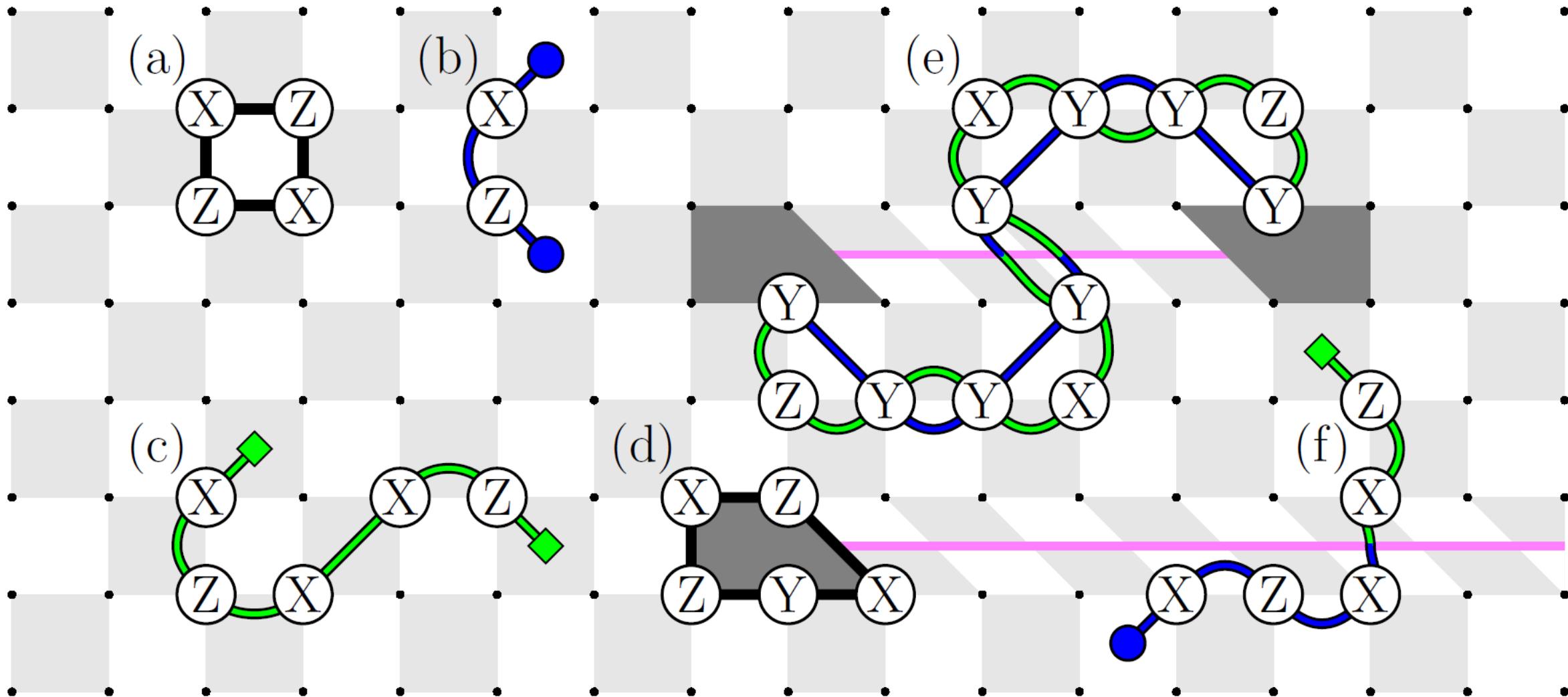












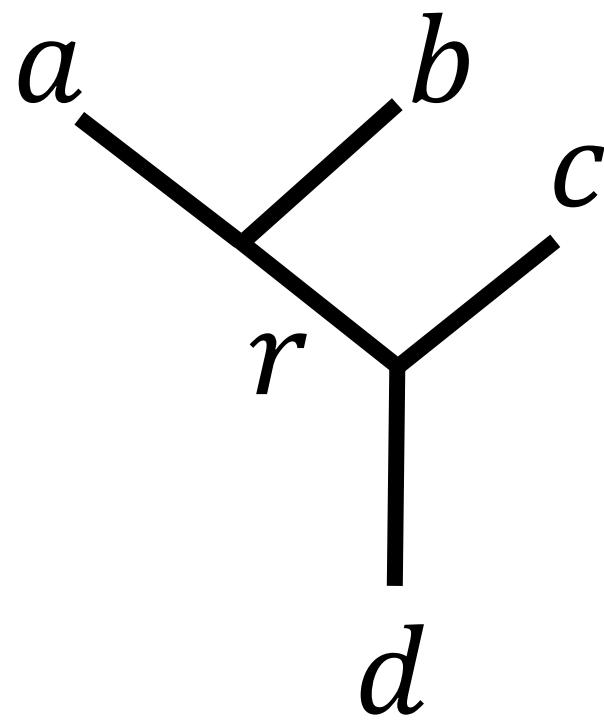
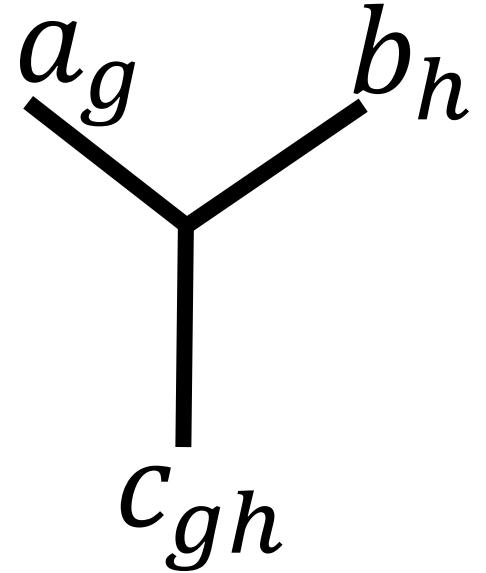
Defects in a topological phase

(Unitary G-Crossed Braided
Fusion Categories)

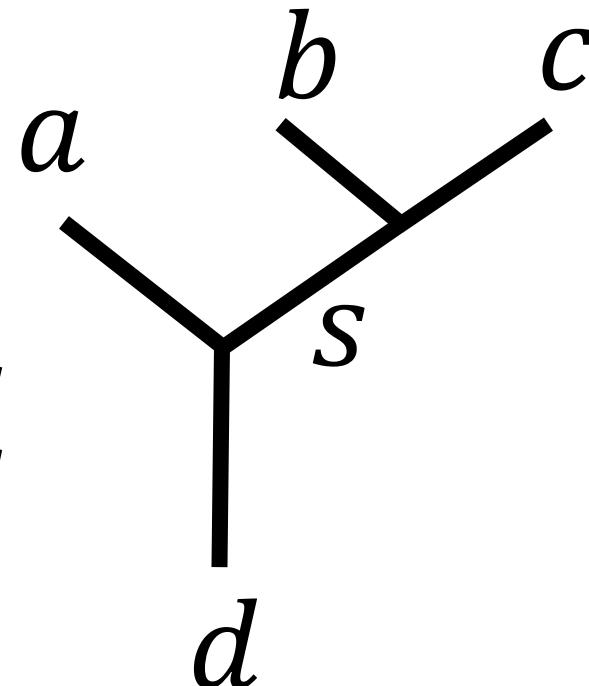
arXiv:1410.4540 Barkeshli, Bonderson, Cheng, Wang

$$a_g \times b_h = c_{gh} + d_{gh} + \dots$$

$$= \sum N_{ab}^c c_{gh}$$

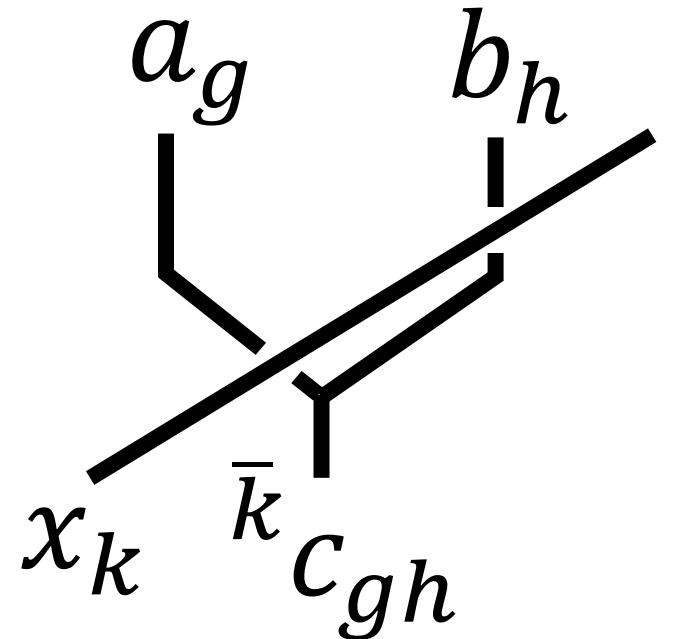


$$= F_{d;rs}^{abc}$$

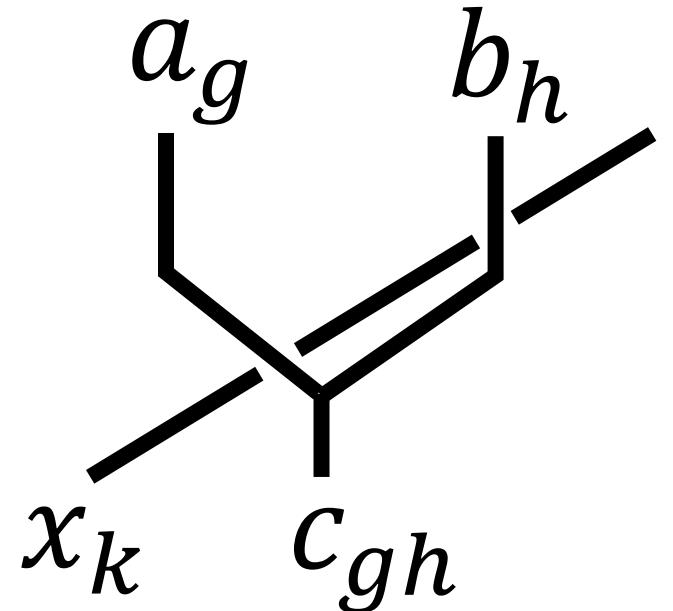
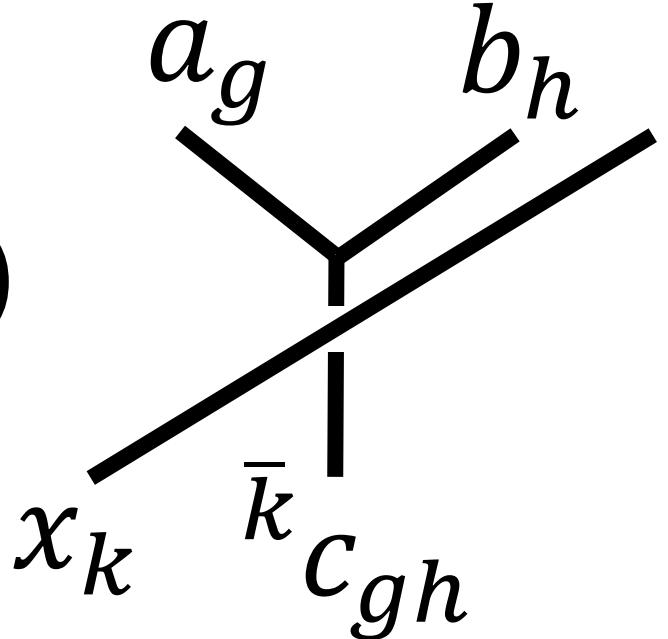


$$R^{ab} = \begin{array}{c} a_g \quad b_h \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ \bar{h} a_g \end{array}$$

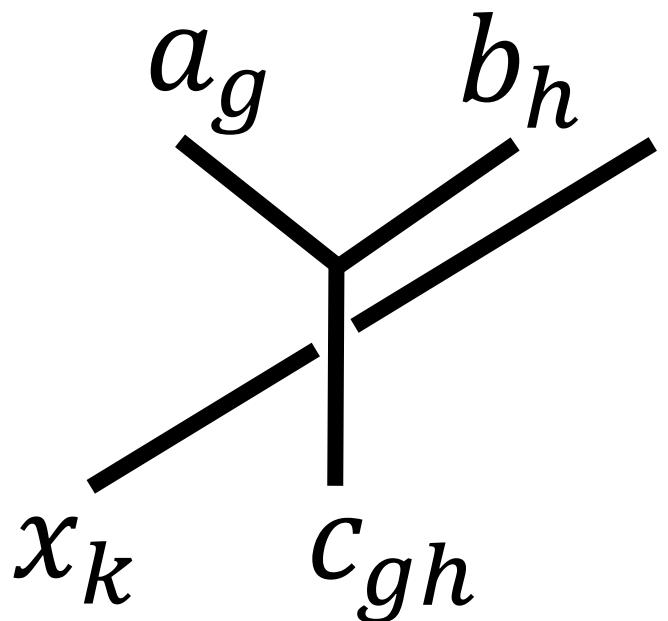
$$\begin{array}{c} a_g \quad b_h \\ \diagup \quad \diagdown \\ \text{---} \quad \text{---} \\ \diagdown \quad \diagup \\ \bar{h} a_g \end{array} = R_c^{ab} \quad \begin{array}{c} a_g \quad b_h \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ c_{gh} \end{array}$$



