Wigner function negativity and contextuality in quantum computation

Nicolas Delfosse
Caltech & UCR

joint work with J. Bermejo-Vega, D. Browne, C. Okay, R. Raussendorf

Coogee - Jan 31, 2017

Motivation

- Understanding what distinguishes quantum mechanics from classical mechanics.
- Why? We must exploit these properties to obtain a quantum superiority for certain tasks.

In this work:

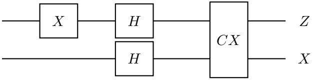
- Contextuality (from foundation)
- Negativity of the Wigner function (from quantum optics)

Outline

- Quopit Wigner functions
  - Negativity is a resource
  - Contextuality is a resource
  - Equivalence

- Generalization to qubits
  - Negativity is NOT a ressource
  - Contextuality is a resource
  - Comparison between negativity and contextuality

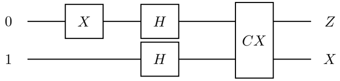
Classical simulation with quopit Wigner functions



Theorem [1]: We can simulate classicaly any Clifford circuit in polynomial time.

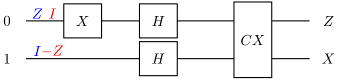
[1] Gotesman (1998)

Simulation of Clifford gates



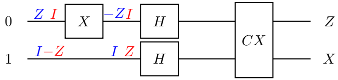
Theorem [1]: We can simulate classicaly any Clifford circuit in polynomial time.

[1] Gotesman (1998)



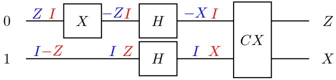
Theorem [1]: We can simulate classicaly any Clifford circuit in polynomial time.

[1] Gotesman (1998)



Theorem [1]: We can simulate classicaly any Clifford circuit in polynomial time.

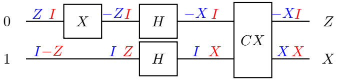
[1] Gotesman (1998)



Theorem [1]: We can simulate classicaly any Clifford circuit in polynomial time.

[1] Gotesman (1998)

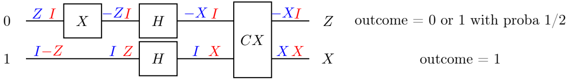
Simulation of Clifford gates



Theorem [1]: We can simulate classicaly any Clifford circuit in polynomial time.

[1] Gotesman (1998)

Simulation of Clifford gates



Theorem [1]: We can simulate classicaly any Clifford circuit in polynomial time.

[1] Gotesman (1998)

Definition of discrete Wigner functions

For $(A_{\mathbf{u}})_{\mathbf{u}\in\mathbf{V}}$ a basis of the space of matrices acting on $(\mathbb{C}^d)^{\otimes n}$.

$$\rho = \sum_{\mathbf{u}\in V} \alpha_{\mathbf{u}} A_{\mathbf{u}}$$

Definition: The Wigner function of ρ associated with the basis $(A_{\mathbf{u}})$ is defined by

$$W_\rho(\mathbf{u}) = \alpha_{\mathbf{u}}.$$

What is a good basis in that context?

Definition of discrete Wigner functions

What is a good basis in that context?

- Inner product over $(m \times m)$-matrices

$$(A|B) = (1/m)\,\mathrm{Tr}(AB^\dagger).$$

- If the basis $(A_{\mathbf{u}})$ is orthonormal, then

$$W_\rho(\mathbf{u}) = (1/m)\,\mathrm{Tr}(A_u \rho).$$

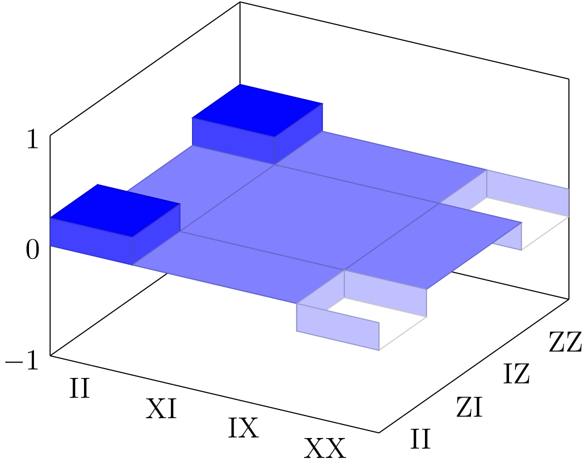
- Simple description of stabilizer states.

