Coherence in logical channels

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Outline

1) Motivation: coherent and incoherent noise
2) Logical noise channels—coherence in stabilizer codes
3) Repetition code calculation
4) The toric code
    1) The theorem
    2) Sketch of proof

Incoherent and coherent noise

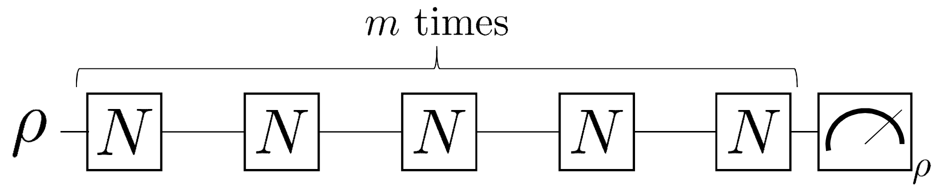
– Incoherent noise means stochastic noise channels, where an error operation is applied with some classical probability.

– By coherent noise, we mean something not incoherent, a channel with some unitary rotation part.

$$\text{Depolarizing:} \quad D_\lambda(\rho) = (1-\lambda)\rho + \tfrac{\lambda}{3}\left(X\rho X + Y\rho Y + Z\rho Z\right)$$

$$\text{Unitary:} \quad U_\theta^X(\rho) = \exp(-iX\theta)\rho\exp(iX\theta)$$

Growth of infidelity



–The average infidelity: $r(N) = 1 - \int_{\text{pure } \rho} \mathrm{Tr}\left(\rho N(\rho)\right) d\rho$

–After m applications of a given noise channel, the average infidelity is given by

$$\begin{aligned} \text{Depolarizing:} \quad r(D^m) &= mr + \text{higher order} \\ \text{Unitary:} \quad r(U_\theta^m) &= m^2 r + \text{higher order} \end{aligned}$$

Diamond distance from identity

–The diamond distance from identity is defined as a max over pure states in a doubled space:

$$|N - id|_{\Diamond} = \max_{\rho} \left|\left((N - id) \otimes id\right)(\rho)\right|_1$$

–The diamond distance from identity is related to the average infidelity differently for coherent and incoherent channels

$$\text{Depolarizing:} \quad \|D_\lambda - id\|_{\Diamond} \quad \propto r$$

$$\text{Unitary:} \quad \left\|Unit_\theta^X - id\right\|_{\Diamond} \quad \propto \sqrt{r}$$

Coherence in channel representations

–Pauli transfer matrix/Liouville representation

$$N(\rho) = N\left(\sum_j \rho_j \sigma^j\right) = \sum_{i,j} N_{i,j}\rho_j\sigma^i$$

$$\{\sigma^i\} \text{ is a basis of } n \text{ qubit Pauli operators}$$

–χ matrix/process matrix representation:

$$N(\rho) = \sum_{i,j} \chi_{i,j}\sigma^i\rho\sigma^j$$

$$(\sigma^i\rho\sigma^j) := \chi_{i,j}$$

–Incoherent components are diagonal in both representations

Error correction

–We will analyze one round of error correction

–We average over syndrome measurements to produce the error correction channel

–We assume perfect syndrome extraction. The errors are all bundled up into the noise channel N

–The logical noise channel is given by

$$\tilde{N} = \text{Decode} \circ N \circ \text{Encode}$$

Logical noise channels

$$\tilde{N} = \text{Decode} \circ N \circ \text{Encode}$$

- Each component of the logical noise channel is a sum of terms from the physical noise channel
- In the χ matrix representation we can write:

Logical χ matrix component $\quad (\tilde{L}_a \tilde{\rho} \tilde{L}_b) = \sum_{s,i,j} (E_s L_a S_i \rho S_j L_b E_s) \quad$ Physical χ matrix terms

$\tilde{L}_a$	Logical *a* operator on encoded qubits	E_s	Standard error for syndrome *s*
$\tilde{\rho}$	State of encoded qubits	L_a	Logical *a* on physical qubits
ρ	State of physical qubits	S_i	Stabilizer operator *i*