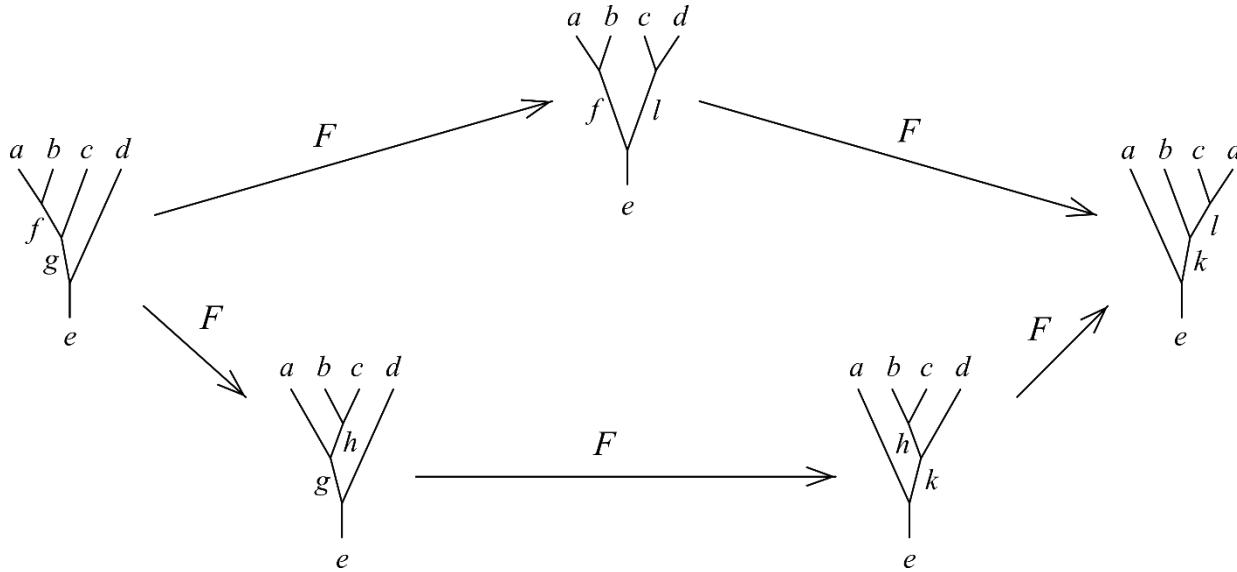
$$= U_k(a, b; c)$$



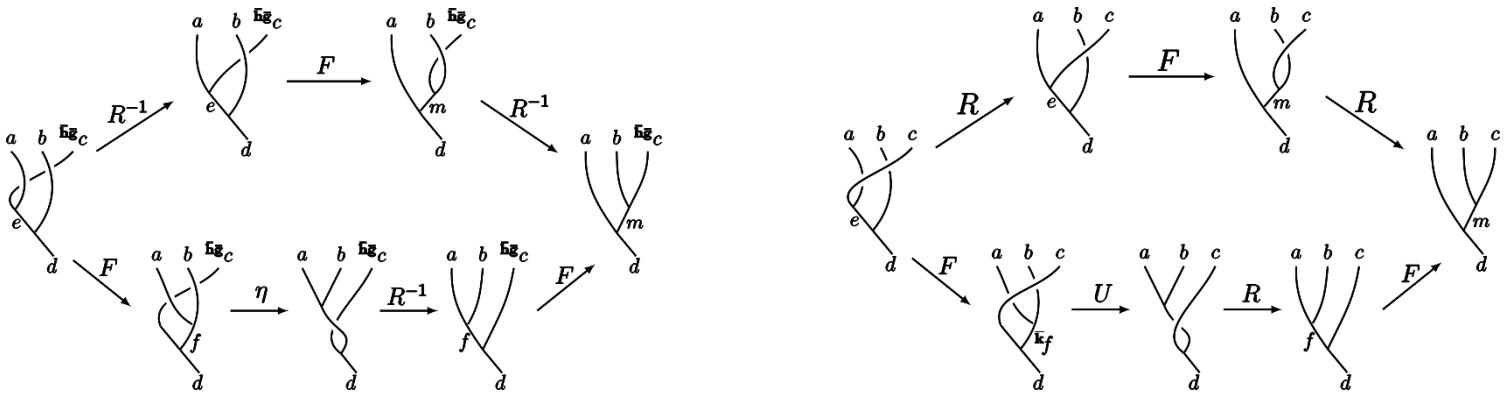
$$= \eta_x(g, h)$$



Pentagon



Heptagons



Toric code with twists $D(\mathbb{Z}_2)_{\mathbb{Z}_2}$

$$\mathcal{C}_{\mathbb{Z}_2} = \{1, e, m, em\} \oplus \{\sigma_+, \sigma_-\}$$

$$\sigma_{\pm} \times e = \sigma_{\mp} = \sigma_{\pm} \times m$$

$$\sigma_{\pm} \times \sigma_{\mp} = e + m$$

$$\sigma_{\pm} \times \sigma_{\pm} = 1 + em$$

$$\sigma_{\pm} \times em = \sigma_{\pm}$$

Double Ising

$$\mathcal{C} = \{1, \psi, \sigma\} \otimes \{1, \bar{\psi}, \bar{\sigma}\}$$

$$\psi \times \psi = 1$$

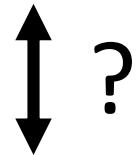
$$\sigma \times \sigma = 1 + \psi$$

$$\sigma \times \psi = \sigma$$

$$\sigma_{\pm} \times \sigma_{\pm} = 1 + em$$

$$\sigma_{\pm} \times em = \sigma_{\pm}$$

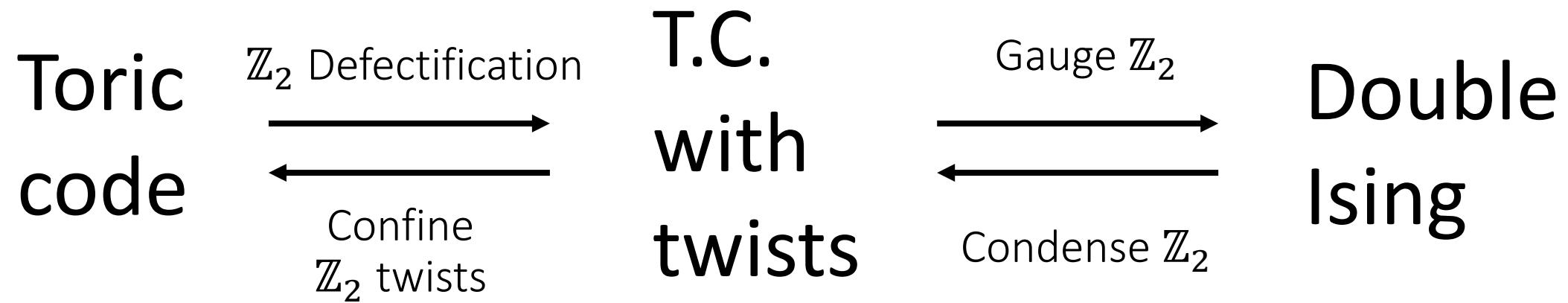
Toric code
twists

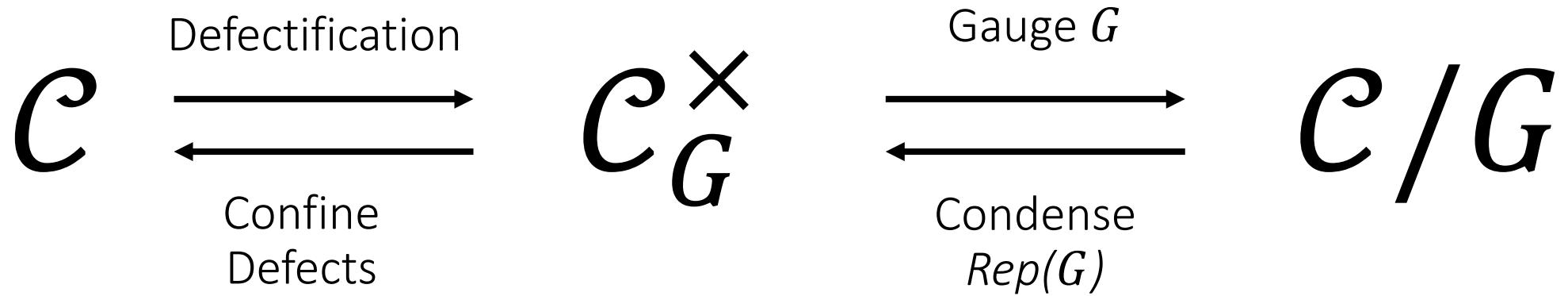


$$\sigma \times \sigma = 1 + \psi$$

$$\sigma \times \psi = \sigma$$

Double Ising





Back to the lattice

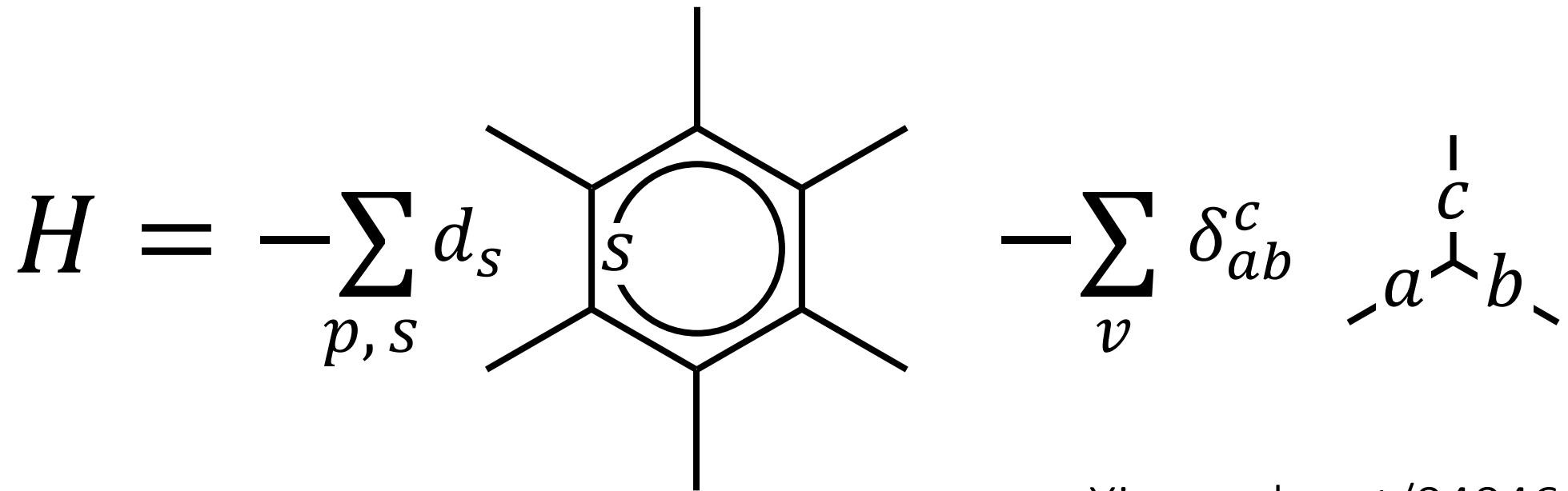
String-net models

$$-s- = |s\rangle$$

$$s \in \mathcal{C}$$

e.g. $\mathcal{C} = \mathbb{Z}_2$

$$= \{1, \psi, \sigma\}$$



Input:

\mathcal{C}

Unitary
Fusion
Category

String-net
Construction



Output:

$$H = -\sum A_v - \sum B_p$$

Commuting
Projector
Hamiltonian

Input:

\mathcal{C}

Unitary
Fusion
Category

Emergent Anyons:

String-net
Construction

Drinfeld
Double

$Z(\mathcal{C})$

Output:

$H = -\sum A_v - \sum B_p$

Sectors
↓
Superselection

Commuting
Projector
Hamiltonian

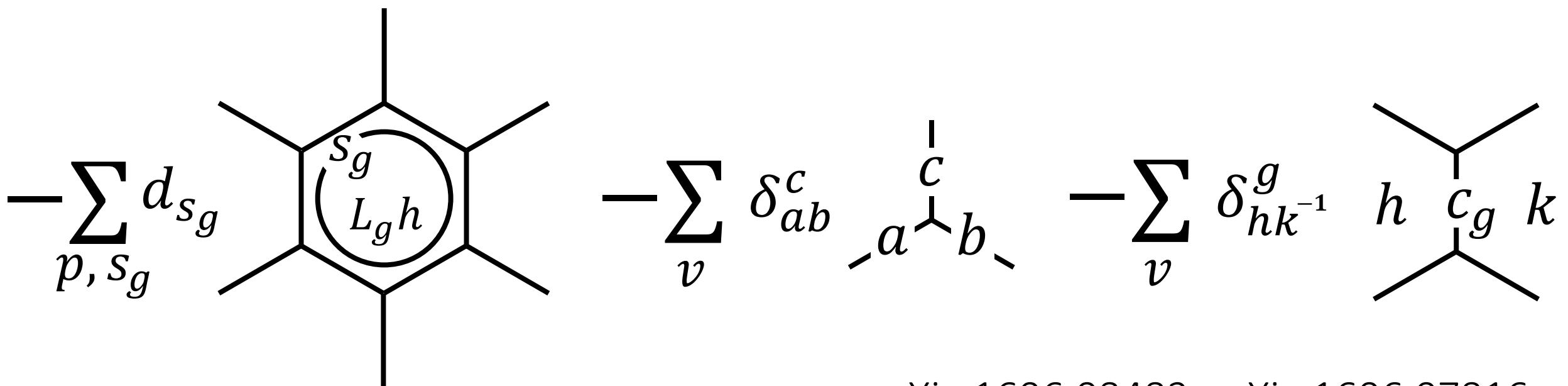
Modular
Tensor
Category

G -graded SET String-net models

$$-s_g- = |s_g\rangle \quad s \in \mathcal{C}_G$$

$$\text{e.g. } \mathcal{C}_{\mathbb{Z}_2} = \{1, \psi\} \oplus \{\sigma\}$$

$$\begin{array}{c} \text{hexagon} \\ h \end{array} = |h\rangle \quad h \in G$$



Input:

$$\mathcal{C}_G \xrightarrow[\text{String-net}]{\text{G-graded}}$$

Hamiltonian:

$$H_{\mathcal{C}_0}^G = -\sum A_v - \sum B_p - \sum C_e$$

Emergent theory:

$$\xrightarrow[\text{sectors}]{\text{Defect supersel.}}$$

$$Z(\mathcal{C}_0)_G$$

Input:

$$\mathcal{C}_G$$

$\xrightarrow{\begin{array}{l} G\text{-graded} \\ \text{String-net} \end{array}}$

$\xrightarrow{\begin{array}{l} \text{String-net} \end{array}}$

Hamiltonian:

$$H_{\mathcal{C}_0}^G = -\sum A_v - \sum B_p - \sum C_e$$

\uparrow restrict \downarrow Gauge G

$$H_{\mathcal{C}_G} = -\sum A_v - \sum B_p$$

Emergent theory:

Defect
supersel.
 $\xrightarrow{\begin{array}{l} \text{sectors} \end{array}}$

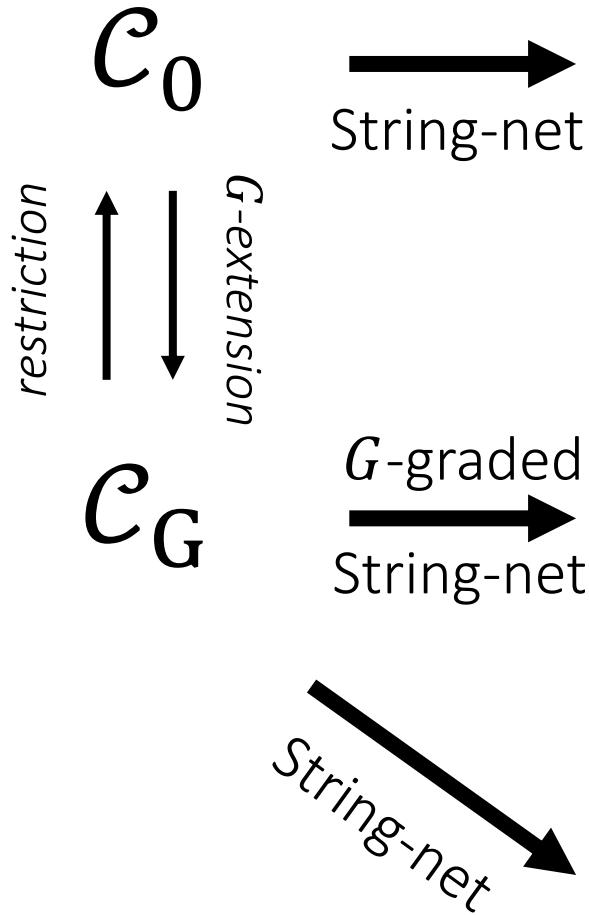
Supersel.
 $\xrightarrow{\begin{array}{l} \text{sectors} \end{array}}$

$$Z(\mathcal{C}_0)_G$$

\uparrow Condense
 \downarrow Rep(G)

$$Z(\mathcal{C}_G)$$

Input:



Hamiltonian:

$$H_{\mathcal{C}_0} = -\sum A_v - \sum B_p$$

project
fluctuate

$$H_{\mathcal{C}_0}^G = -\sum A_v - \sum B_p - \sum C_e$$

restrict
Gauge G

$$H_{\mathcal{C}_G} = -\sum A_v - \sum B_p$$

Emergent theory:

Supersel.
sectors

$$Z(\mathcal{C}_0)$$

Defectify
Confine

$$Z(\mathcal{C}_0)_G$$

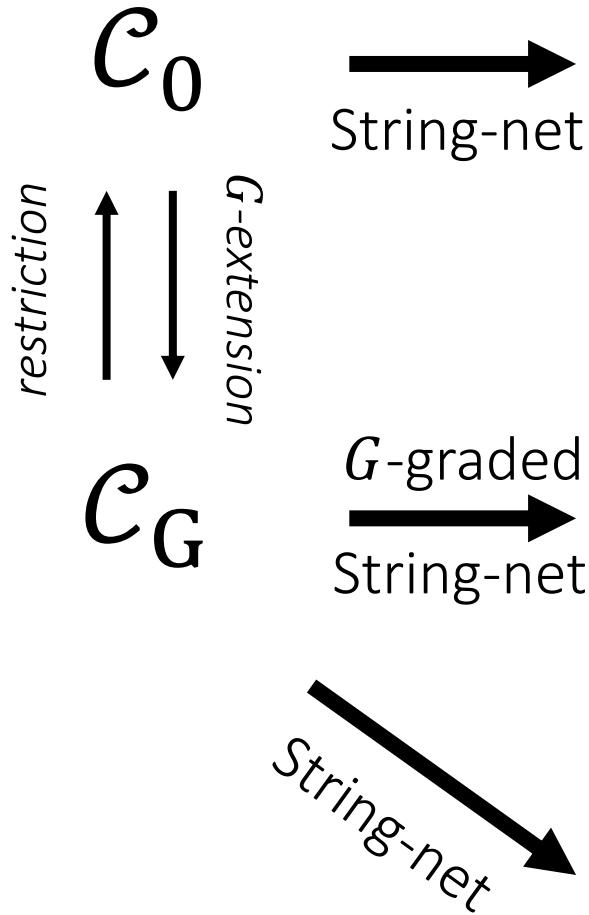
Defect
supersel.
sectors

Gauge G
Condense
Rep(G)

Supersel.
sectors

$$Z(\mathcal{C}_G)$$

Input:



Hamiltonian:

$$H_{\mathcal{C}_0} = -\sum A_v - \sum B_p$$

project
↑ ↓
fluctuate

$$H_{\mathcal{C}_0}^G = -\sum A_v - \sum B_p - \sum C_e$$

restrict
↑ ↓
Gauge G

$$H_{\mathcal{C}_G} = -\sum A_v - \sum B_p$$

Emergent theory:

Supersel.
→
sectors

$$Z(\mathcal{C}_0)$$

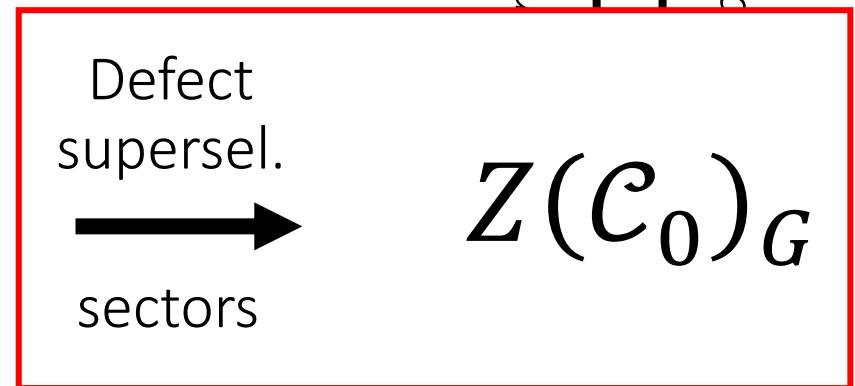
Defect
supersel.
→
sectors

$$Z(\mathcal{C}_0)_G$$

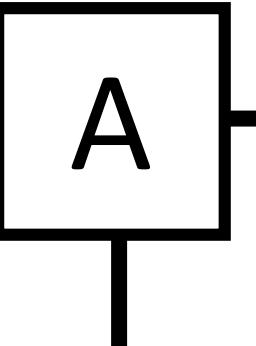
Supersel.
→
sectors

Conder.
Rep(G)
↓
auge G

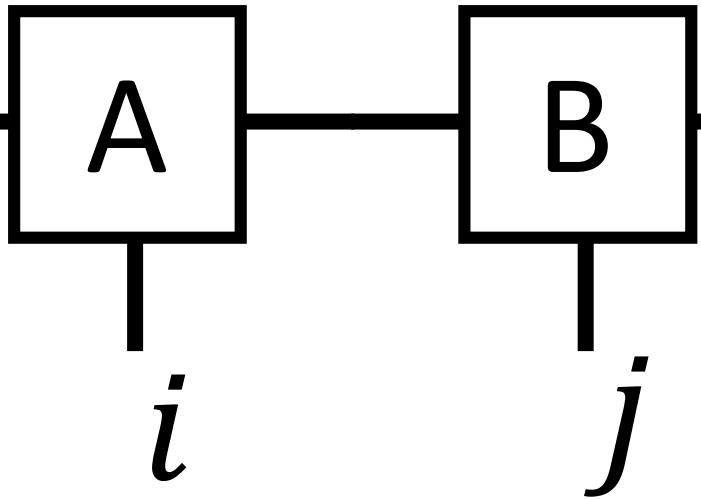
$$Z(\mathcal{C}_G)$$



Tensor Networks

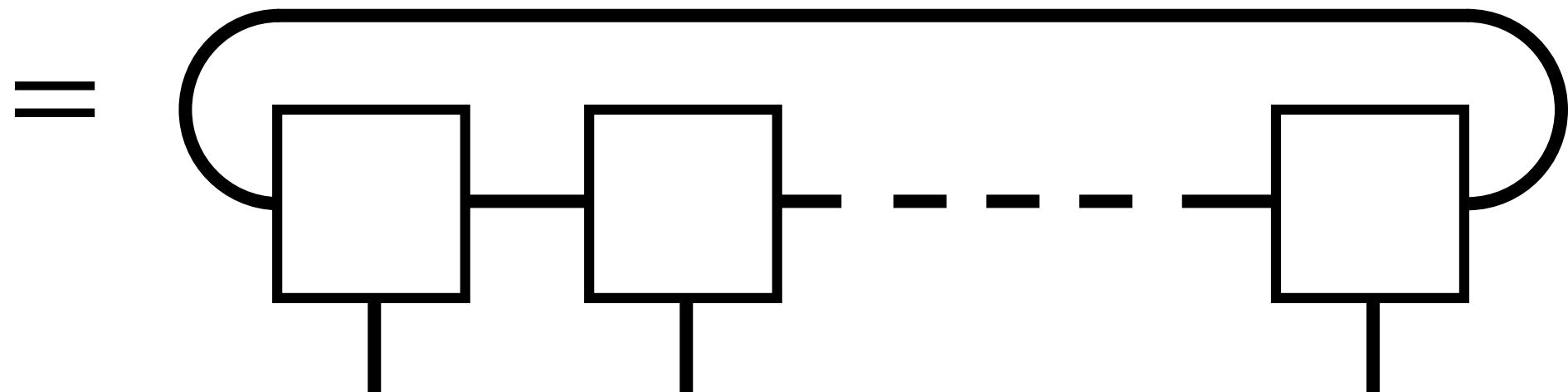
$$\alpha \xrightarrow{\quad A \quad} \beta = (A^i)_{\alpha\beta} \in \mathbb{C}$$


The diagram shows a rectangular box labeled 'A'. To its left is a horizontal line segment ending in an arrowhead pointing right, labeled with the Greek letter α . To its right is another horizontal line segment beginning with an arrowhead pointing left, labeled with the Greek letter β . Below the box 'A' is a short vertical line segment with an arrowhead pointing downwards, labeled with the italicized letter i .