Natural choice: The Pauli basis.

Wigner function associated with Pauli basis

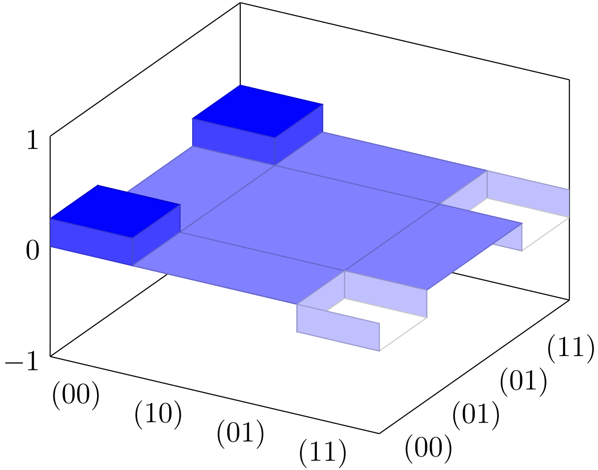
"Pauli Wigner function" of the stabilizer state

$\rho = \mathrm{Fix}(XX, -ZZ)$



Wigner function associated with Pauli basis

"Pauli Wigner function" of the stabilizer state
$\rho = \mathrm{Fix}(XX, -ZZ)$



A better basis

Let $\mathbf{u} = (\mathbf{u}_Z, \mathbf{u}_X)$ and $\omega = e^{2i\pi/p}$.

Two basis:

- (1) Heisenberg-Weyl basis:

$$T_{\mathbf{u}} = \omega^{-\frac{(\mathbf{u}_Z|\mathbf{u}_X)}{2}} Z^{\mathbf{u}_Z} X^{\mathbf{u}_X}$$

- (2) Its Fourier transform (the 'good' choice):

$$A_{\mathbf{u}} = d^{-n} \sum_{\mathbf{v} \in \mathbb{Z}_d^{2n}} \omega^{[\mathbf{u},\mathbf{v}]} T_{\mathbf{v}}$$

Why is (2) good?

- **Theorem[2]: $W_{|\psi\rangle\langle\psi|} \geq 0$ iff $|\psi\rangle\langle\psi|$ is a stabilizer state.**
- Probabilistic interpretation

---

[2] Gross. J. Math. Phys. 2006

A better basis

For n qubits

$$W_\rho : \mathbb{Z}_2^n \times \mathbb{Z}_2^n \longrightarrow \mathbb{R}$$

- Wigner function of $\rho = \mathrm{Fix}(XX, -ZZ)$:

$$W_\rho = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1/4 & 0 & 0 & 1/4 \\ 1/4 & 0 & 0 & 1/4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- $W_\rho \geq 0 \Rightarrow$ probability distribution.
- Interpretation: ρ is in position u with proba $W_\rho(u)$.

Wigner function of a measurement

If (E_m) is measured and $\rho = \sum_u W_\rho(u) A_u$ then

$$\mathbb{P}(m) = \mathrm{Tr}(E_m \rho) = \sum_u W_\rho(u) \, \mathrm{Tr}(E_m A_u).$$

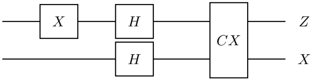
- **Definition: The Wigner function of E_m is**

$$W_{E_m}(u) := \mathrm{Tr}(E_m A_{\mathbf{u}}).$$

- Interpretation:

$$\mathbb{P}(m) = \sum_{\mathbf{u}} W_{E_m}(\mathbf{u}) W_\rho(\mathbf{u}) \approx \sum_{\mathbf{u}} \mathbb{P}(m|\mathbf{u}) \mathbb{P}(\mathbf{u})$$

Example of classical simulation



- input:

$$\begin{bmatrix} 1/4 & 0 & 0 & 1/4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/4 & 0 & 0 & 1/4 \end{bmatrix}$$

- gate $XI = Z^{(00)}X^{10}$ = translation of (00|10):

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1/4 & 0 & 0 & 1/4 \\ 1/4 & 0 & 0 & 1/4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- gate HH = permute X and Z:

$$\begin{bmatrix} 0 & 1/4 & 1/4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1/4 & 1/4 & 0 \end{bmatrix}$$