Structure of coherent components

– In any stabilizer code, the logical coherent components are given by

$$(\tilde{L}_a \tilde{\rho}) = \sum_{s,i,j} (E_s L_a S_i \rho S_j E_s)$$

– Each of these physical noise terms can be mapped to a logical string

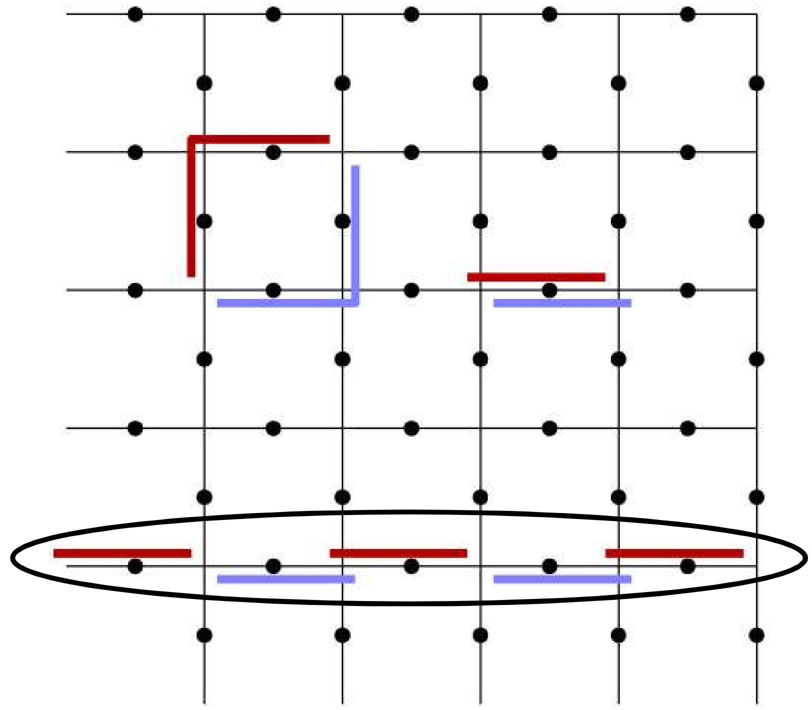
$(E_s L_a S_i \rho S_j E_s)$ ⟶ Logical String: $L_a S_i S_j$

Logical String: $\mathcal{L}$ ⟶ Noise term

$(O_U \rho O_C) : O_U O_C = \mathcal{L}$

Coherent connected part

– We define the connected part of the noise term in the coherent logical noise component:



Connected part of $(E_s L_a S_i \rho S_j E_s)$

$E_s L_a S_i$

$E_s S_j$

Structure of incoherent components

– The noise terms that enter into the incoherent noise components include both coherent and incoherent physical terms

$$(\tilde{L}_a \tilde{\rho} \tilde{L}_a) = \sum_{s,i,j} (E_s L_a S_i \rho S_j L_a E_s)$$

– Each physical noise term maps to two different logical strings, and many noise terms map to the same string