$$\alpha - \boxed{A} - \boxed{B} - \gamma = \sum_{\beta} (A^i)_{\alpha\beta} (B^j)_{\beta\gamma}$$


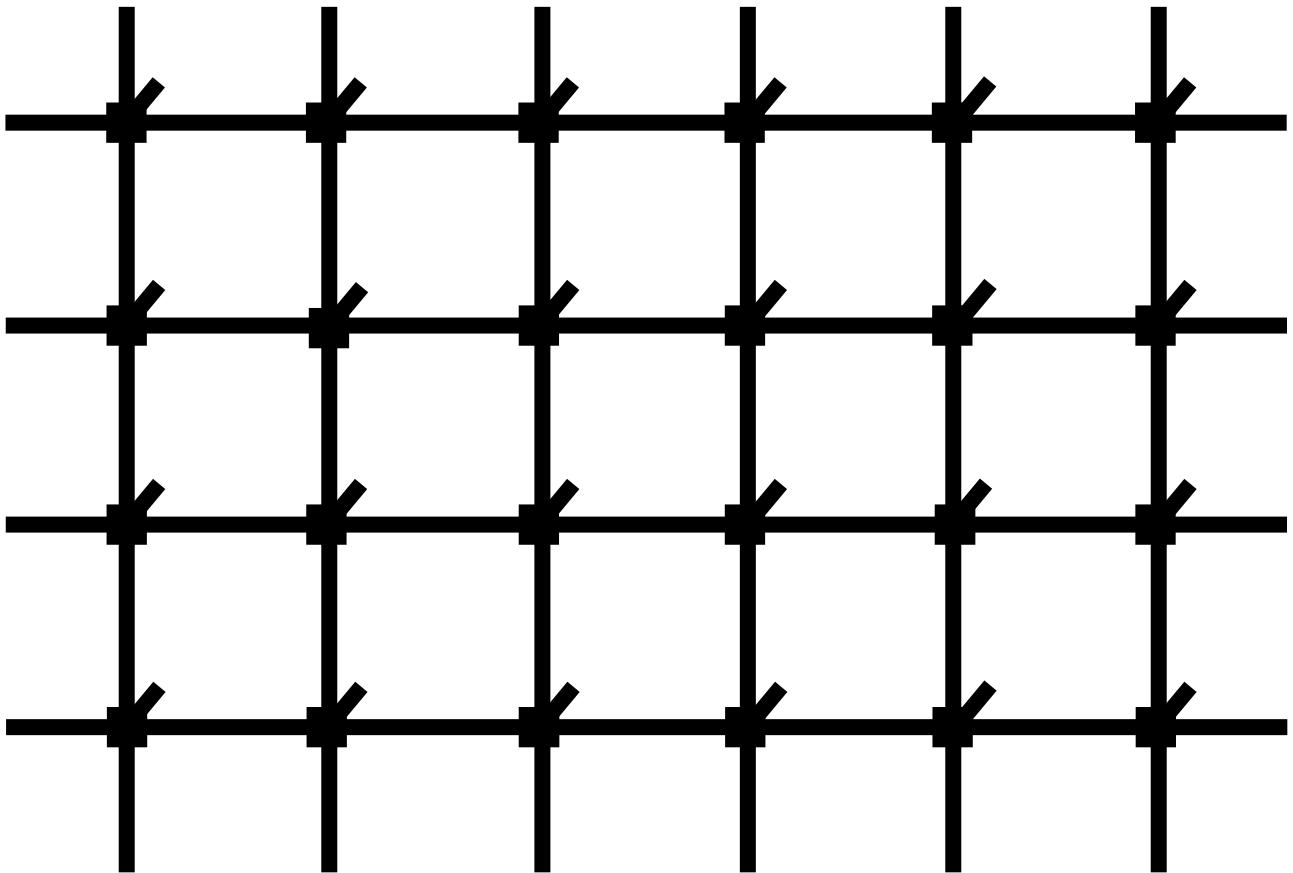
The diagram illustrates a sequence of operations. Two rectangular boxes, labeled 'A' and 'B', are connected by a horizontal line. Below box A is the index 'i', and below box B is the index 'j'. On the far left, an incoming line is labeled with the Greek letter 'alpha' (α). On the far right, an outgoing line is labeled with the Greek letter 'gamma' (γ). This visual representation corresponds to the mathematical expression above it.

$$|MPS_N(A)\rangle = \sum_{i_1 \dots i_N} Tr[A^{i_1} \dots A^{i_N}] |i_1 \dots i_N\rangle$$

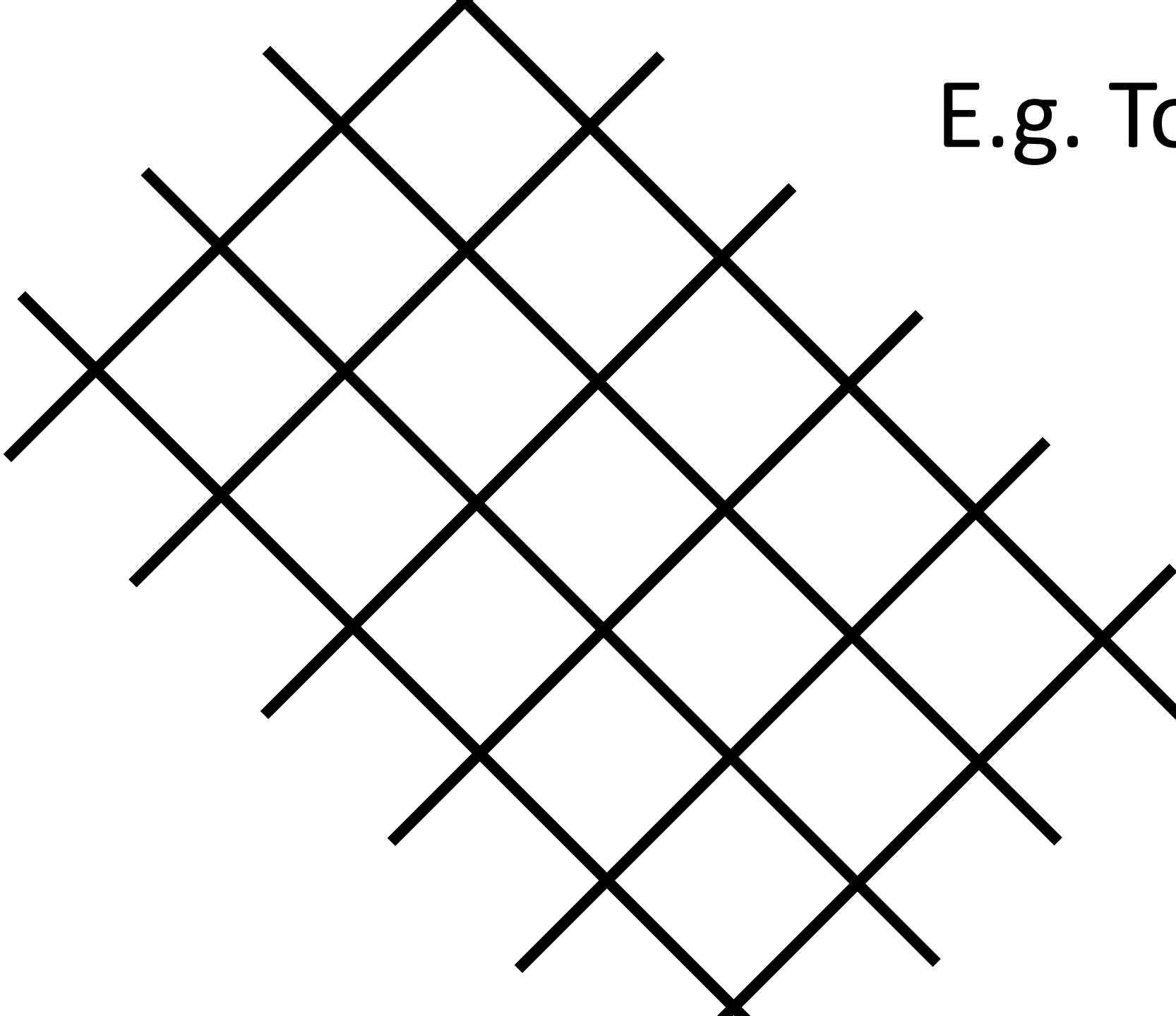


$|PEPS(A)\rangle =$

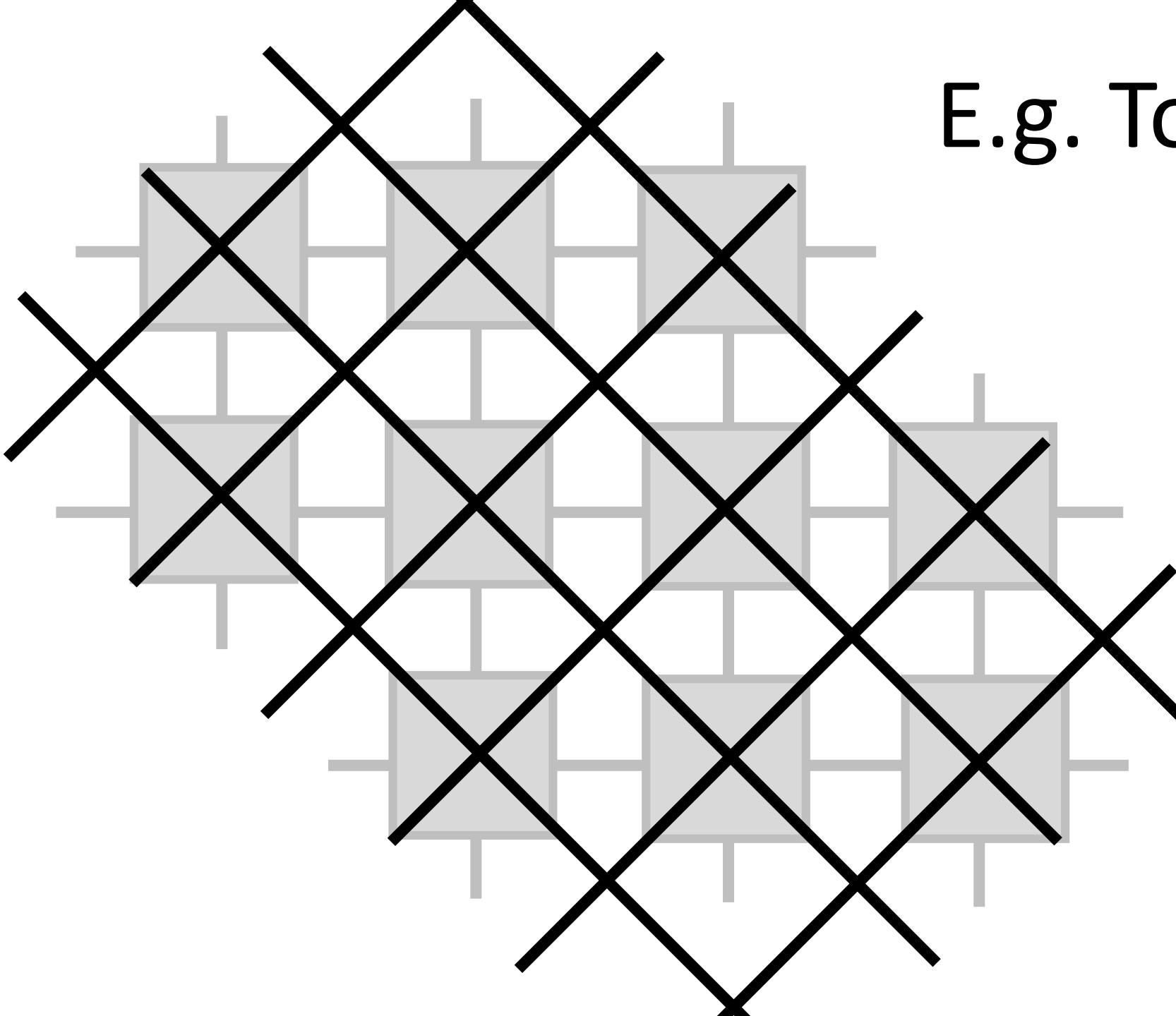
$$\begin{array}{c} \beta \\ \alpha -+ i \\ \gamma \\ \delta \\ = (A^i)_{\alpha\beta\gamma\delta} \end{array}$$



E.g. Toric code



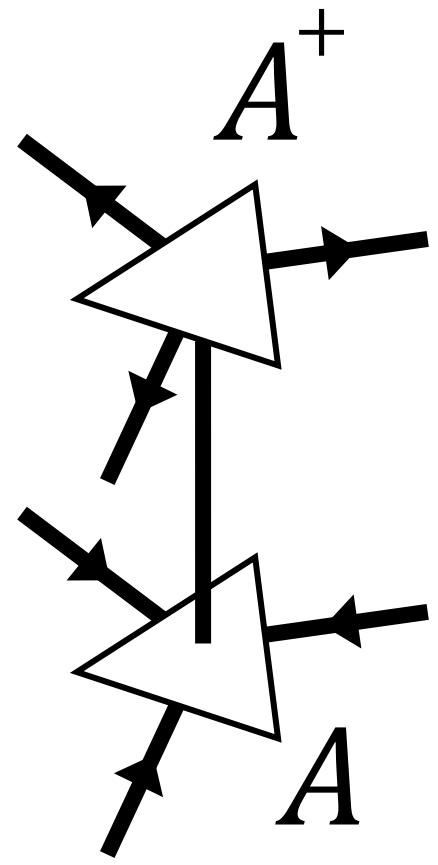
E.g. Toric code



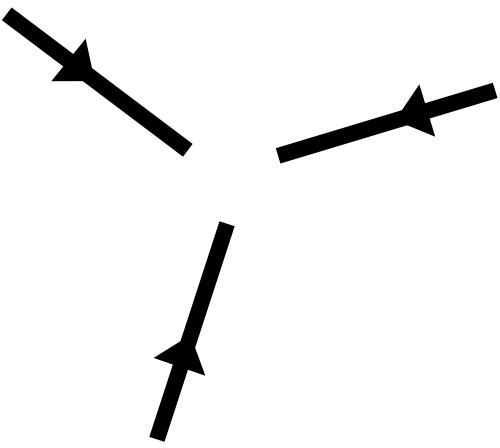
T.C.

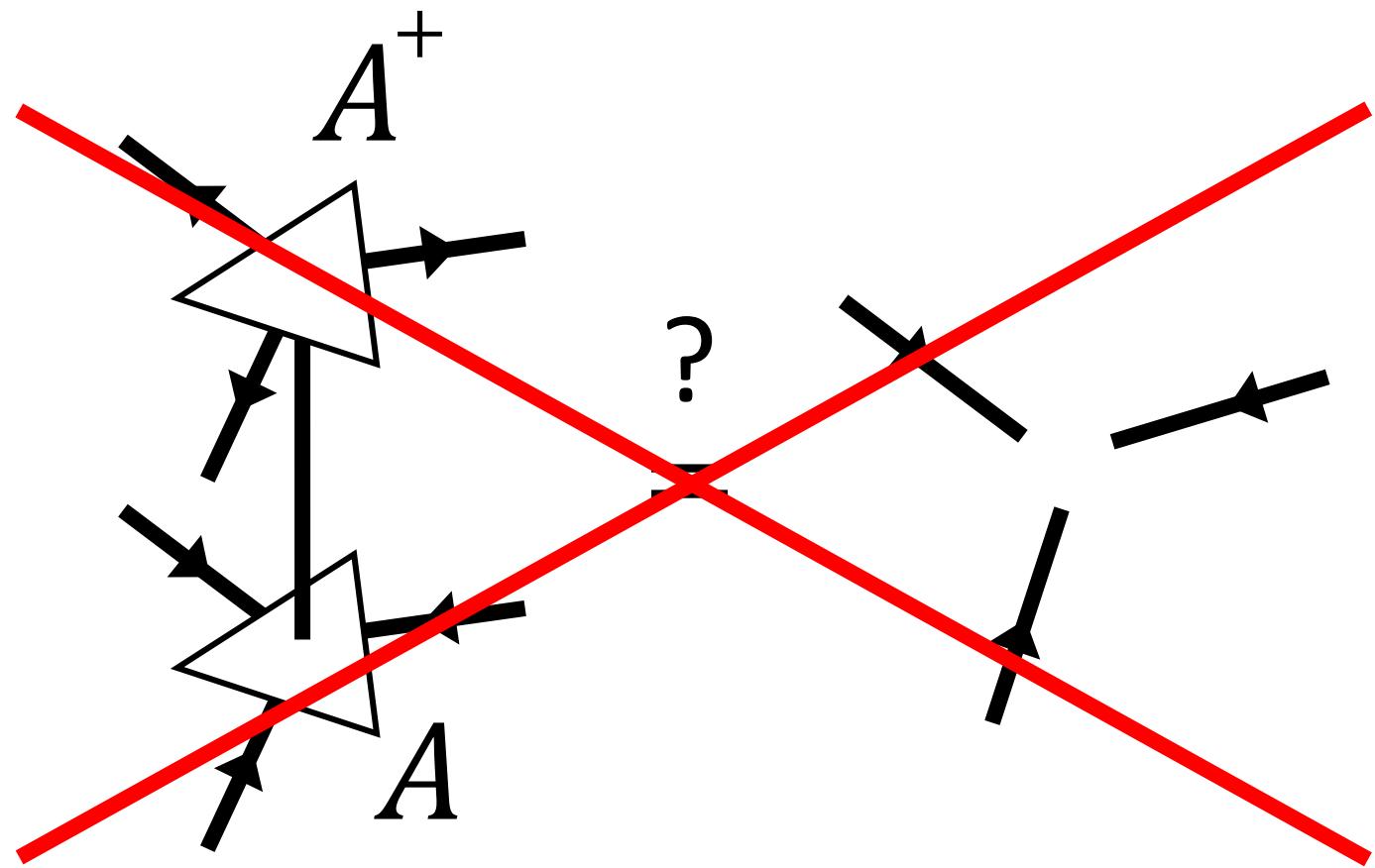
$$\text{Diagram} = \sum_{ijkl} i+k \quad \begin{matrix} i+j \\ k+l \end{matrix} \quad j+l$$

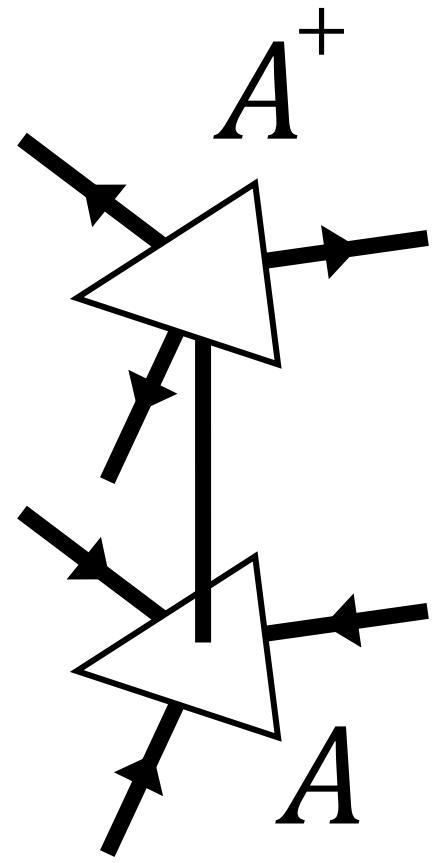
The diagram consists of a central gray square containing four white circles at its corners. Two thick black diagonal lines cross through the square, connecting opposite corners. To the left of the square is a horizontal bar with two vertical lines extending upwards from its ends. To the right is another horizontal bar with two vertical lines extending downwards from its ends. The entire diagram is positioned to the left of an equals sign, followed by a summation symbol with indices i, j, k, l below it. To the right of the summation is a second gray square with indices i, j at the top and k, l at the bottom. This is followed by a third horizontal bar with two vertical lines extending downwards from its ends. Below the first and third bars is the expression $j + l$. Below the second bar is the expression $k + l$.



