Multi-qubit Clifford orbits fail gracefully to be spherical 4-design

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arXiv:1609.08172
arXiv:1609.08595
arXiv:1610.08070

see also: Jonas Helsen, Joel Wallman, Stephanie Wehner
arXiv:1609.08188

Outline

- Introduction: 4-designs
- Justification: 4th moments do matter:
    - randomized benchmarking
    - distinguishing quantum states
    - state tomography via compressed sensing
- Technical part: 4th moments of Clifford orbits
- Implications:
    - distinguishing quantum states
    - state tomography via compressed sensing
    - entropic uncertainty relations

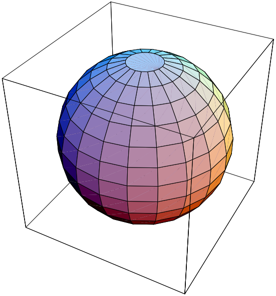
Spherical t-designs

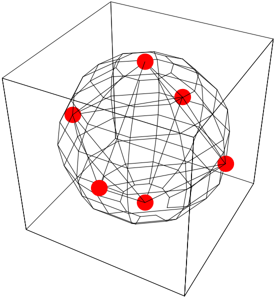
- Many results in quantum info rely on randomized constructions/analysis
- Haar-random states/measurements obey

$$\mathbb{E}\left[(|\psi\rangle\langle\psi|)^{\otimes k}\right] \propto P_{\mathrm{Sym}^k} \quad k \in \mathbb{N} \qquad (1)$$

Definition 1 (Spherical designs)

A t-design $\{\psi_i\}_{i=1}^{N} \subset \mathbb{C}^d$ obeys Eq. (1) up to $k = t$.





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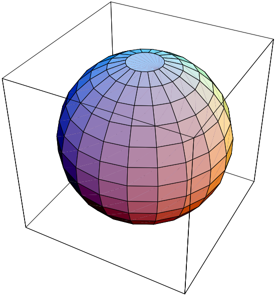

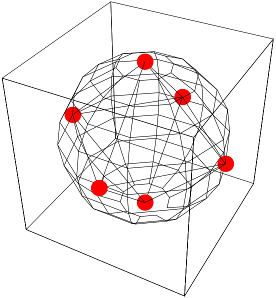

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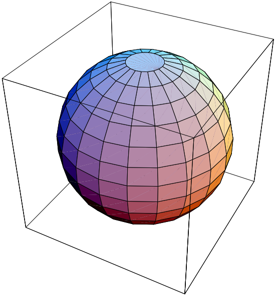

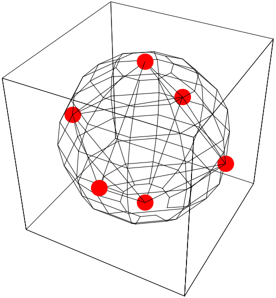

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- SICs, MUBs, Clifford orbits form 2-designs
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- 2-design property of Clifford group essential for twirling

⇒ error channel reduces to depolarizing channel

- 4th moments allow to control variance

⇒ better concentration

⇒ substantial improvement in sequence length, cf. Joel's talk, Stephanie's talk

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- **Task:** distinguish two pure quantum states with a single measurement
- Helstrom's theorem:

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- **Twist:** (Ambainis, Emerson; Matthews, Wehner, Winter) fix a POVM $\mathcal{M}$:

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- **Task:** recover rank-r states ρ of d-level systems ($r \ll d$)
- Construct a POVM $\mathcal{A}$ that contains $N \gtrsim rd\log(d)$ random elements of a 4-design
- w.h.p any rank-r ρ can be recovered from frequencies $f = \mathcal{A}(\rho)$ via solving (RiK, Rauhut, Terstiege)

$$\underset{Z \geq 0}{\text{minimize}} \|\mathcal{A}(Z) - f\|_{\ell_2}$$

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4th moments of Clifford orbits

- 4-designs often yield essentially optimal results
- No concrete/nice examples of 4-designs
- multi-qubit Clifford orbits ($d = 2^n$):

$$\{\psi_i\}_{i=1}^N = \{C|\phi\rangle : \; C \in \mathrm{Cl}(d)\}$$

- $|\phi\rangle = |0\rangle^{\otimes n} \Rightarrow$ stabilizer states
- these orbits form 3-designs (Zhu, Webb, RiK, Gross)