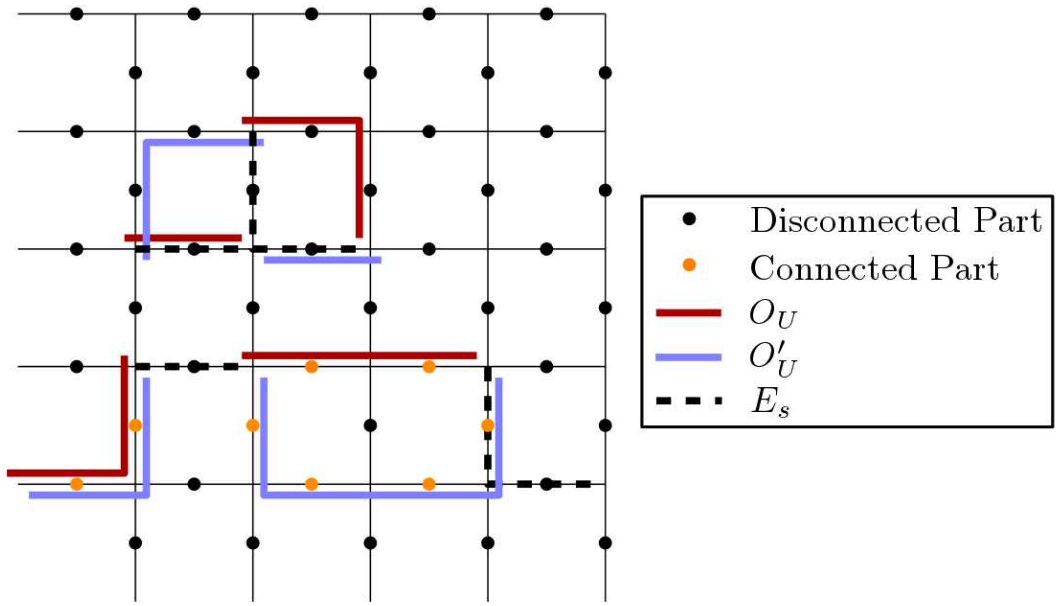
$$(E_s L_a S_i \rho S_j L_a E_S)$$

$$\underline{L_a S_i} \qquad L_a S_j$$

$$\mathcal{L} \longrightarrow O_U \longrightarrow O'_U$$

Incoherent connected part

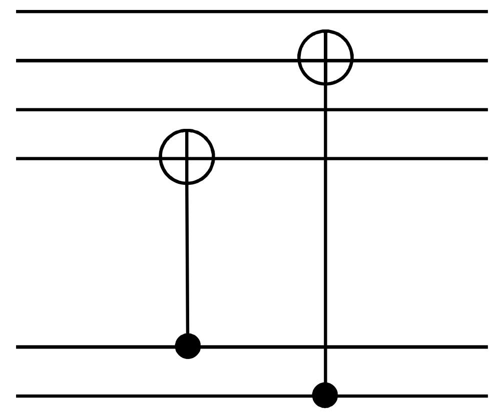
–We again define the connected part. The definition is slightly different for the noise terms that enter into the incoherent logical noise components



The noise model: unitary noise

- We are interested in how error correction transforms coherent noise
- It is easy to show that incoherent (Pauli) channels are mapped to incoherent logical channels
- Therefore, we will study full coherent (unitary) noise channels
- These could be single-qubit or multi-qubit unitaries

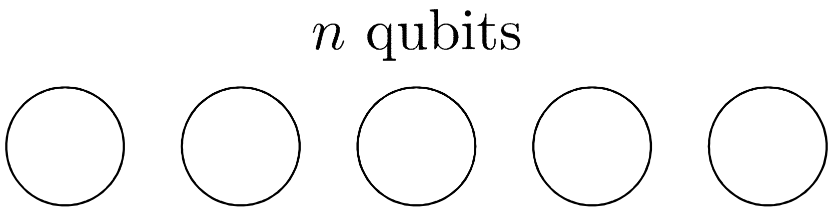
The noise model: ideal error correction



- –We will assume ideal error correction
- –We expect that realistic error correction will add significant noise to any implementation of error correcting codes
- –Most of this noise will not be coherent
- –Exception is single qubit coherent rotations caused by the two qubit entangling gates. This noise fits into the noise model we study

Repetition code calculation 1

–Consider an n qubit bit flip code where n is odd



Check operators $\{Z_i \otimes Z_{i+1}\}$

Logical X operator $L_1 = \bigotimes_{i=1}^{n} X_i$

–Let our noise model consist of single qubit rotations about the X axis

$$U = \cos\theta\, I + i \sin\theta\, X \qquad N(\rho) = U^{\otimes n} \rho\, U^{\dagger \otimes n}$$

Repetition code calculation 2

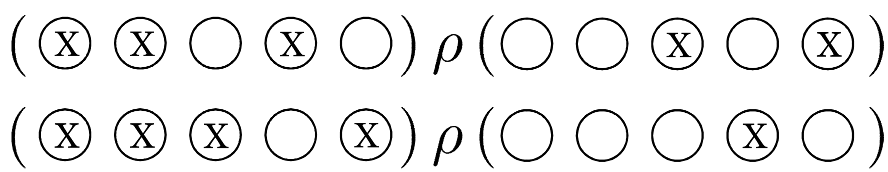
–Compute the coherent logical channel component $\tilde{\chi}_{1,0}$

$$\tilde{\chi}_{X,I} = \sum_s \left(E_s L_x \rho E_s\right)$$

s is a syndrome

L_x is the logical X operator

–Each term in the sum corresponds to a partitioning of the logical operator into two:



Repetition code calculation 3

- Each syndrome and correction is a set of fewer than half of the n qubits. Together with the phases that come from the factors of $i \sin \theta$ in the unitary, we have

$$\tilde{\chi}_{X,I} = \sum_{j=0}^{(n-1)/2} \binom{n}{j} (-1)^j (i \sin \theta \cos \theta)^n$$
$$= \binom{n-1}{\frac{n-1}{2}} i (\sin \theta \cos \theta)^n$$

- Notice that the sum is alternating
- Cancellations are crucial to the suppression of coherence

Repetition code calculation 4

–Now let us compute the incoherent logical channel component

$$\tilde{\chi}_{X,X} = \sum_s \left(E_s L_x \rho L_x E_s\right)$$

–The same logical operator appears on both sides of ρ.

$$\begin{aligned}\tilde{\chi}_{X,X} &= \sum_{j=0}^{(n-1)/2} \binom{n}{j} (\sin\theta)^{2n-2j} (\cos\theta)^{2j} \\ &= \binom{n}{\frac{n-1}{2}} (\sin\theta)^{n+1} (\cos\theta)^{n-1} + \ldots\end{aligned}$$

Repetition code calculation 5

–We have computed exactly the coherent and incoherent components of the logical noise channel. Now compare them:

$$\left(\tilde{L}_1 \tilde{\rho}\right) = \frac{i \cos\theta}{2 \sin\theta} \left(\tilde{L}_1 \tilde{\rho} \tilde{L}_1\right)$$

–As a function of the code size n, the two components are related by a constant. This allows us to prove the following statement about the growth of infidelity:

$$r(\tilde{N}^m) \leq m r(\tilde{N}) + O(r(\tilde{N})^2) m^2$$

Different rotation angles

– Instead of rotating each qubit by a fixed angle θ, we can rotate qubit i by an angle θ_i

– Write coherent and incoherent components as functions of two particular rotation angles, θ_i and θ_j:

$$\tilde{\chi}_{X,I}(\theta_i, \theta_j) = \alpha \sin\theta_i \sin\theta_j$$

$$\tilde{\chi}_{X,X}(\theta_i, \theta_j) = a \sin^2\theta_i/2 \sin^2\theta_j/2 + b(\sin^2\theta_i/2 + \sin^2\theta_j/2) + c$$

– Coherence is maximized when all angles are equal

– Can also rotate by an axis other than the X-axis

Correlations

–We can allow for correlations between qubits

–We use a Hamiltonian to model the correlations

$$H = \sum_k h_1 X_k + \sum_{i,j} h_2 X_i X_j$$

$$U = \exp(-iH)$$

–The same two body term couples every pair of qubits along a logical string

–Coherence is still suppressed in this case

Correlations (cont.)

- Instead of the simple expressions we had for the magnitude and phases of each error, we now have a sum over all possible combinations of one and two body terms

$$\frac{\sin^n \theta}{2} \longrightarrow \left(h_1^n + \sum_{i,j|i<j} h_1^{n-2} h_2 + \ldots \right)$$

- The sum is much more complicated, but it can still be evaluated
- We find

$$\tilde{\chi}_{X,X} > \frac{2n}{n+1} h_1 \tilde{\chi}_{X,I}$$

The toric code

Theorem: For the toric code with minimal-weight decoding, as long as a condition on the single qubit rotation angles is satisfied, then coherent and incoherent components are related by

$$\left(\tilde{L}_1 \tilde{\rho}\right) = \frac{i(-1)^{n+1}\cos\theta}{2\sin\theta}\left(\tilde{L}_1 \tilde{\rho} \tilde{L}_1\right)$$

–Statements about diamond distance from identity and growth of average infidelity follow from this