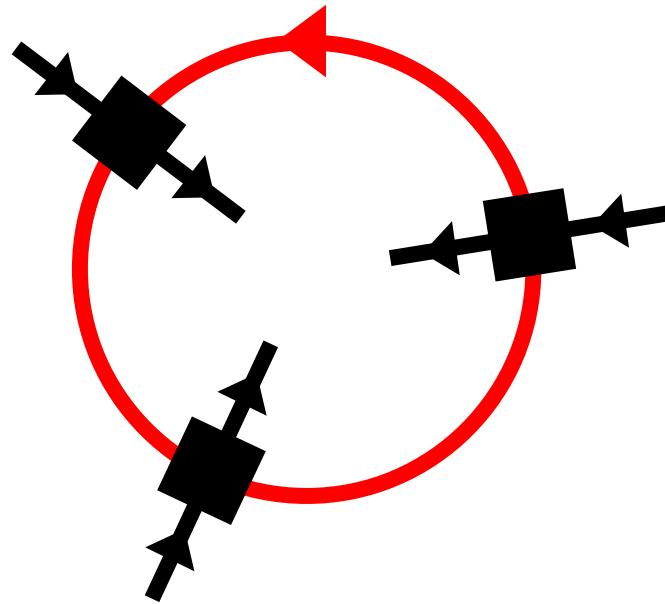
? =



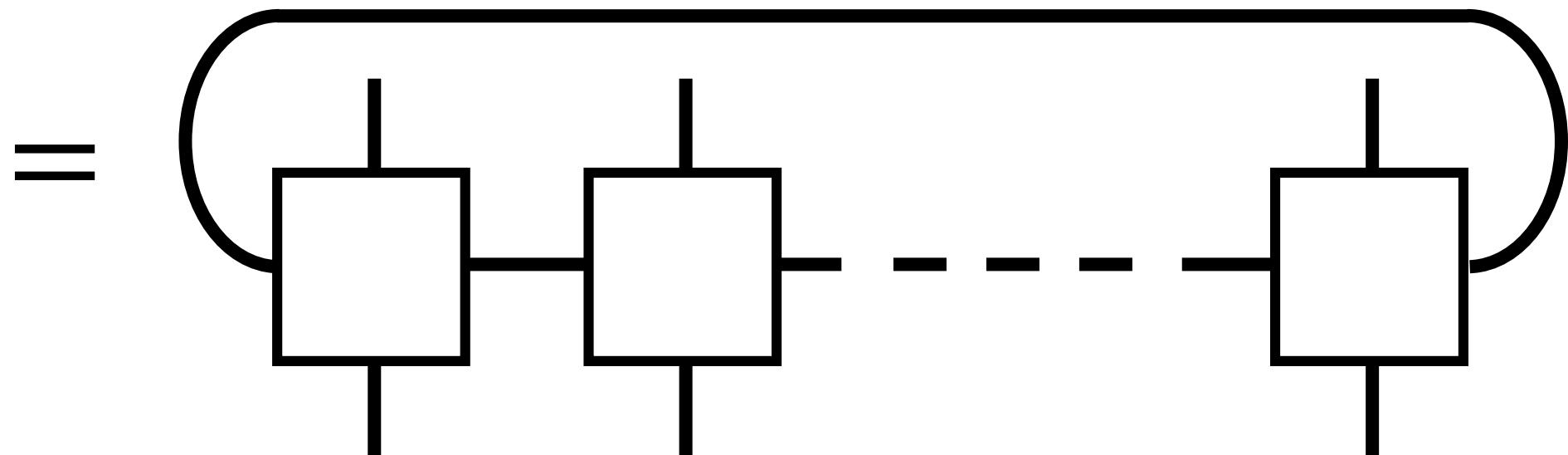


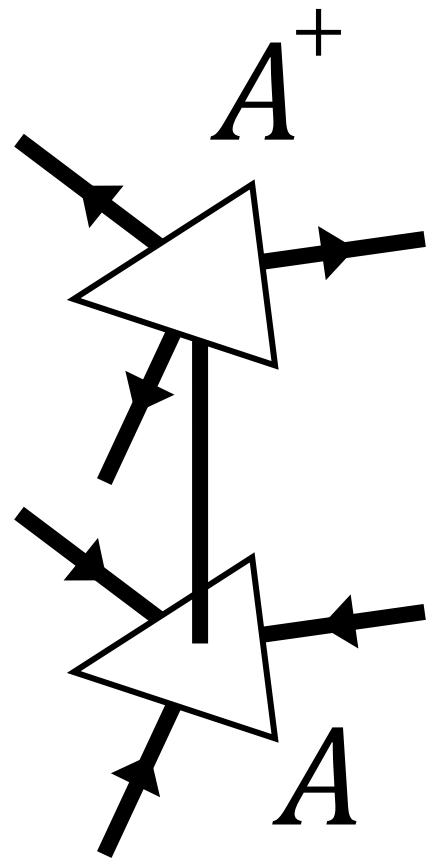


=

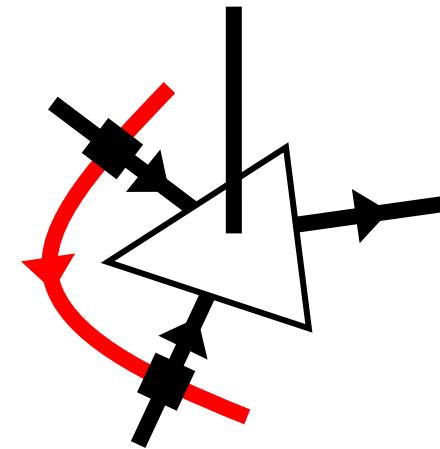
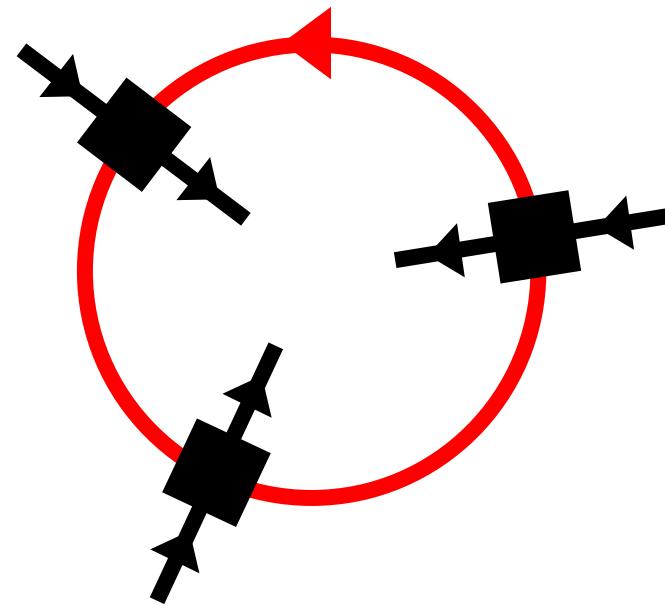


$$MPO_N(A) = \sum_{\substack{i_1 \dots i_N \\ j_1 \dots j_N}} Tr[A^{i_1 j_1} \dots A^{i_N j_N}] |i_1 \dots i_N\rangle \langle j_1 \dots j_N|$$

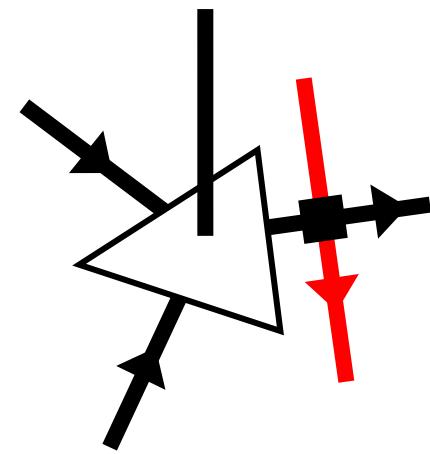


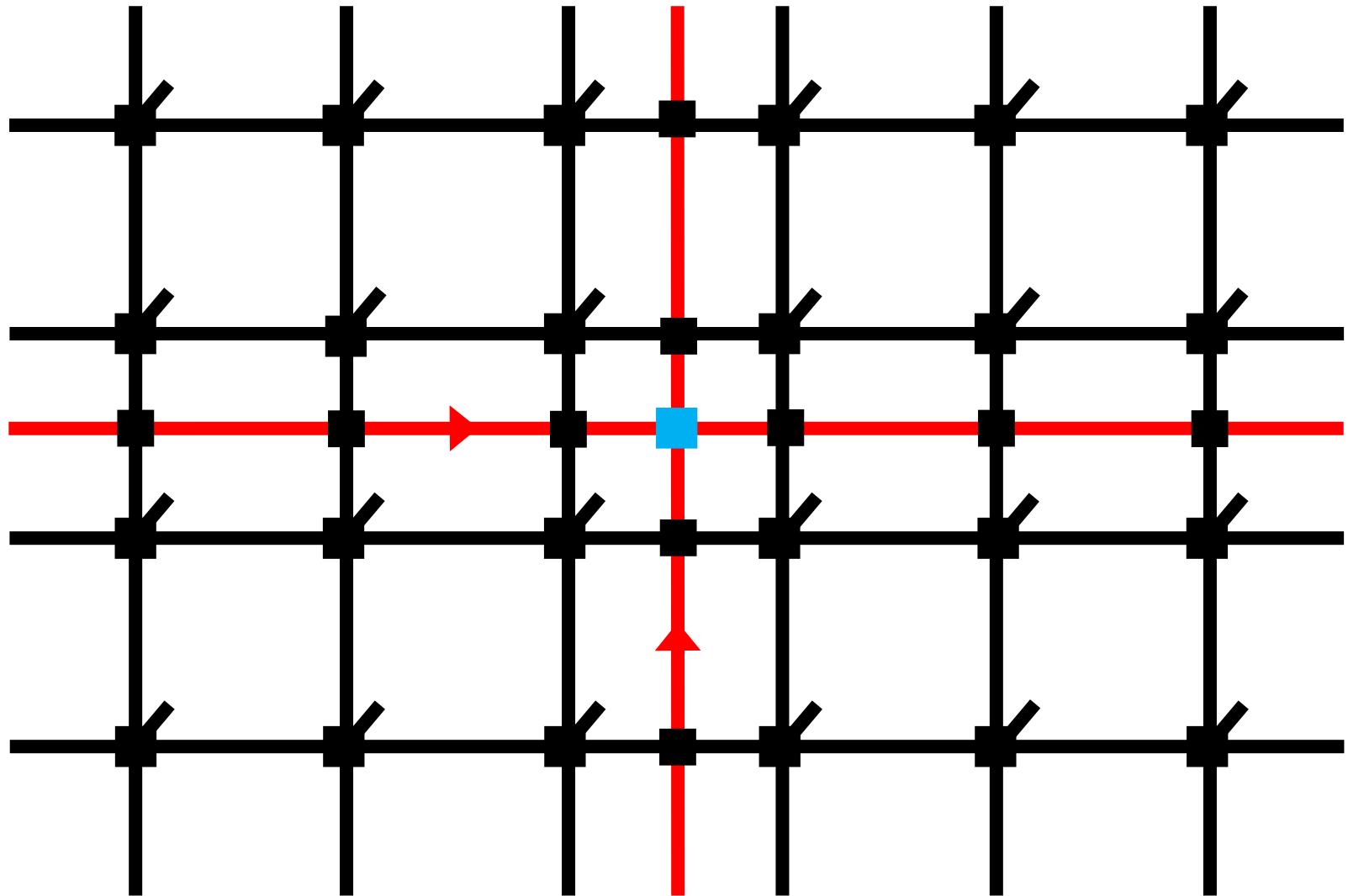


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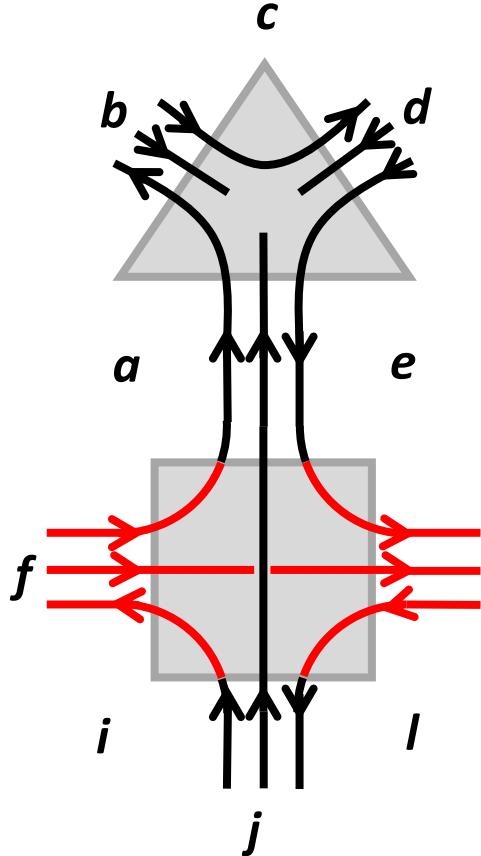
$|\Psi_0^{\square}\rangle =$ 

T.C.

$$\text{[Diagram of a 2x2 tensor network node with four ports and four white circles]} = \sum_{ijkl} i+k \text{ [Diagram of a 2x2 tensor network node with indices i,j,k,l] } j+l$$

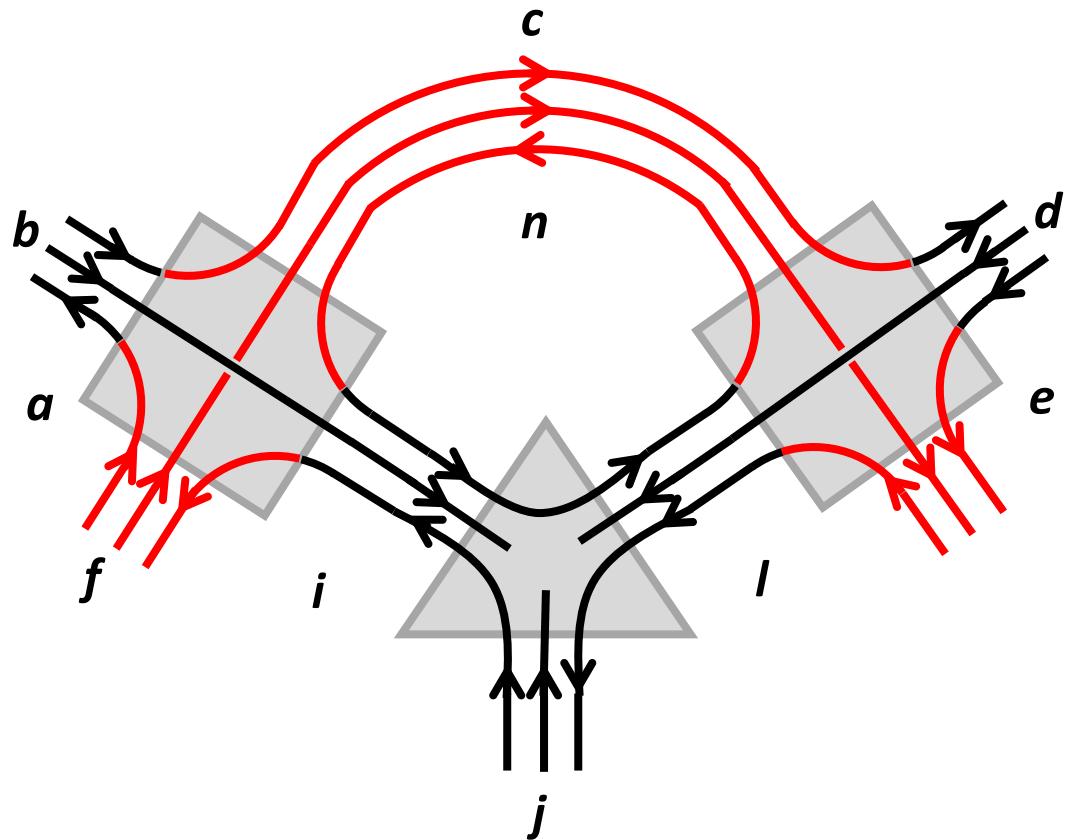
$$z \text{ [Diagram of a 2x2 tensor network node with four ports and four white circles]} = \text{[Diagram of a 2x2 tensor network node with four ports and four white circles]} z$$