- gate CZ = $(ab|cd) \to (a+b, b|c, c+d)$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1/4 & 1/4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1/4 & 1/4 \end{bmatrix}$$

Extention of Gottesman-Knill theorem

The classical simulation algorithm:
Input: $\rho = \otimes_{i=1}^{n} \rho_i$ with $W_{\rho_i} \geq 0$.
Output: Output distribution of the circuit.
1. Pick u with proba $W_\rho(\mathbf{u})$
2. Update $\mathbf{u}$ after Clifford gates
3. Return m with proba $W_{E_m}(\mathbf{u})$ when the state is $\mathbf{u}$
4. Repeat

- Key ingredients: $W_\rho \geq 0$ and $W_{E_m} \geq 0$.
- Can simulate mixed states.
- Can simulate non-stabilizer states.

Theorem[3]: Clifford circuits with inputs $\rho = \otimes_{i=1}^{n} \rho_i$ such that $W_{\rho_i} \geq 0$ are efficiently classically simulable.

[3] Veitch, Ferrie, Gross, Emerson - NJP 2012

Equivalence with contextuality

Theorem[4]: $W_\rho \geq 0$ iff ρ is non-contextual for stabilizer measurements.

- Technical and non explicit proof.
- For one quopit only. For odd prime dimension only.

[4] Howard, Wallman, Veitch, Emerson - Nature 2014

Equivalence between negativity and contextuality

Hidden variable model

Problem: Can we explain classicaly the randomness in quantum measurements?

Idea: Extend the description of the quantum state ρ:

- **State:** ρ is a probabilistic mixture

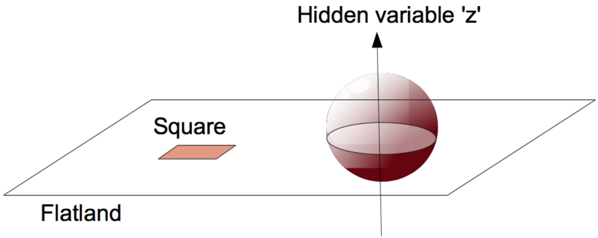
$$\rho = u \quad \text{with proba } \mathbb{P}(u)$$

  where u is the state of the hidden variable model.
- **Measurement:** In position u, every observable A has a fixed value $\lambda_u(A)$.
- **Prediction:** Reproduces quantum mechanics

$$< A >_\rho = \sum_u \mathbb{P}(u)\lambda_u(A)$$


Flatland

Flatlant



Can we properly define a pre-existing value $\lambda_u(A)$ for every observable A?

- **Non-contextuality**:

$$[A, B] = 0 \Rightarrow \lambda_u(A)\lambda_u(B) = \lambda_u(AB).$$

Value assignments

$$\lambda_\nu : T_u \longmapsto \lambda_\nu(T_u) \in \mathbb{C}$$

- Non-contextuality means

$$[u, v] = 0 \quad \Rightarrow \quad \lambda_\nu(T_u T_v) = \lambda_\nu(T_u)\lambda_\nu(T_v)$$

- Character means

$$\lambda_\nu(T_u T_v) = \lambda_\nu(T_u)\lambda_\nu(T_v)$$

Value assignments

$$\lambda_\nu : T_u \longmapsto \lambda_\nu(T_u) \in \mathbb{C}$$

- Non-contextuality means

$$[u, v] = 0 \quad \Rightarrow \quad \lambda_\nu(T_u T_v) = \lambda_\nu(T_u)\lambda_\nu(T_v)$$

- Character means

$$\lambda_\nu(T_u T_v) = \lambda_\nu(T_u)\lambda_\nu(T_v)$$

Lemma: If d is odd and $n \geq 2$, then value assignments on $(\mathbb{C}^d)^{\otimes n}$ are characters.