–Similar statements continue to hold when we have correlated unitary noise

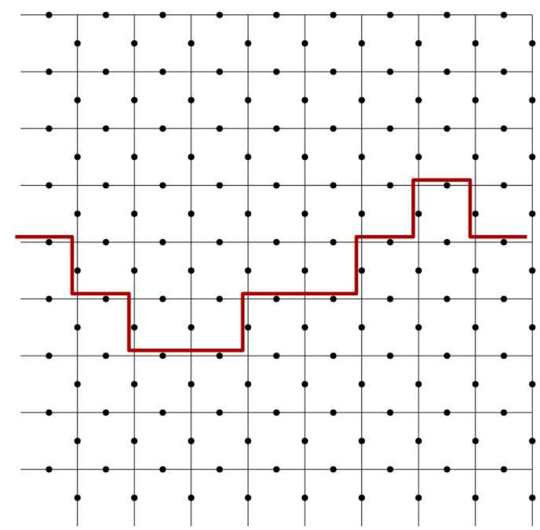
Proof sketch

– The basic plan will be to apply something like the repetition code calculation for each logical string

$$\text{coherent}/\text{incoherent} = \sum_{\mathcal{L}}(\ldots)$$

– The disconnected parts of the syndrome will be factored out

– We will compare coherent and incoherent contributions for each string



An X type logical string in the toric code

Which logical strings?

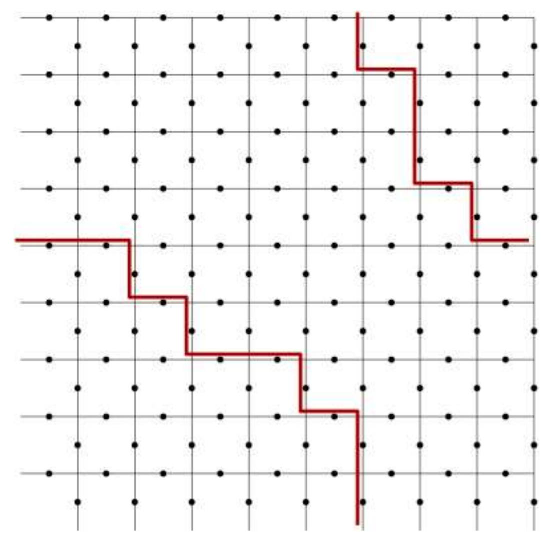
–We can neglect certain logical noise components

$$(\tilde{L}_a \tilde{\rho} \tilde{L}_b)$$

–Neglect $a \neq b$ with both non-trivial

–Neglect a or b logical Y-type operator

–Neglect a or b that act on both encoded qubit



The reason is always that these noise components are much higher order in the local noise strength