$$F_{j;be}^{acd}$$



E.g. String-net

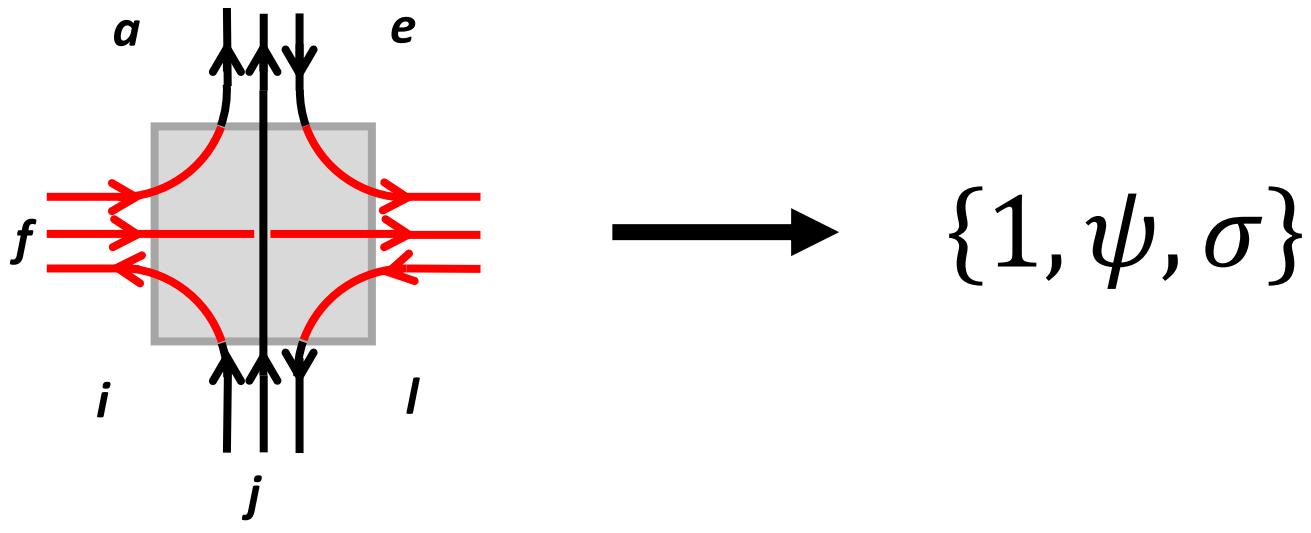
=



$$F_{l;ei}^{jaf}$$

$$FF = \sum FFF$$

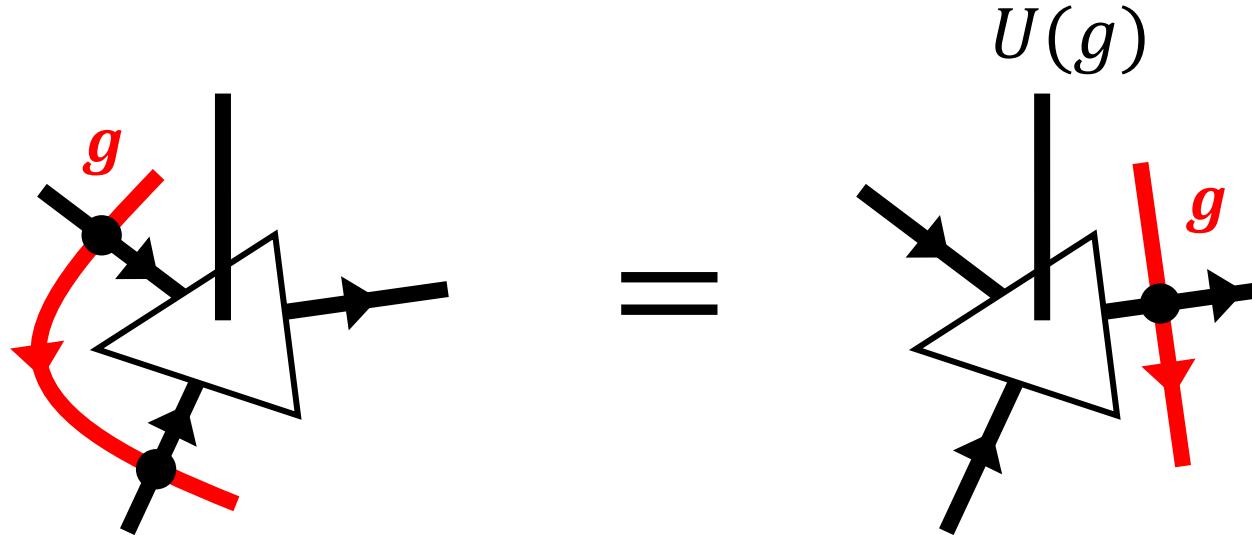
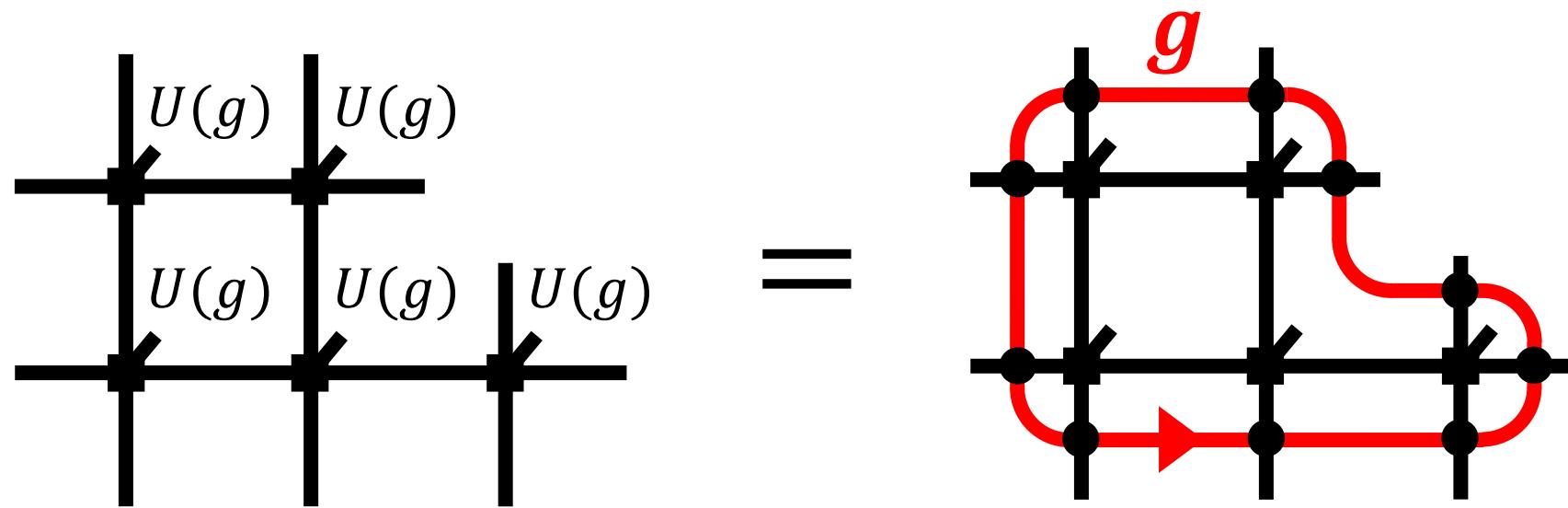
Double Ising