Elementary proof of the equivalence

Let us prove that if ρ has a NCHVM then $W_\rho \geq 0$.

$$
\begin{aligned}
W_\rho(u) &= d^{-n}\,\mathrm{Tr}(A_u \rho) && \text{def. of } W_\rho \\
&= d^{-2n} \sum_{v\in V} \omega^{[u,v]}\,\mathrm{Tr}(T_v \rho) && \text{linearity of Tr} \\
&= d^{-2n} \sum_{v\in V} \omega^{[u,v]} \sum_{\nu\in S} \lambda_\nu(T_v) q_\rho(\nu) && \text{def. of the HVM} \\
&= d^{-2n} \sum_{\nu\in S} \left(\sum_{v\in V} \omega^{[u,v]} \lambda_\nu(T_v) \right) q_\rho(\nu) && \text{permute the sums}
\end{aligned}
$$

Since $\omega^{[u,\cdot]}\lambda_\nu$ is a character $\sum_{v\in V} \omega^{[u,v]}\lambda_\nu(T_v)$ is either 0 or d^{2n}. This proves that

$$W_\rho(u) \geq 0$$

The Wigner function is the only NCHVM

If (S, q_ρ, λ) is a NCHVM for ρ then there exists an bijective map $\sigma : S \to V$ such that

$$q_\rho(\nu) = W_\rho(\sigma(\nu))$$

for all $\nu \in S$.

The Wigner function is the only NCHVM

- Recover Howard et al.
- Extention to any odd dimension
- Extention to multiple qudits
- Description of all NCHVM

Quopit summary

To have a quantum speed-up with quopits:

- Negativity is required
- Contextuality is required
- They are equivalent (difficult to prove)

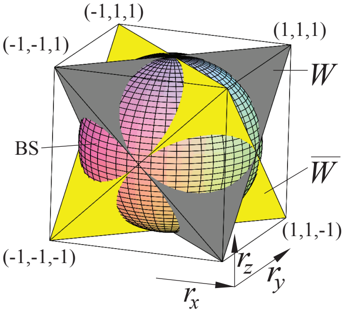
Problems with qubits:

- For ≥ 2 qubits, every state is contextual
- No Hudson theorem
- No Clifford covariance

Qubits Wigner functions

The problem with qubits

- There is no good choice of Wigner function for n-qubits
- For a single qubit, there is a NCHVM
- For more than 2 qubits, every state is contextual



Assume that we can define the values

$$\begin{array}{cc} a & b \\ X \otimes I & I \otimes X \\ \\ c & d \\ I \otimes Z & Z \otimes I \end{array}$$

Mermin square

Assume that we can define the values

a	b	ab
$X \otimes I$	$I \otimes X$	$X \otimes X$
c	d	cd
$I \otimes Z$	$Z \otimes I$	$Z \otimes Z$

Mermin square

Assume that we can define the values

a	b	ab
$X \otimes I$	$I \otimes X$	$X \otimes X$
c	d	cd
$I \otimes Z$	$Z \otimes I$	$Z \otimes Z$
ac	bd	
$X \otimes Z$	$Z \otimes X$	

Mermin square

Assume that we can define the values

a	b	ab
$X \otimes I$	$I \otimes X$	$X \otimes X$
c	d	cd
$I \otimes Z$	$Z \otimes I$	$Z \otimes Z$
ac	bd	
$X \otimes Z$	$Z \otimes X$	$Y \otimes Y$

Mermin square

Assume that we can define the values

$$\begin{array}{ccc} a & b & ab \\ X \otimes I & I \otimes X & X \otimes X \\ & & \\ c & d & cd \\ I \otimes Z & Z \otimes I & Z \otimes Z \\ & & \\ ac & bd & \\ X \otimes Z & Z \otimes X & Y \otimes Y \end{array}$$

This implies $\lambda_u(Y \otimes Y) = abcd = -abcd$.

- Every state ρ is contextual!
- Contextuality cannot be a resource

The problem with qubits

By analogy with quopits, we must pick a phase $\varphi(\mathbf{u})$ for

$$T_u = \varphi(\mathbf{u}) Z^{\mathbf{u}_Z} X^{\mathbf{u}_X}.$$

For quopits
$\varphi(\mathbf{u}) = \omega^{-\frac{(\mathbf{u}_Z|\mathbf{u}_X)}{2}}$ But:

- We cannot divide by 2.
- For quopits: $T_u T_v = T_{u+v}$ when they commute. This is impossible for qubits.
- No Hudson's theorem.
- No Clifford covariance

Qubit Wigner functions

- Pick a phase for each Pauli:

$$T^{\gamma}_{\mathbf{u}} = i^{\gamma(u)} Z^{\mathbf{u}_Z} X^{\mathbf{u}_X},$$

where $\gamma : V \to \mathbb{Z}_4$.