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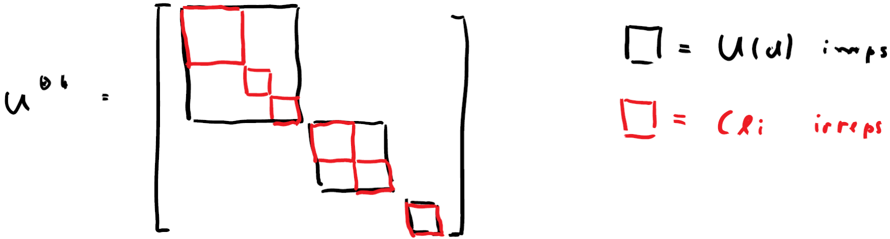
Technical part

$$\mathbb{E}\left[(|\psi\rangle\langle\psi|)^{\otimes 4}\right] \simeq \sum_{C \in \mathrm{Cl}(2^n)} C^{\otimes 4} (|\phi\rangle\langle\phi|)^{\otimes 4} \left(C^{\dagger}\right)^{\otimes 4}$$

$$\overset{\text{Schur}}{=} \sum_{\alpha} c_{\alpha} P_{\alpha}$$

$\Rightarrow$ find irreps in $C \mapsto C^{\otimes 4}$ of $\mathrm{Cl}(2^n)$



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- **Task:** find invariant subspaces under $C \mapsto C^{\otimes 4}$
- key insight: $i^4 = (-1)^4 = (-i)^4 = 1^4 = 1$

$\Rightarrow$ $\left[P^{\otimes 4}, Q^{\otimes 4}\right] = 0$ for all Pauli's $P, Q \in \mathrm{P}(d)$ (Pauli matrices)

$\Rightarrow$ $\left\{P^{\otimes 4} : P \in \mathrm{P}(d)\right\}$ defines stabilizer code

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Main technical result

Theorem 2 (Zhu, RiK, Grassl, Gross 2016)

- *All irreps of the 4th tensor power of the Clifford group are obtained by intersecting an irrep of $U(d)$ with V, or $V^{\perp}$.*
- *the stabilizer group of V is given by the 4th tensor power of all Pauli matrices.*

- **Corollary:** Every Clifford orbit $\{\psi_i\}_{i=1}^{N} = \{C|\phi\rangle : C \in \mathrm{Cl}(d)\}$ obeys

$$\mathbb{E}\left[(|\psi\rangle\langle\psi|)^{\otimes 4}\right] = \alpha_1(\phi)P_1 + \alpha_2(\phi)P_2 \quad P_1 + P_2 = \mathrm{P}_{\mathrm{Sym}^4}$$

$\Rightarrow$ choosing $\phi \in \mathbb{C}^d$ such that $\alpha_1(\phi) = \alpha_2(\phi)$ results in 4-designs

Main technical result

Theorem 2 (Zhu, RiK, Grassl, Gross 2016)

- *All irreps of the 4th tensor power of the Clifford group are obtained by intersecting an irrep of $U(d)$ with V, or $V^{\perp}$.*
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Applications

- Randomized benchmarking, cf. Joel+Stephanie
- Distinguishing quantum states
- state tomography/compressed sensing
- entropic uncertainty relations

Distinguishing pure quantum states

$$\Pr[\text{success}] \leq \frac{1}{2} + \frac{1}{4}\|\mathcal{M}(\psi - \phi)\|_{\ell_1} \leq \frac{1}{2} + \frac{1}{4}\|\phi - \psi\|_1$$

- $\|\mathcal{M}_{4\mathrm{d}}(\psi - \phi)\|_{\ell_1} \simeq \|\psi - \phi\|_1$ (optimal)
- $\|\mathcal{M}_{2\mathrm{d}}(\psi - \phi)\|_{\ell_1} \simeq \frac{1}{d}\|\psi - \phi\|_1$ (bad)

Theorem 3 (RiK, Zhu, Gross 2016)

Set $d = 2^n$ and let $\mathcal{M}$ be any Clifford orbit (e.g. stabilizer states). Then

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This result becomes worse for highly mixed states.

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State tomography/compressed sensing

Task: recover rank-r states of d-level systems ($r \ll d$)

Theorem 4 (RiK, Zhu, Gross 2016)

Fix $d = 2^n$, $r \leq d$ Let $\mathcal{A}$ be a POVM that contains $N \gtrsim r^3 d \log(d)$ random elements of a Clifford orbit. Then w.h.p. any rank-r ρ can be recovered from frequencies $f = \mathcal{A}(\rho)$ via solving

$$\underset{Z \geq 0}{\mathit{minimize}} \quad \|\mathcal{A}(Z) - f\|_{\ell_2}.$$

This reconstruction is stable under noise corruption and relaxation of the rank-r constraint.

- For pure states ($r = 1$) the associated sample complexity is optimal up to log-factors.