$F_{l;ei}^{jaf}$

Form a matrix product operator algebra

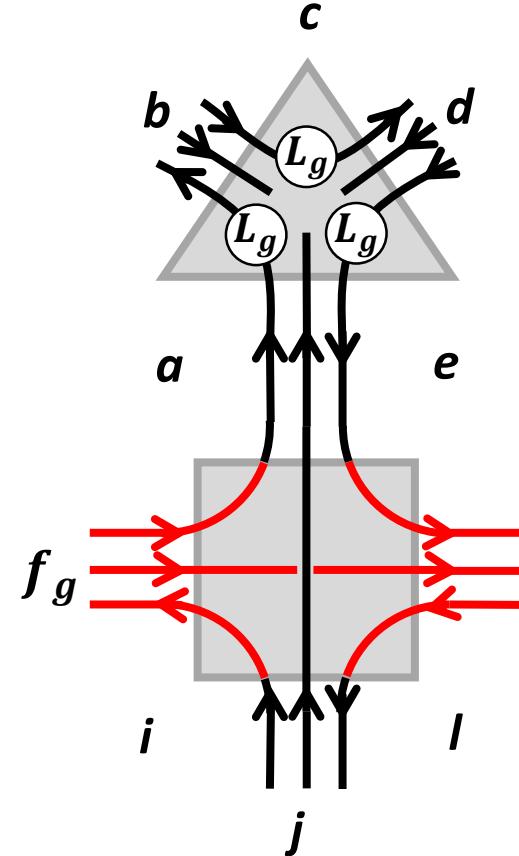
With a global symmetry



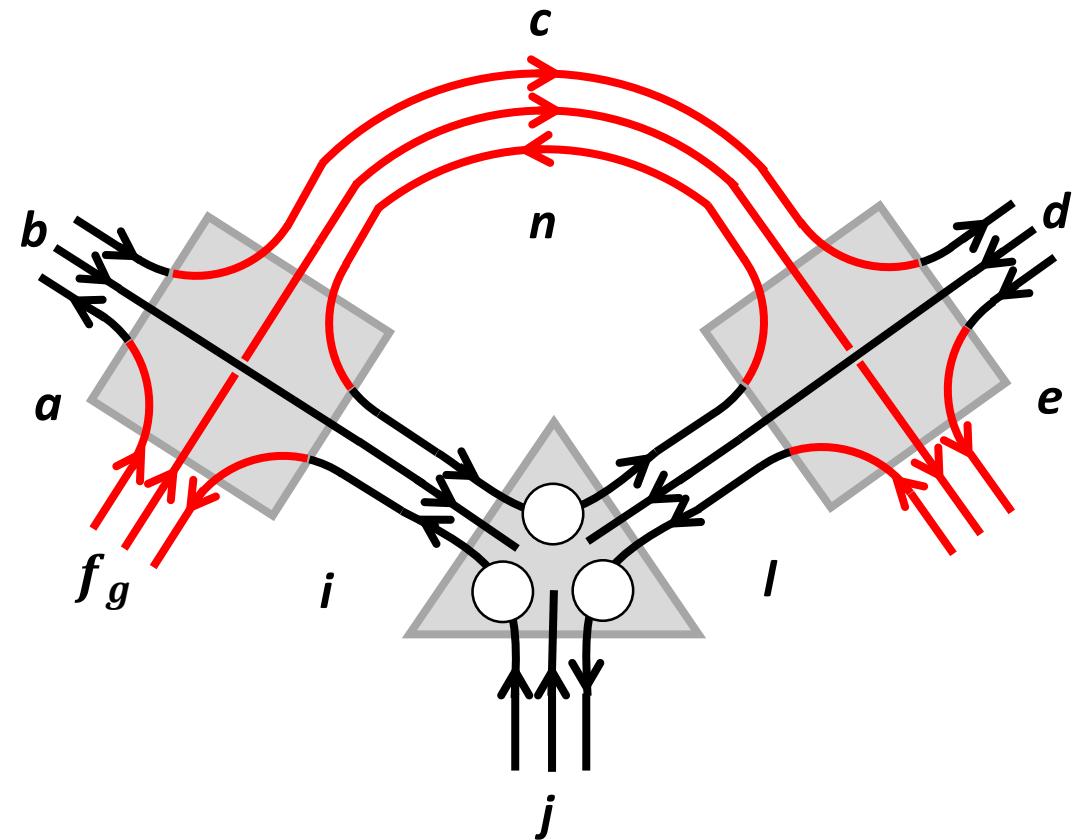
$$a_h - \textcircled{h} - a_h$$

G -Graded String-net

$$F_{j;be}^{acd}$$



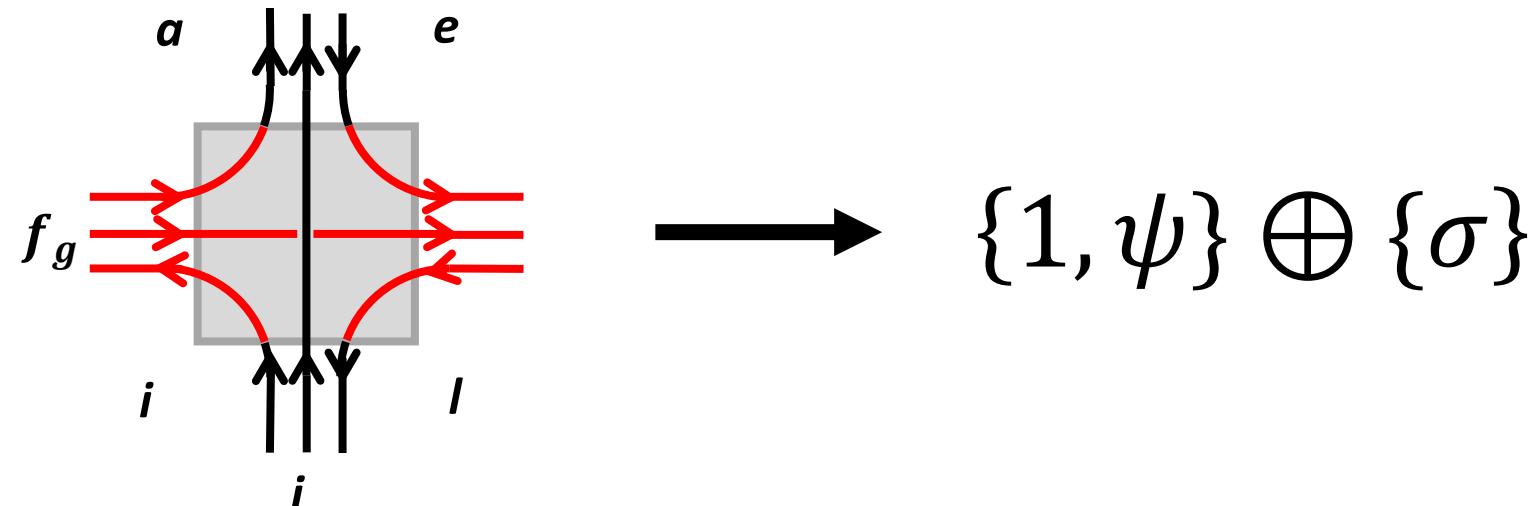
=



$$F_{l;ei}^{jaf}$$

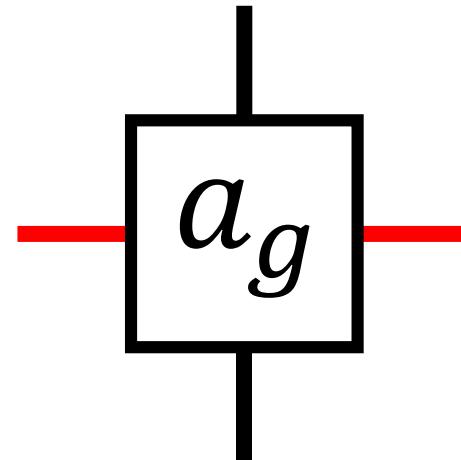
$$\textcolor{red}{FF} = \sum FFF$$

Toric code w/ twists

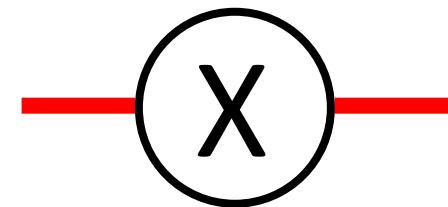


$F_{l;ei}^{jaf}$

Form a \mathbb{Z}_2 -graded matrix product operator algebra



Objects

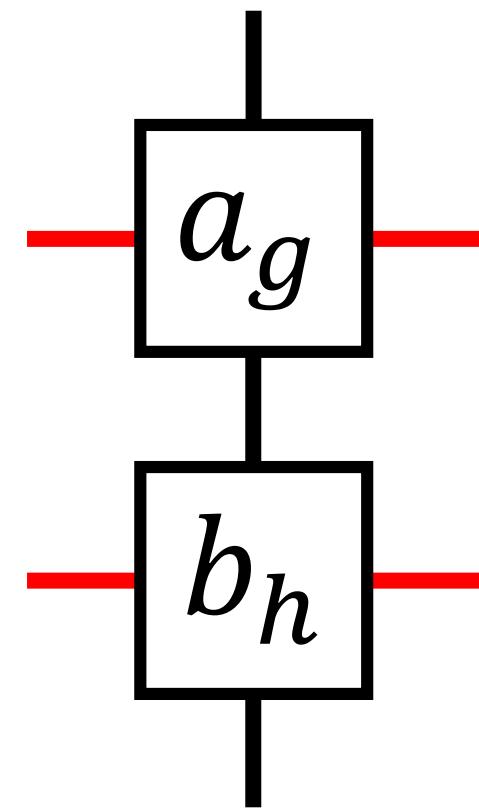


morphisms

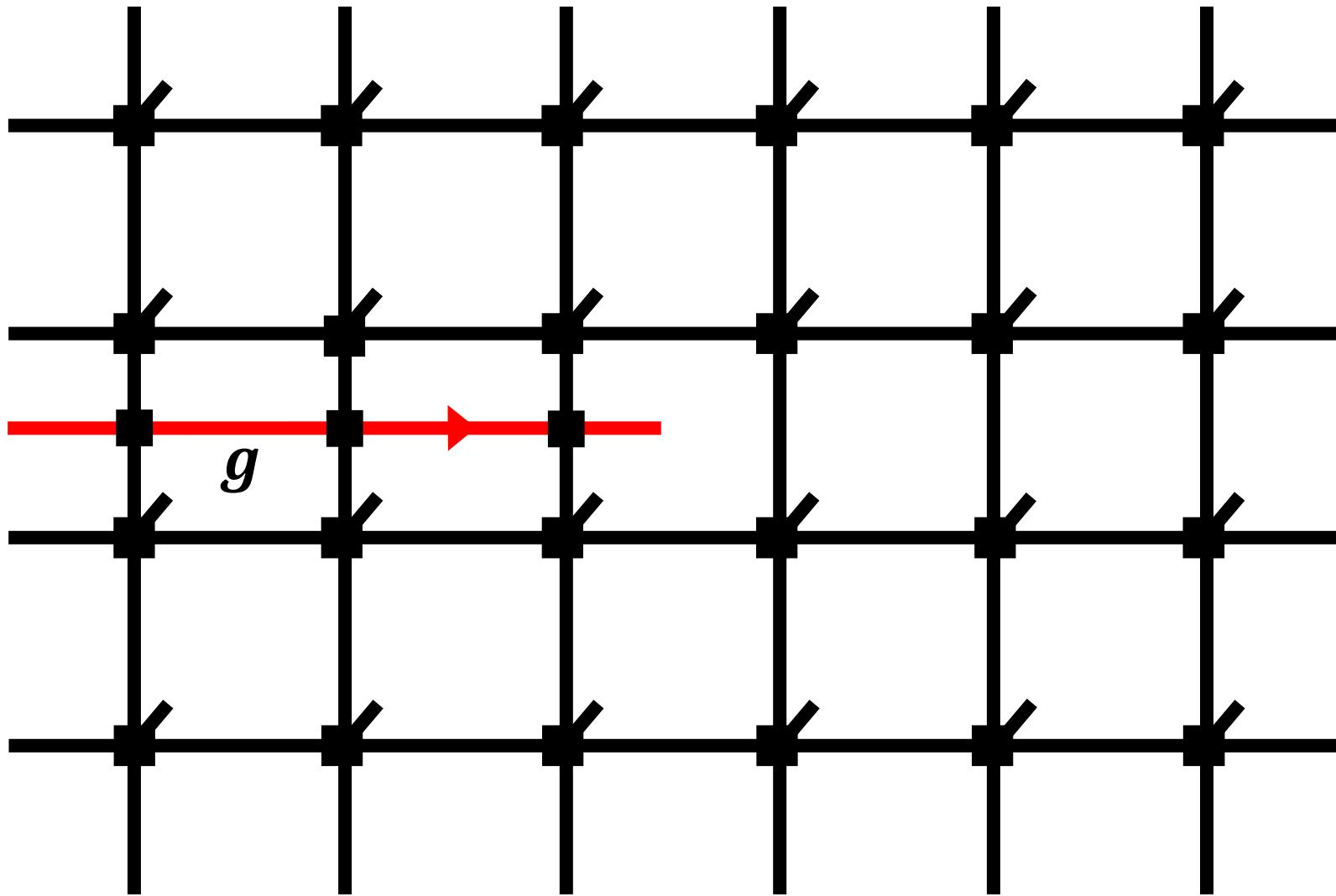
G -graded fusion category \mathcal{C}_G

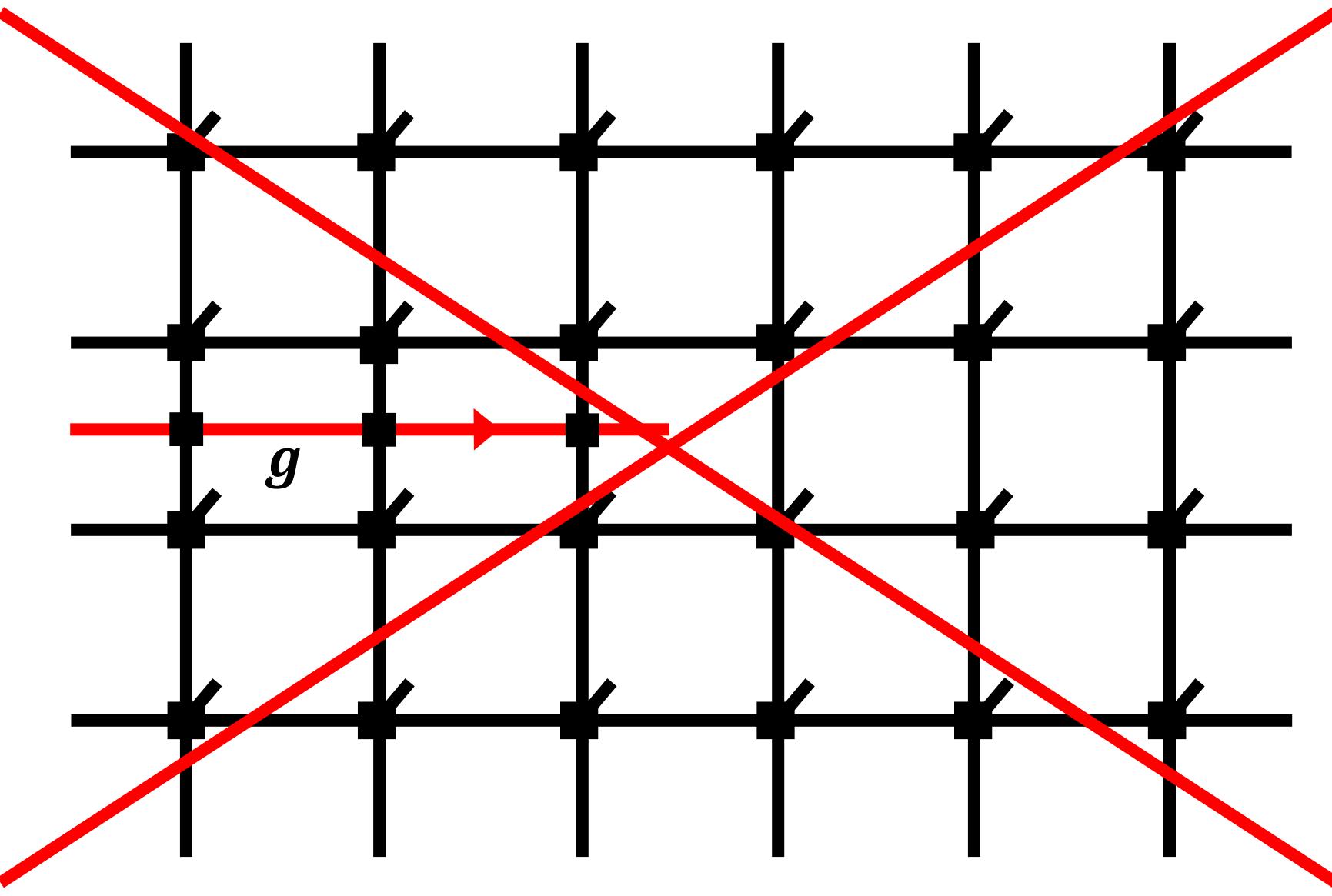
G -graded
tensor product

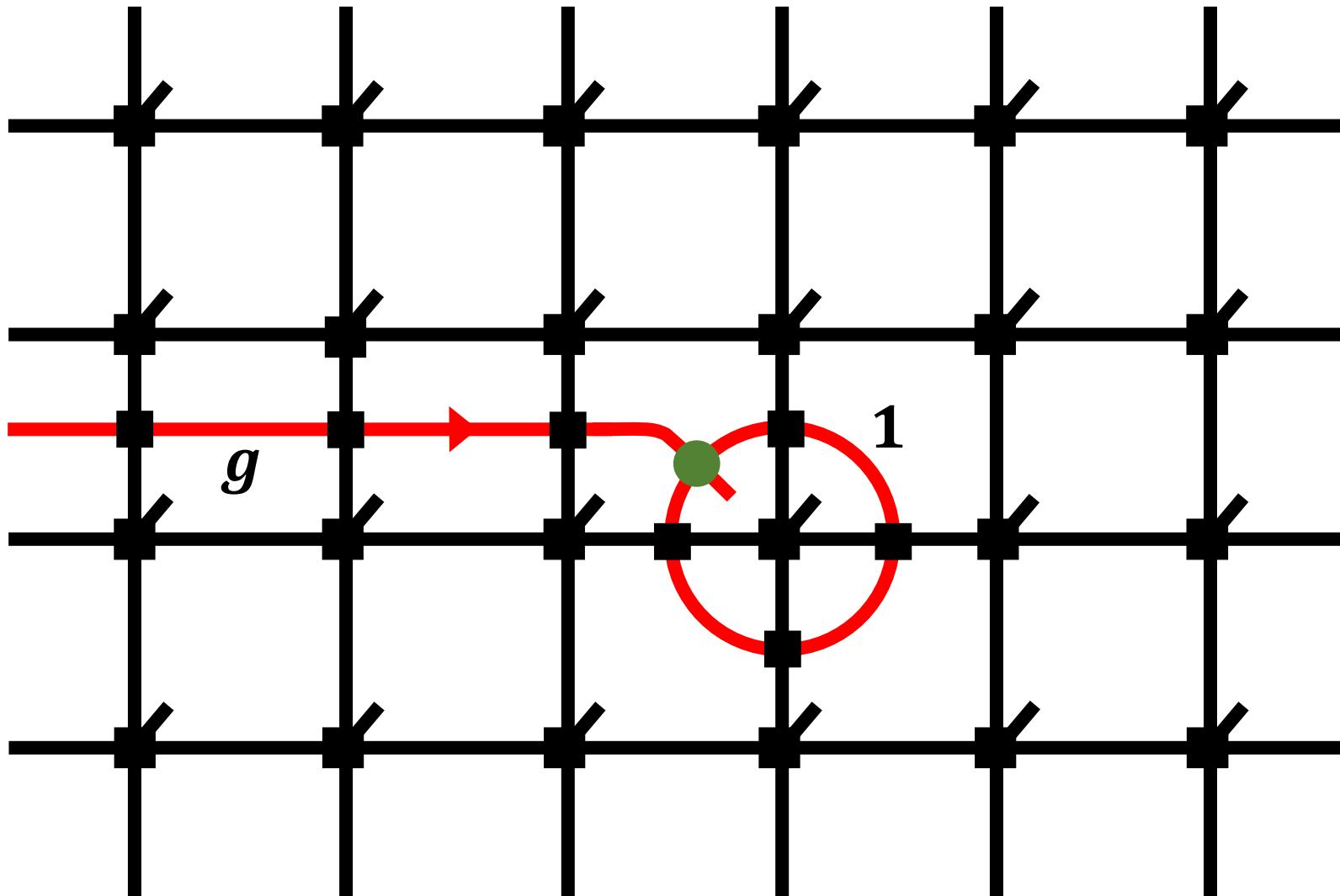
$$a_g \otimes b_h$$

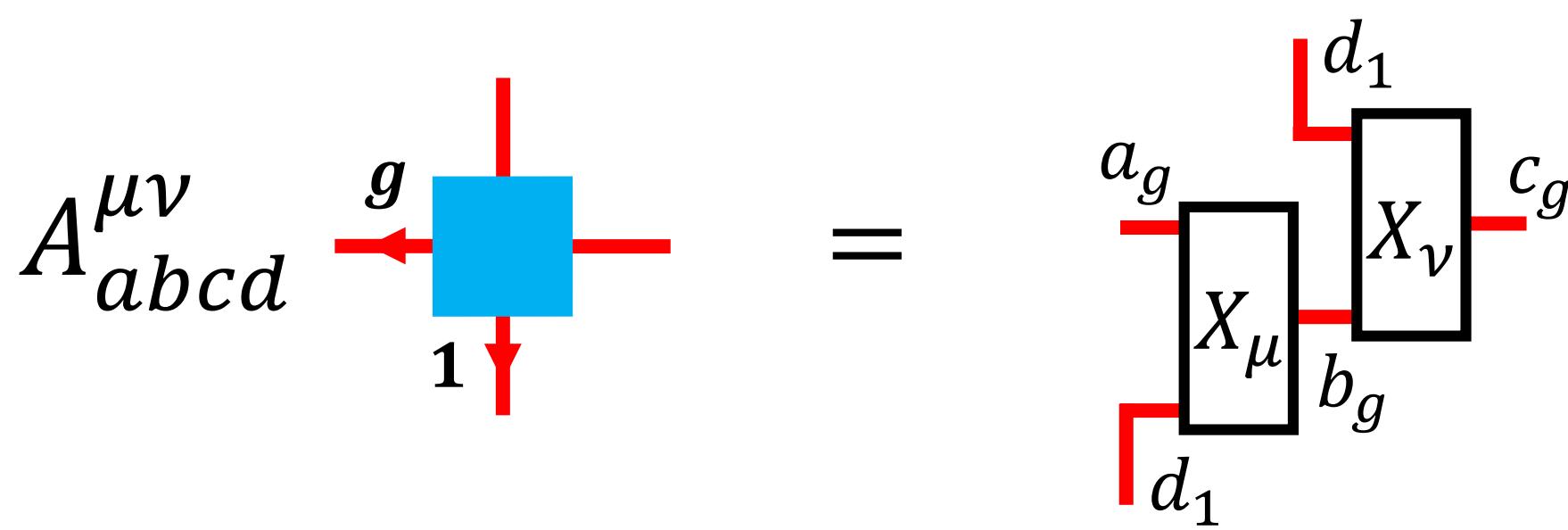
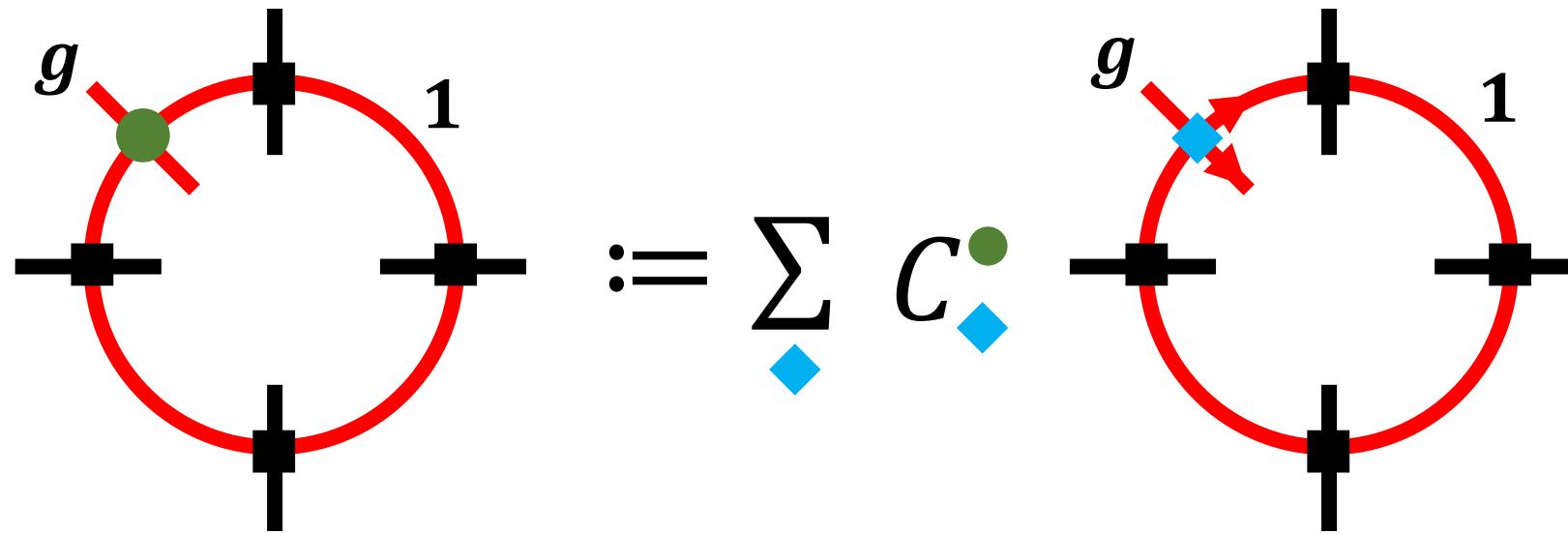


What about anyons and
defects?





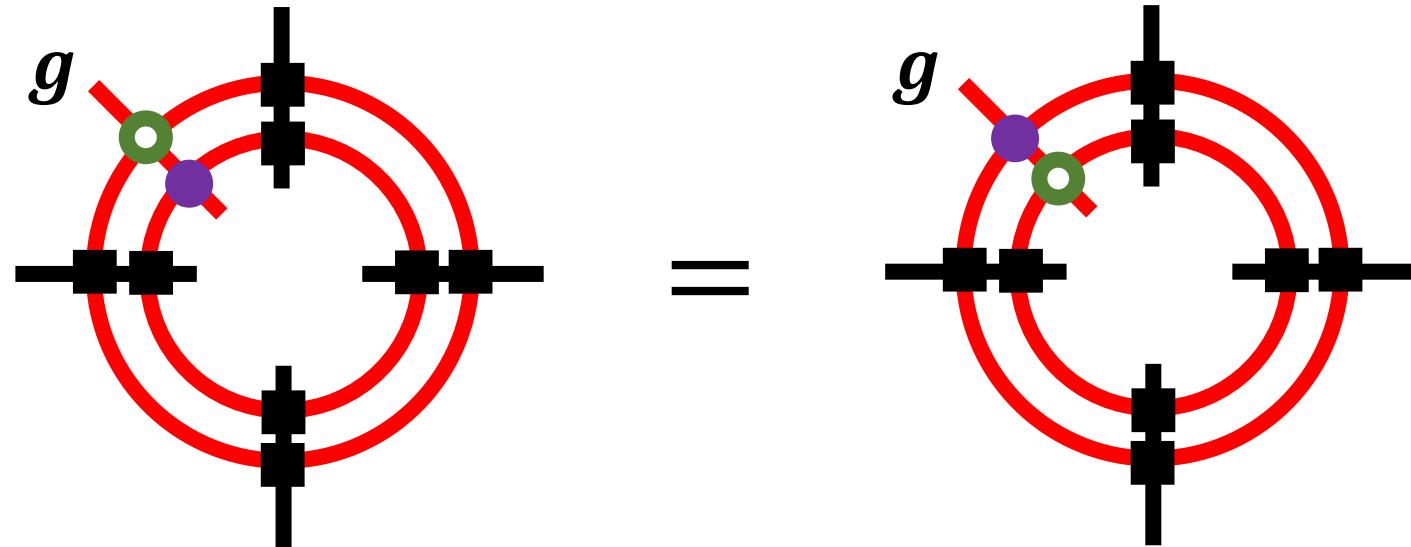




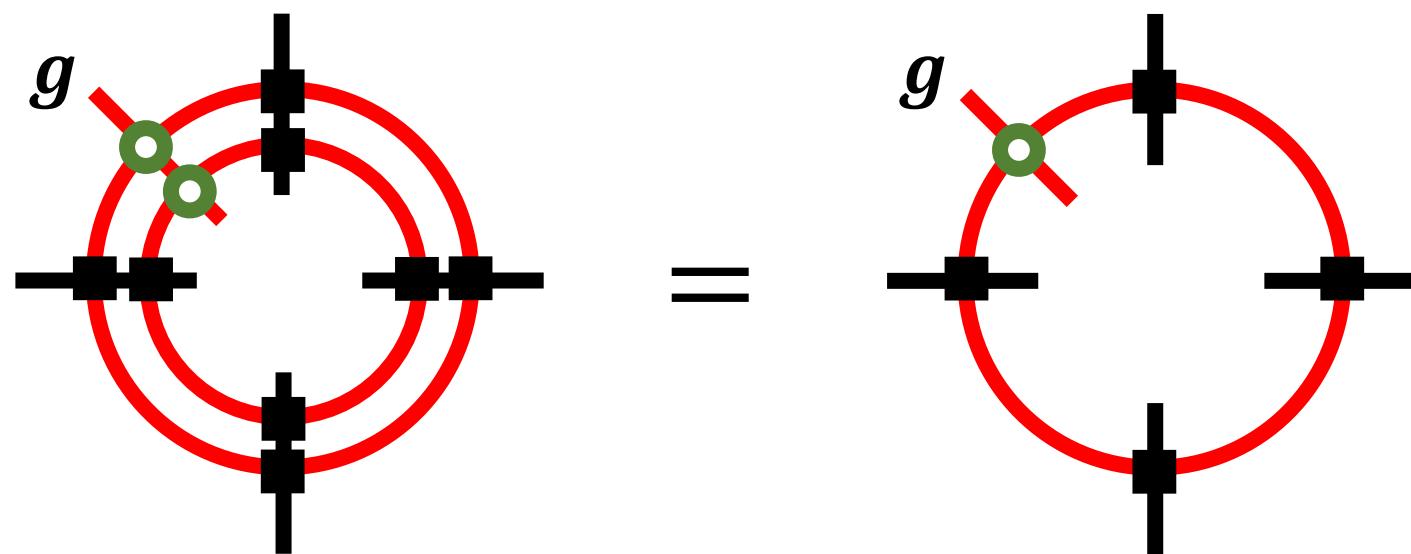
$$\left(\begin{array}{c} g \\ \text{---} \\ \text{---} \end{array} \right) = \left(\begin{array}{c} g \\ \text{---} \\ \text{---} \end{array} \right)$$

$$\left(\begin{array}{c} g \\ \text{---} \\ \text{---} \end{array} \right) = \left(\begin{array}{c} g \\ \text{---} \\ \text{---} \end{array} \right)$$