- Then define $W^{\gamma}_{\rho}(u) = \frac{1}{2^n} \operatorname{Tr}(A_u \rho)$ where

$$A_{\mathbf{u}} = \frac{1}{2^n} \sum_{\mathbf{v} \in V} (-1)^{[\mathbf{u},\mathbf{v}]} T^{\gamma}_{\mathbf{v}}$$

The rebit example

Pick $\gamma(u) = 0$ for all u. Then

$$T_{\mathbf{u}} = Z^{\mathbf{u}_Z} X^{\mathbf{u}_X} \qquad A_{\mathbf{u}} = \frac{1}{2^n} \sum_{(\mathbf{v}_Z|\mathbf{v}_X)=0} (-1)^{[\mathbf{u},\mathbf{v}]} T_{\mathbf{v}}$$

and $W_\rho(\mathbf{u}) = \frac{1}{2^n} \operatorname{Tr}(A_{\mathbf{u}}\rho)$.

- Hudson: $W_{|\psi\rangle\langle\psi|} \geq 0$ iff it is a real stabilizer states (CSS).
- Gates preserving real states $\leftrightarrow$ linear maps.
- Real measurements preserve non-negativity.
- No Peres-Mermin square.

Theorem[5]: Negativity of this Wigner function is required for a quantum speed-up with rebits (real states).

[5] Delfosse, Allard, Bian, Raussendorf - Phys. Rev. X 2015

Quantum computing scheme associated with γ

Pick γ and define W^γ. The corresponding scheme is based on

- Input states = $|\psi\rangle$ such that $W^\gamma_{|\psi\rangle\langle\psi|} \geq 0$
- Measurements = Pauli measurements that do not introduce negativity in W^γ
- Gates = preserve the set of measurements
- Inject magic states for universality

Theorem[6]: Negativity of magic states for W^γ is required for a quantum speed-up.

[6] Raussendorf, Browne, Delfosse, Okay, Bermejo-Vega - arxiv:1511.08506

Classical simulation from NCHVM

New approach: Directly use the HVM for classical simulation.

Given a NCHVM for all γ-measurements for a state ρ.
The NCHVM can be updated

- after a γ-gate
- after a γ-measurement

Theorem[7]: Contextuality for γ-measurements is required for a quantum speed-up and for universality.

Assumption: The HVM can be sampled from efficienty.

[7] Raussendorf, Browne, Delfosse, Okay, Bermejo-Vega - arxiv:1511.08506

Measurement update

After measurement of $T_a \in \mathcal{O}$ update the NCHVM by:

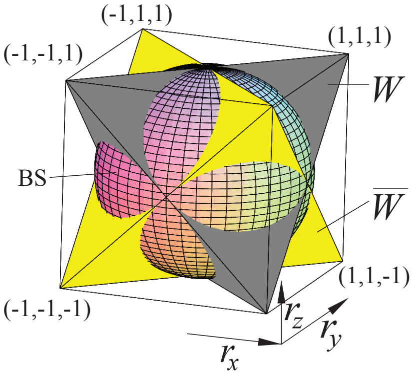
- $S' = S \cup (S + a)$,
- $\lambda_{\nu+a}(T_v) = \lambda_\nu(T_v)(-1)^{[a,v]}$
- $q_{\rho'}(\nu) = \frac{\delta_{\lambda_\nu(T_a),(-1)^s}}{\mathbb{P}(s)} \cdot \frac{q_\rho(\nu)+q_\rho(\nu+a)}{2}$.

Inspired by Wigner functions update:

$$W_{\rho'}(u) = \frac{\delta_{(-1)^{[u,a]},(-1)^s}}{\mathbb{P}(s)} \cdot \frac{W_\rho(u) + W_\rho(u+a)}{2}$$

Neagtivity is not a resource

- We can simulate gates that introduce negativity.
- The negativity can be large but of a particular form.
- Comparison with Pashayan, Wallman, Bartlett?



Theorem[8]: ρ has a NCHVM iff $\rho = \sum_i p_i \rho_i$ such that ρ_i has a non-negative Wigner function $W_{\rho_i}^{\gamma_i} \geq 0$.

[8] Raussendorf, Browne, Delfosse, Okay, Bermejo-Vega - arxiv:1511.08506

Conclusion

Overall

- Quopits are simple, the qubit case is still partially open!

Next step:

- Prove that negativity is sufficient for quopits.
- Classify our QC schemes.
- Other models of computation.
- Continous variable.