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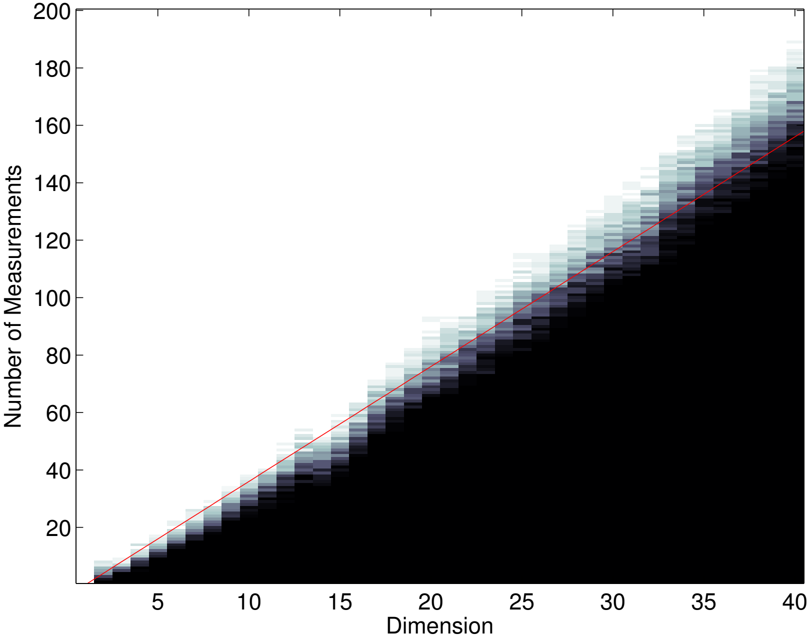
200
180
160
140
120
100
80
60
40
20
Number of Measurements
5
10
15
20
25
30
35
40
Dimension

Entropic uncertainty relations

- A maximal set of MUBs obeys

$$\frac{1}{d+1}\sum_{k=1}^{d+1} H\left(\mathcal{B}_k|\rho\right) \geq \log_2(d+1) - 1$$

- Derivation only uses 2-design property
- **Wehner, Winter:** exploit higher moments?
- Stabilizer states also form ONBs + “kind of” 4-designs

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Entropic uncertainty relations for stabilizer states

- Ansatz:

$$\frac{1}{M}\sum_{k=1}^{m} H(\mathcal{B}_k|\rho) \gtrsim -\frac{1}{\epsilon}\log_2\left(\mathbb{E}\left[\langle\psi_k|\rho|\psi_k\rangle^{1+\epsilon}\right]\right) \quad \alpha = 1+\epsilon$$

- $\langle\psi_k|\rho|\psi_k\rangle$ is a r.v. on $[0,1]$

$\Rightarrow$ replace $\langle\psi_k|\rho|\psi_k\rangle$ by solution of

$$\begin{aligned} \underset{\mu:[0,1]\to[0,1]}{\text{maximize}} \quad & \int_0^1 \mu(x)x^{1+\alpha}\mathrm{d}x \\ \text{subject to} \quad & \int_0^1 \mu(x)x^k = c_k \quad k=1,2,3,4 \\ & \int_0^1 \mu(x)\mathrm{d}x = 1,\ \mu(x)\geq 0 \end{aligned}$$

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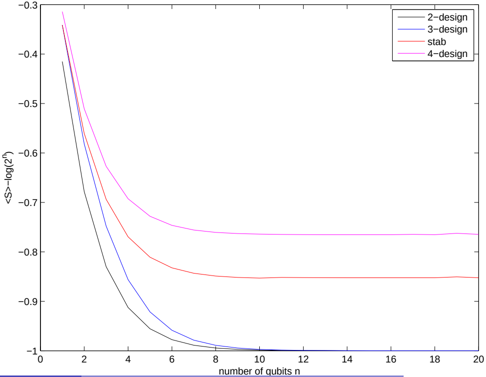
Theorem 5 (RiK, Zhu, Gross 2016)

For $d = 2^n$, stabilizer bases $\{\mathcal{B}_k\}_{k=1}^{M}$ obey

$$\frac{1}{M}\sum_{k=1}^{M} H\left(\mathcal{B}_k|\rho\right) \geq \log_2(d) - c(d) \qquad \lim_{d\to\infty} c(d) \simeq 0.854 < 1.$$



Summary

- We have characterized the 4th moments of Clifford orbits in $d = 2^n$
- The $t = 4$-case really matters!
- Applications include
  - randomized benchmarking
  - quantum state discrimination
  - state tomography (compressed sensing)
  - entropic uncertainty relations