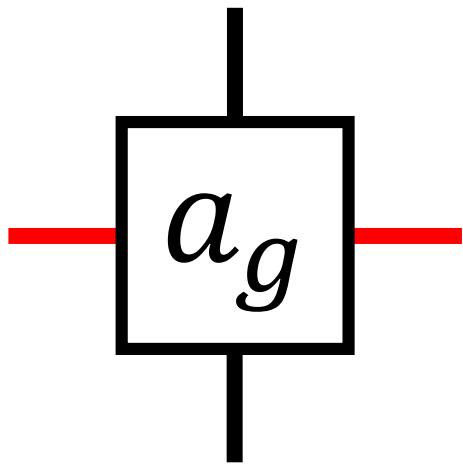
Algebra



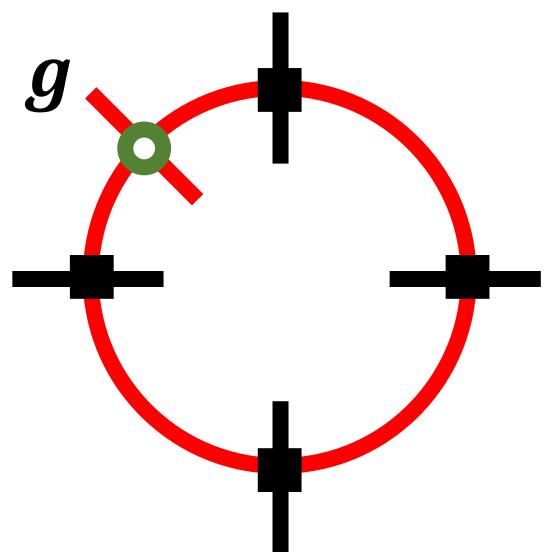
Central



Idempotent



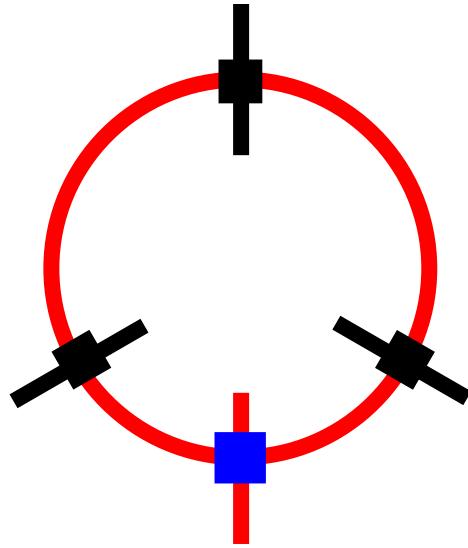
: \mathcal{C}_G



: $Z(\mathcal{C}_0)_G$

Examples

Toric Code

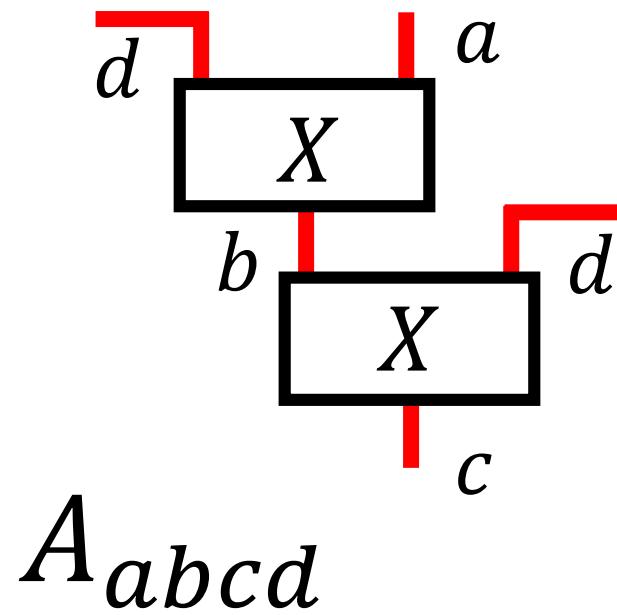


$$0 \begin{array}{c} | \\ - \\ | \end{array} 0 = I$$

$$1 \begin{array}{c} | \\ - \\ | \end{array} 1 = Z$$

1	0 0 0	$0 = 1$	0 1 0	$1 = 1$
e	0 0 0	$0 = 1$	1 0 0	$1 = -1$
m	0 1 1	$0 = 1$	1 1 1	$1 = 1$
em	0 1 1	$0 = 1$	1 1 1	$1 = -1$

Double Ising



$$(1, 1) = \frac{1}{4} (A_{1111} + 2^{3/4} A_{1\sigma 1\sigma} + A_{1\psi 1\psi})$$

$$(\sigma, 1) = \frac{1}{4} (A_{\sigma\sigma\sigma 1} + 2^{1/4} e^{\frac{\pi i}{8}} A_{\sigma 1\sigma\sigma} + 2^{1/4} e^{-\frac{3\pi i}{8}} A_{\sigma\psi\sigma\sigma} + e^{\frac{\pi i}{2}} A_{\sigma\sigma\sigma\psi})$$

$$(1, \bar{\sigma}) = \frac{1}{4} (A_{\sigma\sigma\sigma 1} + 2^{1/4} e^{-\frac{\pi i}{8}} A_{\sigma 1\sigma\sigma} + 2^{1/4} e^{\frac{3\pi i}{8}} A_{\sigma\psi\sigma\sigma} + e^{-\frac{\pi i}{2}} A_{\sigma\sigma\sigma\psi})$$

$$(\psi, 1) = \frac{1}{4} (A_{\psi\psi\psi 1} + 2^{3/4} e^{\frac{\pi i}{2}} A_{\psi\sigma\psi\sigma} - A_{\psi 1\psi\psi})$$

$$(1, \psi) = \frac{1}{4} (A_{\psi\psi\psi 1} + 2^{3/4} e^{-\frac{\pi i}{2}} A_{\psi\sigma\psi\sigma} - A_{\psi 1\psi\psi})$$

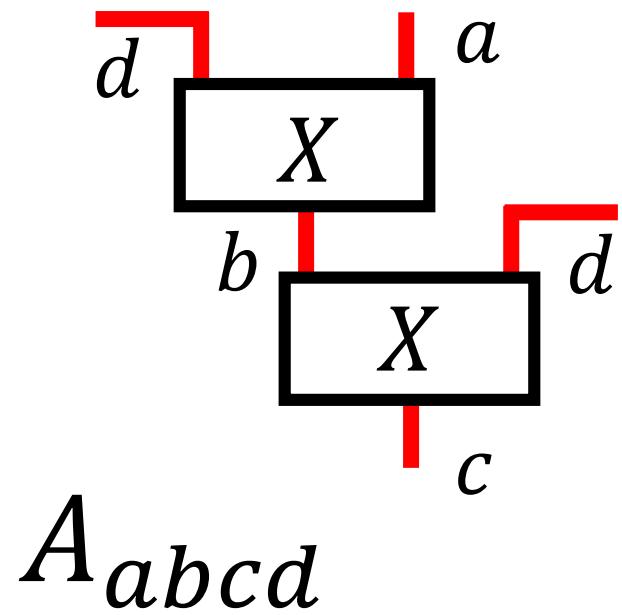
$$(\sigma, \bar{\psi}) = \frac{1}{4} (A_{\sigma\sigma\sigma 1} + 2^{1/4} e^{-\frac{7\pi i}{8}} A_{\sigma 1\sigma\sigma} + 2^{1/4} e^{\frac{5\pi i}{8}} A_{\sigma\psi\sigma\sigma} + e^{\frac{\pi i}{2}} A_{\sigma\sigma\sigma\psi})$$

$$(\psi, \bar{\sigma}) = \frac{1}{4} (A_{\sigma\sigma\sigma 1} + 2^{1/4} e^{\frac{7\pi i}{8}} A_{\sigma 1\sigma\sigma} + 2^{1/4} e^{-\frac{5\pi i}{8}} A_{\sigma\psi\sigma\sigma} + e^{-\frac{\pi i}{2}} A_{\sigma\sigma\sigma\psi})$$

$$(\psi, \bar{\psi}) = \frac{1}{4} (A_{1111} - 2^{3/4} A_{1\sigma 1\sigma} + A_{1\psi 1\psi})$$

$$(\sigma, \bar{\sigma}) = \frac{1}{2} (A_{1111} + A_{\psi\psi\psi 1} - A_{1\psi 1\psi} + A_{\psi 1\psi\psi})$$

Condensing \mathbb{Z}_2



$$(1, 1) = \frac{1}{4} (A_{1111} + 2^{3/4} \cancel{A_{1\sigma 1\sigma}} + A_{1\psi 1\psi})$$

$$(\sigma, 1) = \frac{1}{4} (A_{\sigma\sigma\sigma 1} + 2^{1/4} e^{\frac{\pi i}{8}} \cancel{A_{\sigma 1\sigma\sigma}} + 2^{1/4} e^{-\frac{3\pi i}{8}} \cancel{A_{\sigma\psi\sigma\sigma}} + e^{\frac{\pi i}{2}} A_{\sigma\sigma\sigma\psi})$$

$$(1, \bar{\sigma}) = \frac{1}{4} (A_{\sigma\sigma\sigma 1} + 2^{1/4} e^{-\frac{\pi i}{8}} \cancel{A_{\sigma 1\sigma\sigma}} + 2^{1/4} e^{\frac{3\pi i}{8}} \cancel{A_{\sigma\psi\sigma\sigma}} + e^{-\frac{\pi i}{2}} A_{\sigma\sigma\sigma\psi})$$

$$(\psi, 1) = \frac{1}{4} (A_{\psi\psi\psi 1} + 2^{3/4} e^{\frac{\pi i}{2}} \cancel{A_{\psi\sigma\psi\sigma}} - A_{\psi 1\psi\psi})$$

$$(1, \psi) = \frac{1}{4} (A_{\psi\psi\psi 1} + 2^{3/4} e^{-\frac{\pi i}{2}} \cancel{A_{\psi\sigma\psi\sigma}} - A_{\psi 1\psi\psi})$$

$$(\sigma, \bar{\psi}) = \frac{1}{4} (A_{\sigma\sigma\sigma 1} + 2^{1/4} e^{-\frac{7\pi i}{8}} \cancel{A_{\sigma 1\sigma\sigma}} + 2^{1/4} e^{\frac{5\pi i}{8}} \cancel{A_{\sigma\psi\sigma\sigma}} + e^{\frac{\pi i}{2}} A_{\sigma\sigma\sigma\psi})$$

$$(\psi, \bar{\sigma}) = \frac{1}{4} (A_{\sigma\sigma\sigma 1} + 2^{1/4} e^{\frac{7\pi i}{8}} \cancel{A_{\sigma 1\sigma\sigma}} + 2^{1/4} e^{-\frac{5\pi i}{8}} \cancel{A_{\sigma\psi\sigma\sigma}} + e^{-\frac{\pi i}{2}} A_{\sigma\sigma\sigma\psi})$$

$$(\psi, \bar{\psi}) = \frac{1}{4} (A_{1111} - 2^{3/4} \cancel{A_{1\sigma 1\sigma}} + A_{1\psi 1\psi})$$

$$(\sigma, \bar{\sigma}) = \frac{1}{2} (A_{1111} + A_{\psi\psi\psi 1} - A_{1\psi 1\psi} + A_{\psi 1\psi\psi})$$

Condensing \mathbb{Z}_2

1	$(1, 1)$	\sim	$(\psi, \bar{\psi})$
e, m	$(\sigma, \bar{\sigma})$	\rightarrow	Splits
em	$(\psi, 1)$	\sim	$(1, \psi)$
<hr/>	<hr/>	<hr/>	<hr/>
σ_+	$(\sigma, 1)$	\sim	$(\sigma, \bar{\psi})$
σ_-	$(1, \bar{\sigma})$	\sim	$(\psi, \bar{\sigma})$



$$(1, 1) = \frac{1}{4} (A_{1111} + 2^{3/4} \cancel{A_{1\sigma 1\sigma}} + A_{1\psi 1\psi})$$

$$(\sigma, 1) = \frac{1}{4} (A_{\sigma\sigma\sigma 1} + 2^{1/4} e^{\frac{\pi i}{8}} \cancel{A_{\sigma 1\sigma\sigma}} + 2^{1/4} e^{-\frac{3\pi i}{8}} \cancel{A_{\sigma\psi\sigma\sigma}} + e^{\frac{\pi i}{2}} A_{\sigma\sigma\sigma\psi})$$

$$(1, \bar{\sigma}) = \frac{1}{4} (A_{\sigma\sigma\sigma 1} + 2^{1/4} e^{-\frac{\pi i}{8}} \cancel{A_{\sigma 1\sigma\sigma}} + 2^{1/4} e^{\frac{3\pi i}{8}} \cancel{A_{\sigma\psi\sigma\sigma}} + e^{-\frac{\pi i}{2}} A_{\sigma\sigma\sigma\psi})$$

$$(\psi, 1) = \frac{1}{4} (A_{\psi\psi\psi 1} + 2^{3/4} e^{\frac{\pi i}{2}} \cancel{A_{\psi\sigma\psi\sigma}} - A_{\psi 1\psi\psi})$$

$$(1, \psi) = \frac{1}{4} (A_{\psi\psi\psi 1} + 2^{3/4} e^{-\frac{\pi i}{2}} \cancel{A_{\psi\sigma\psi\sigma}} - A_{\psi 1\psi\psi})$$

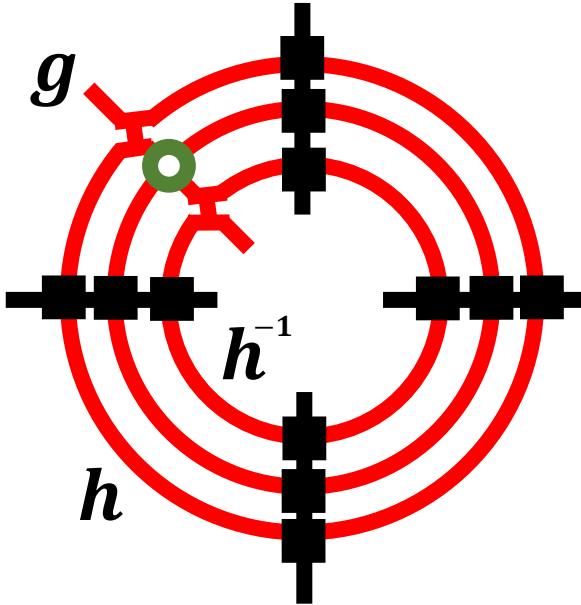
$$(\sigma, \bar{\psi}) = \frac{1}{4} (A_{\sigma\sigma\sigma 1} + 2^{1/4} e^{-\frac{7\pi i}{8}} \cancel{A_{\sigma 1\sigma\sigma}} + 2^{1/4} e^{\frac{5\pi i}{8}} \cancel{A_{\sigma\psi\sigma\sigma}} + e^{\frac{\pi i}{2}} A_{\sigma\sigma\sigma\psi})$$

$$(\psi, \bar{\sigma}) = \frac{1}{4} (A_{\sigma\sigma\sigma 1} + 2^{1/4} e^{\frac{7\pi i}{8}} \cancel{A_{\sigma 1\sigma\sigma}} + 2^{1/4} e^{-\frac{5\pi i}{8}} \cancel{A_{\sigma\psi\sigma\sigma}} + e^{-\frac{\pi i}{2}} A_{\sigma\sigma\sigma\psi})$$

$$(\psi, \bar{\psi}) = \frac{1}{4} (A_{1111} - 2^{3/4} \cancel{A_{1\sigma 1\sigma}} + A_{1\psi 1\psi})$$

$$(\sigma, \bar{\sigma}) = \frac{1}{2} (\underline{A_{1111}} + \underline{A_{\psi\psi\psi 1}} - \underline{A_{1\psi 1\psi}} + \underline{A_{\psi 1\psi\psi}})$$

U_g :



$$A_{1111}$$

$$A_{\psi\psi\psi 1}$$

$$A_{1\psi 1\psi}$$

$$A_{\psi 1\psi\psi}$$



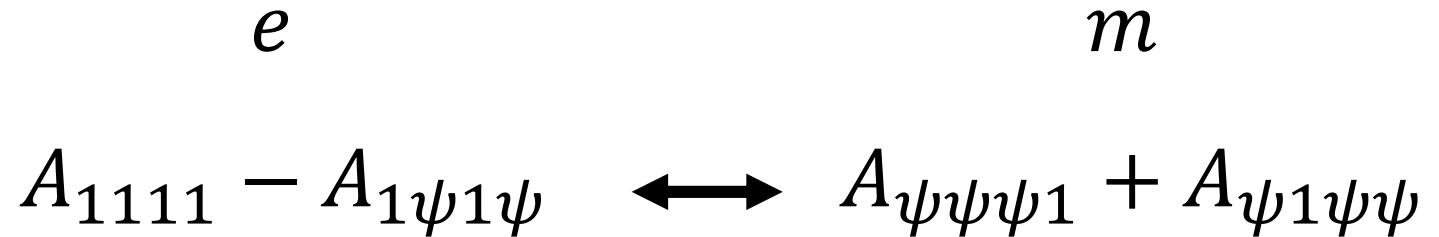
$$A_{1111} + A_{\psi\psi\psi 1} + A_{1\psi 1\psi} + A_{\psi 1\psi\psi}$$

$$A_{1111} + A_{\psi\psi\psi 1} - A_{1\psi 1\psi} - A_{\psi 1\psi\psi}$$

$$A_{1111} - A_{\psi\psi\psi 1} + A_{1\psi 1\psi} - A_{\psi 1\psi\psi}$$

$$A_{1111} - A_{\psi\psi\psi 1} - A_{1\psi 1\psi} + A_{\psi 1\psi\psi}$$

$U:$



$$A_{1111}$$

$$A_{\psi\psi\psi 1}$$

$$A_{1\psi 1\psi}$$

$$A_{\psi 1\psi\psi}$$



$$A_{1111} + A_{\psi\psi\psi 1} + A_{1\psi 1\psi} + A_{\psi 1\psi\psi}$$

$$A_{1111} + A_{\psi\psi\psi 1} - A_{1\psi 1\psi} - A_{\psi 1\psi\psi}$$

$$A_{1111} - A_{\psi\psi\psi 1} + A_{1\psi 1\psi} - A_{\psi 1\psi\psi}$$

$$A_{1111} - A_{\psi\psi\psi 1} - A_{1\psi 1\psi} + A_{\psi 1\psi\psi}$$

- Tensor networks describing SET states have matrix product operator symmetries
- By studying these MPOs we extract a graded fusion algebra
- From this we can construct a dube algebra to extract (all) information about the SET order in the system

Questions?

Questions?