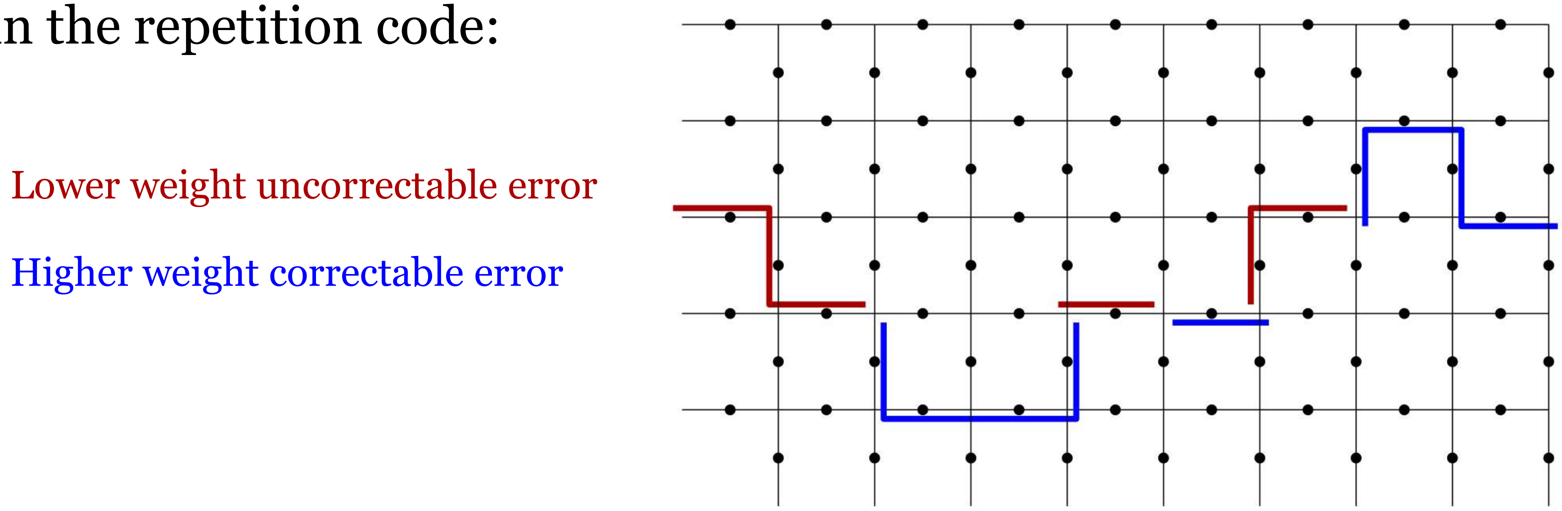
Sum over partitions

– The contribution to $(\tilde{L}_a \tilde{\rho})$ from a logical string $\mathcal{L}$ is $\sum (O_U \rho O_C)$

$$O_U \text{ uncorrectable, } O_C \text{ correctable, } O_U O_C = \mathcal{L}$$

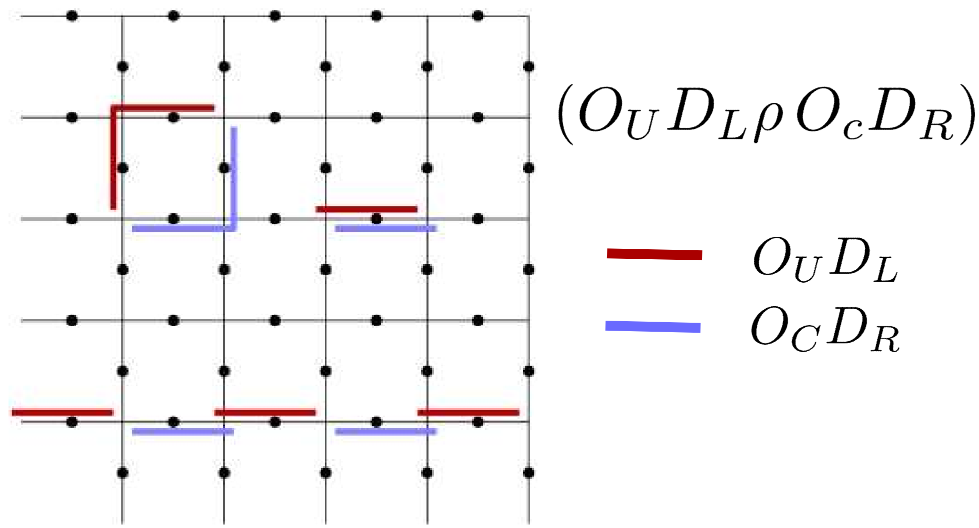
– This is a sum over ways of dividing the logical string into an uncorrectable and correctable error. We call these partitions

– The sum over partitions for the connected logical string is not as simple as in the repetition code:



Disconnected part

- We take a connected noise term and dress it with additional errors to produce a generic noise term
- Consider a partition of a logical string in conjunction with a distant closed loop
- Incoherent-type added errors
- Coherent-type added errors



Phases in incoherent components

–The incoherent components of the logical noise channel now involve physical coherent terms that become incoherent under error correction:

$$\tilde{\chi}_{X_1 X_1} = \sum (O_U D_L \rho O'_U D_R)$$

–Many of the terms on the right are off-diagonal.