Normal form

$$A^i \sim \begin{pmatrix} A_0^i & M_{01}^i & \cdots \\ 0 & A_1^i & \ddots \\ \vdots & & \ddots \end{pmatrix}$$

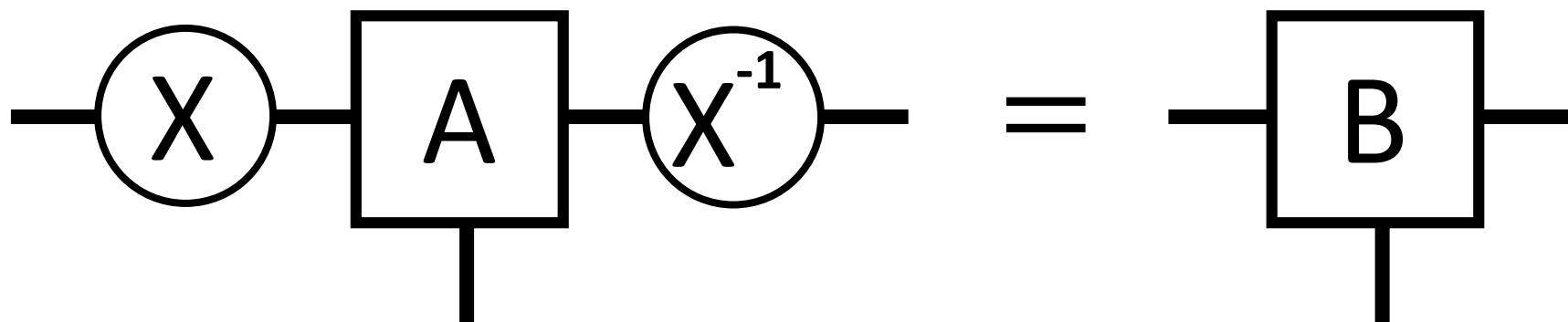
s.t. $\{A_k^i\}_i$ generates
an irred. algebra

$$|MPS_N(A)\rangle = \sum_k |MPS_N(A_k)\rangle$$

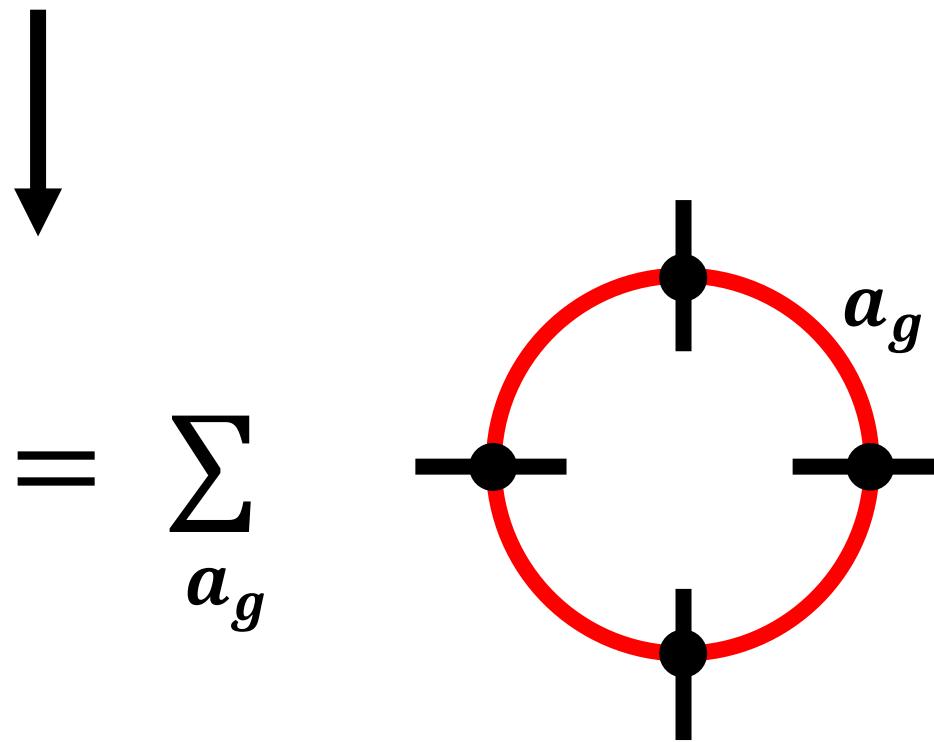
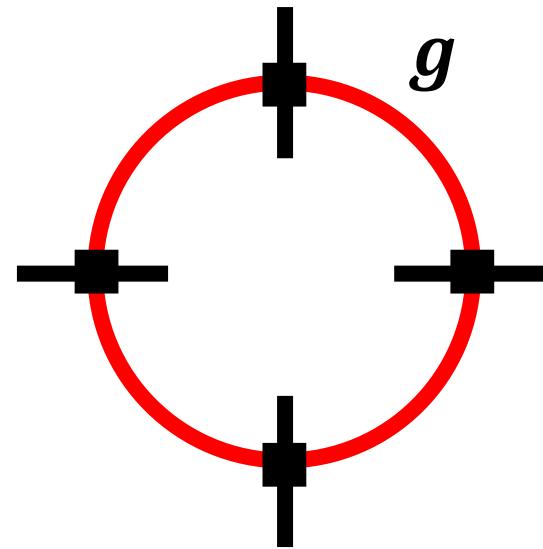
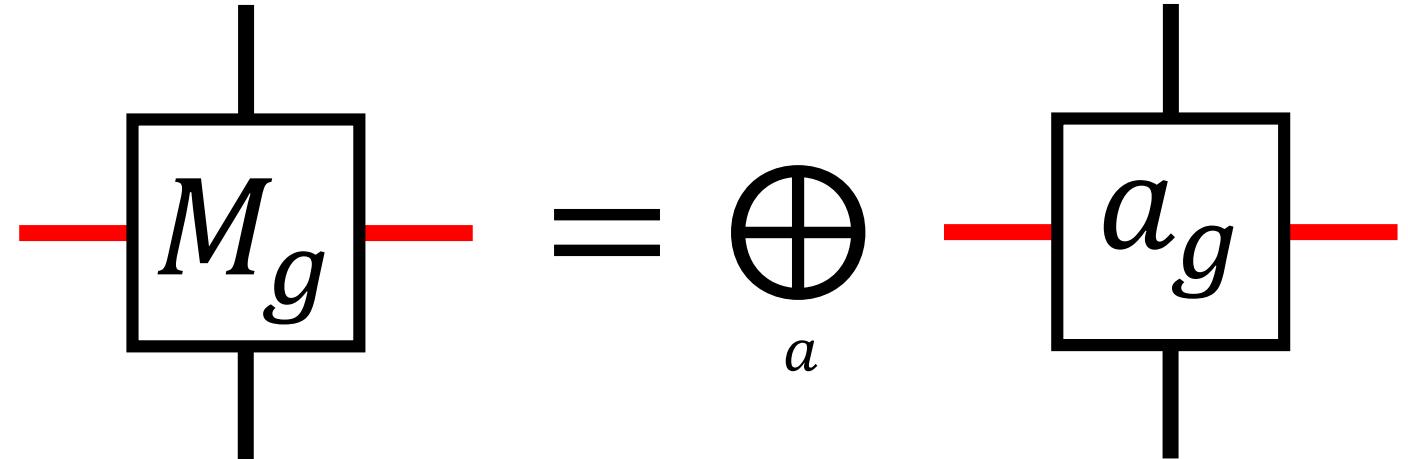
Fundamental Thm. of MPS

$$|MPS_N(A)\rangle = |MPS_N(B)\rangle \quad \forall N$$

$\Rightarrow \exists X$ s.t.



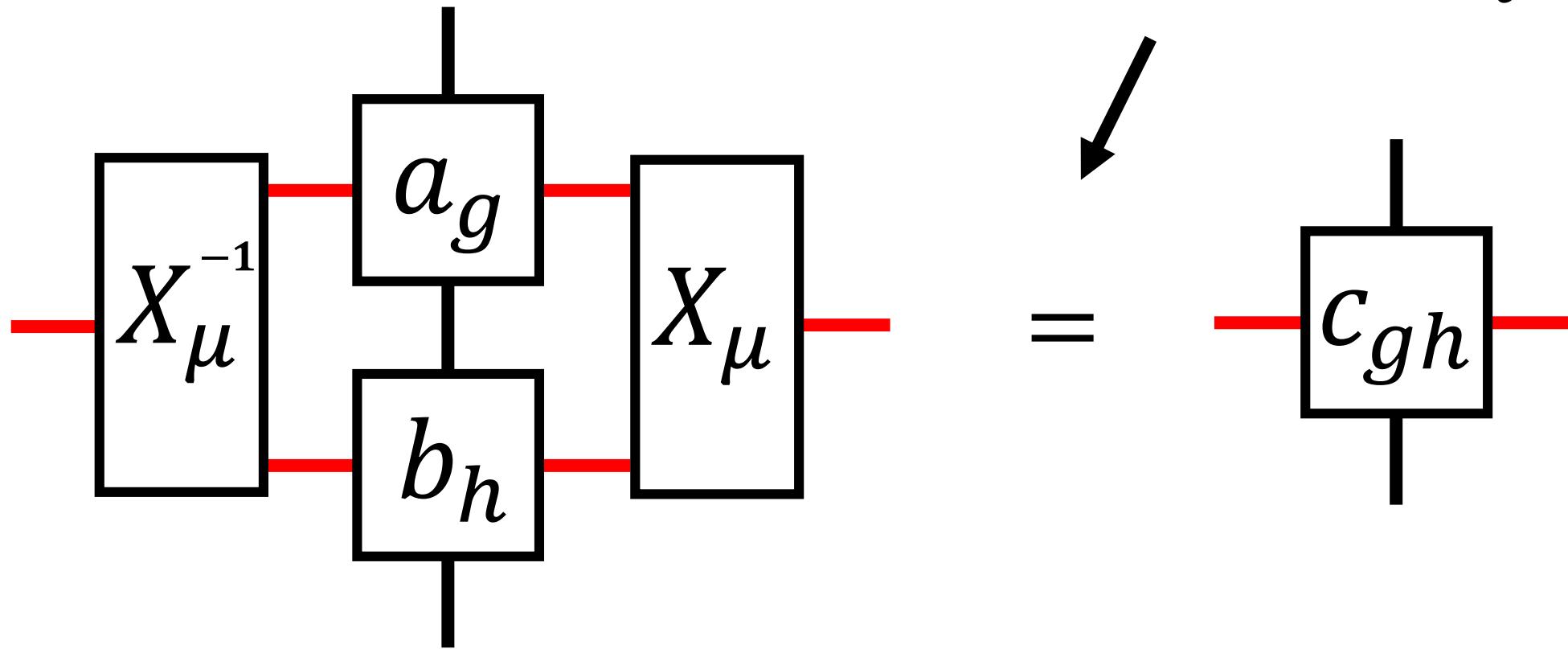
Normal form:



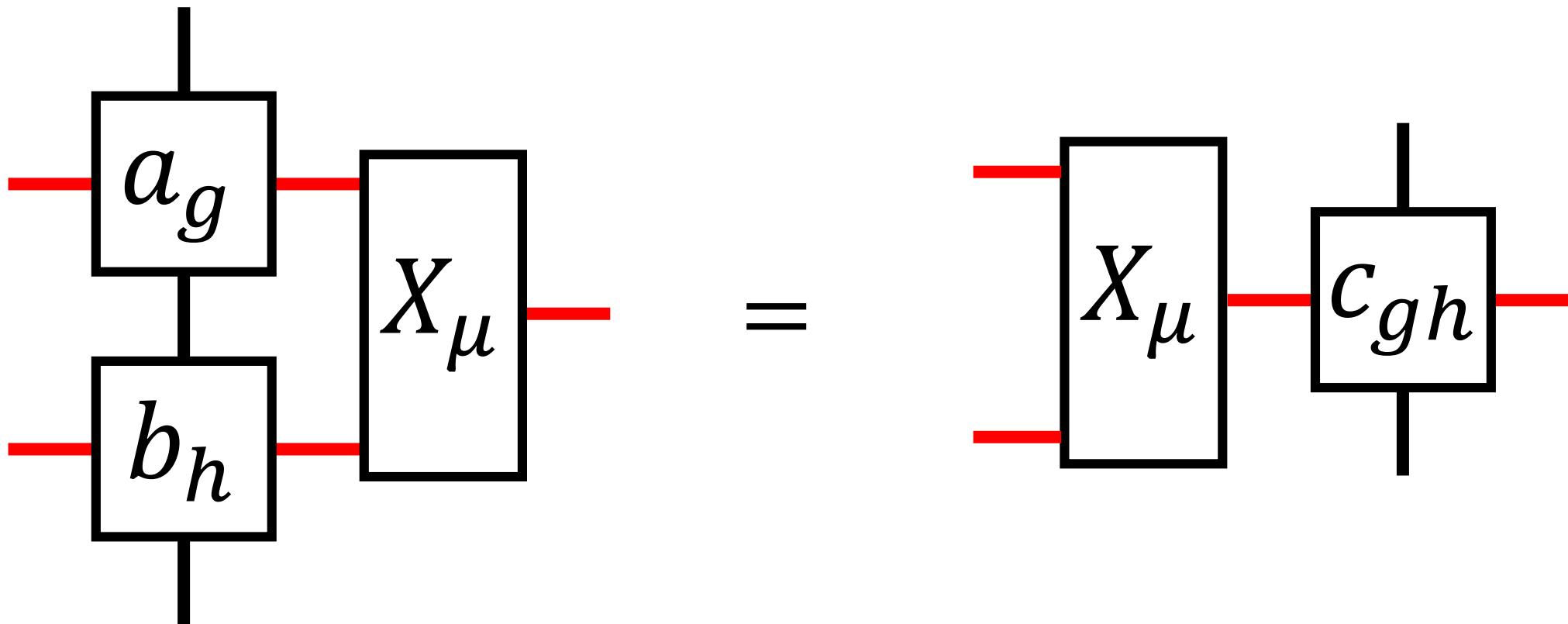
$$M_g(L)M_h(L) = M_{gh}(L)$$



$$M_{ag} M_{bh} = \sum_c N_{ab}^c M_{cgh}$$



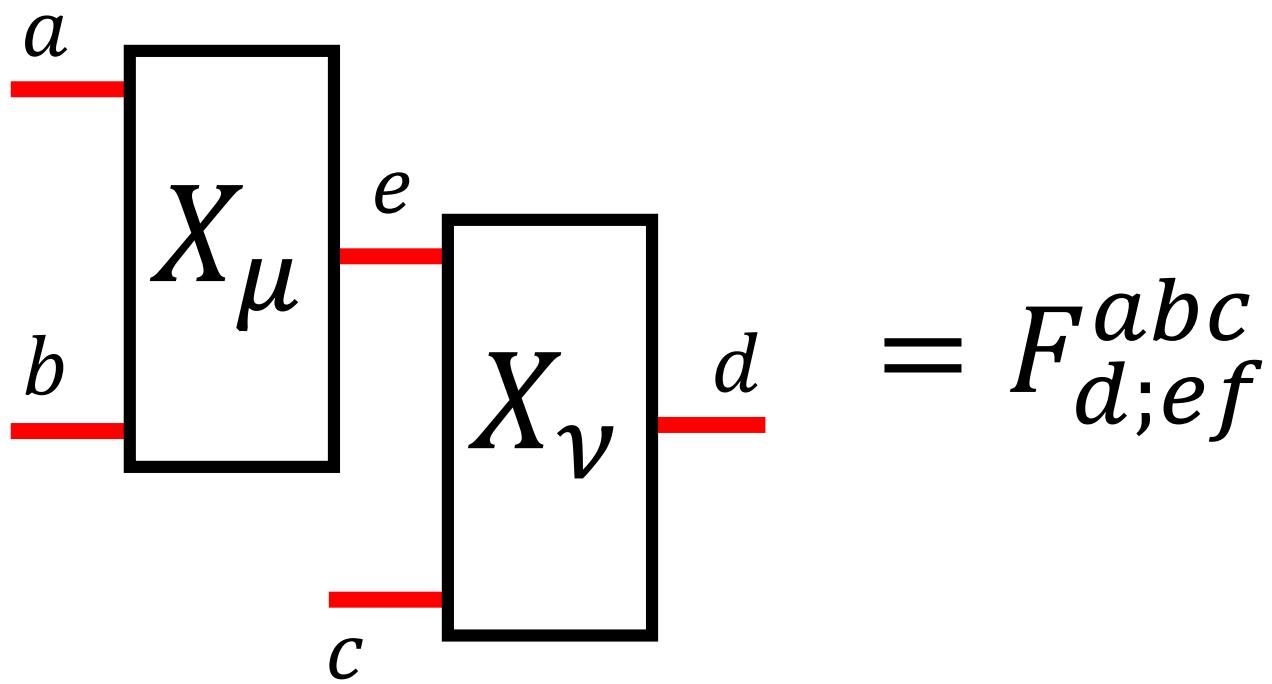
The Zipper condition



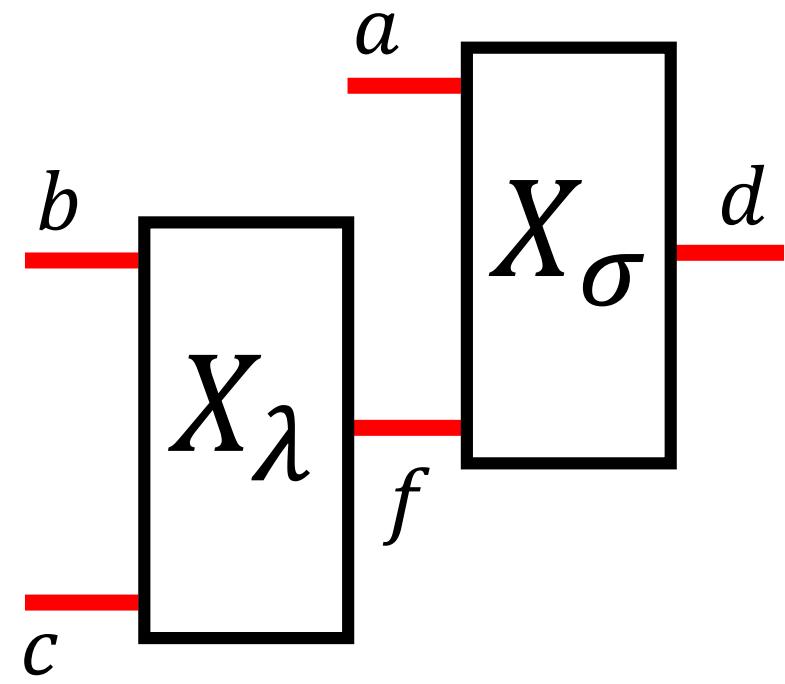
$$\alpha \begin{array}{c} \beta \\ \diagup \\ \gamma \end{array} i \delta = \begin{array}{c} \text{Diagram of a red circle with four black cross caps at the vertices. The number '1' is placed to the right of the circle.} \end{array}$$

Zipper \Rightarrow

$$\begin{array}{c} \text{Diagram showing a red circle with four black cross caps. A vertical red line labeled 'g' passes through the center. The number '1' is placed to the right of the circle.} \end{array} = \begin{array}{c} \text{Diagram showing a red circle with four black cross caps. A horizontal red line labeled 'g' is at the top, and another horizontal red line labeled 'g' is at the bottom. The number '1' is placed to the right of the circle.} \end{array}$$



$$= F_{d;ef}^{abc}$$



$$\sum_{de\mu\nu} \begin{array}{c} a \\ b \\ c \end{array} \begin{array}{c} X_\mu \\ | \\ X_\nu \end{array} e \begin{array}{c} d \\ \bullet \end{array} \begin{array}{c} X_\nu^+ \\ | \\ X_\mu^+ \end{array} e \begin{array}{c} a \\ b \\ c \end{array} = \sum_{df\sigma\lambda} \begin{array}{c} a \\ b \\ c \end{array} \begin{array}{c} X_\lambda \\ | \\ X_\sigma \end{array} f \begin{array}{c} X_\sigma^+ \\ | \\ X_\lambda^+ \end{array} f \begin{array}{c} d \\ \bullet \end{array} \begin{array}{c} a \\ b \\ c \end{array}$$

$$\begin{array}{c} a \\ b \\ c \end{array} \begin{array}{c} X_\mu \\ | \\ X_\nu \end{array} d = \sum_{d'f\sigma\lambda} \begin{array}{c} a \\ b \\ c \end{array} \begin{array}{c} X_\lambda \\ | \\ X_\sigma \end{array} f \begin{array}{c} d' \\ \bullet \\ X_\sigma^+ \end{array} \begin{array}{c} a \\ b \\ f \end{array} \begin{array}{c} X_\lambda^+ \\ | \\ X_\mu \end{array} b \begin{array}{c} e \\ \bullet \\ X_\nu \end{array} d$$

$$\begin{array}{c} a \\ b \\ c \end{array} \begin{array}{c} X_\mu \\ | \\ X_\nu \end{array} d \otimes \underline{d} = \sum_{f\sigma\lambda} \begin{array}{c} a \\ b \\ c \end{array} \begin{array}{c} X_\lambda \\ | \\ X_\sigma \end{array} f \begin{array}{c} d \\ \otimes \\ X_\sigma^+ \end{array} \begin{array}{c} a \\ b \\ f \end{array} \begin{array}{c} X_\lambda^+ \\ | \\ X_\mu \end{array} b \begin{array}{c} e \\ \bullet \\ X_\nu \end{array} d$$