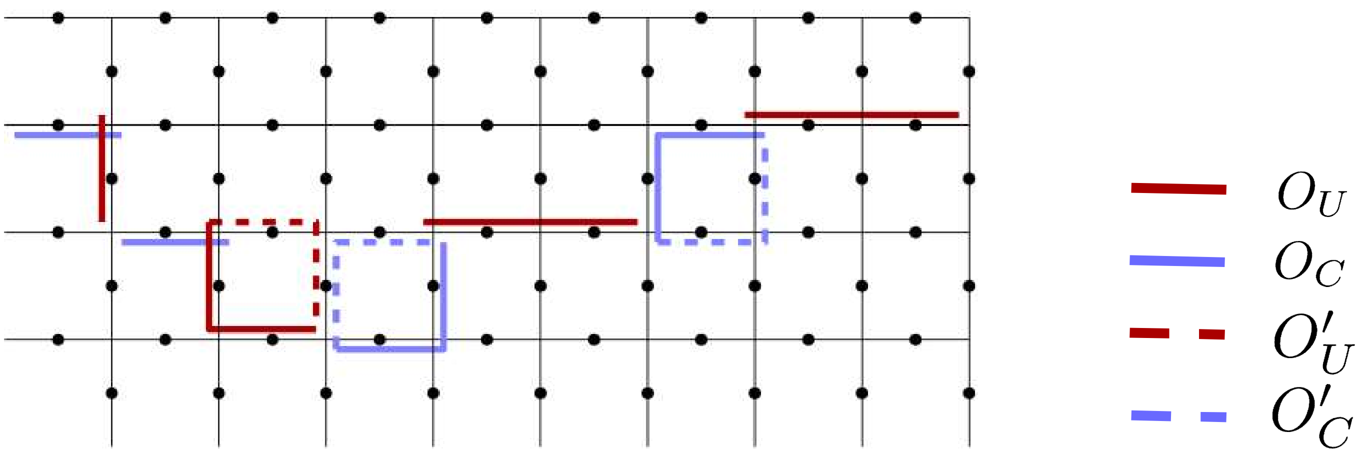
–These terms have different signs

Over-counting factor in incoherent components

– We must lower bound the contribution of each logical string to the incoherent logical noise

– This contribution is given by a combinatorial factor

$$\sum_{\mathcal{L}} \sum_{O_U} \frac{|\{O'_U\}|}{|\{O'_C\}|}$$



Truncation

- The main tool in the proof is to use the path counting expression to truncate the length of logical strings we consider
- If the angle of rotation θ is $< {}^{1}/_{L}$, then we can neglect the strings longer than $L + 2k$ for some constant k
- The error is exponentially small in k
- This is not the most physically relevant case

Arbitrary Angles

- So far we have considered a noise model in which every qubit is rotated by the same unitary
- We can show that this maximizes the coherence of the logical noise channel within a region around the point where all rotations are equal
- Consider the logical coherent and incoherent noise components as functions of the individual rotation angles as we did for the repetition code

Correlations

- Consider the model of correlations we introduced earlier for the repetition code
- We can apply our repetition code calculation with correlations to the short logical strings in the toric code
- The ratio of coherent to incoherent contributions from each logical string is bounded by the same upper bound

Results

- Toric code without boundary
- Minimum weight decoding
- Single qubit unitary noise with equal rotation angles θ

Theorem: Suppose that $\theta < 1/L$ where L is the code size. Then, the following bounds hold:

$$D_{\Diamond}(N - id)^2 \leq c\tilde{r}^2 \text{ for a constant is given by } c \propto \left(\frac{1}{(\sin\theta)^2}\right)$$

Let $\tilde{r}_m$ be the infidelity after m applications, then

$$\tilde{r}_m \leq m\tilde{r}\left(1 + \frac{d_L}{2(d_L+1)\sin\theta} m\tilde{r}\right)$$

- Similar statements continue to hold for correlated unitary noise and for different rotation angles on each qubit within a region.

Remaining difficulties

- –Coherent components: sum over partitions
- –Incoherent components: combinatorial over-counting factor and minus signs
- –Factoring disconnected piece
- –Self-avoiding random walk counting lets us truncate logical strings at length $L(1+\alpha)$
- –Each of these counting problems become more difficult as the length of logical strings increases

Future Work

- Extend our proof to the physically reasonable case where noise strength is constant
- For now, our proof applies only to the toric code with minimum weight decoding. We expect that a similar theorem holds for any stabilizer code and reasonable decoding scheme
- Numerics are probably needed to test how tight our bound is on the logical coherence for a particular code size
- Can we find a more physical model for correlations that is tractable?

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