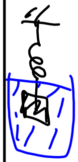
Today - using complex algebra to solve DEs
- Intro to chaos

Examples Damped oscillator



assume damping force $\propto$ speed

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt} \quad (3)$$

Pretend x is Real part of complex fn.

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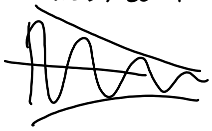
So $x(t) = \mathrm{Re}\left[z(t)\right]$

Solve $m\frac{d^2z}{dt^2} = -kz - b\frac{dz}{dt} \quad (4)$

Guess soln



$$z(t) = A e^{-Bt} e^{i(\omega' t + \phi)}$$

$$= A e^{-Bt + i\omega' t + i\phi}$$

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So $\frac{dz}{dt} = A\underline{(-B+i\omega')}e^{t(-B+i\omega')+i\phi}$

$$= (-B+i\omega')z(t)$$

$$\frac{d^2z}{dt^2} = (-B+i\omega')^2 z(t)$$

$$= \left[B^2 - \omega'^2 - 2iB\omega'\right] z(t)$$

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sub into (4)

$$m(B^2 - \omega'^2 - 2B\omega' i)z + b(-B+i\omega')z + kz = 0$$

LHS=0 provided real & imag. parts of "something" are zero

some algebra $\rightarrow$ [something] $z = 0$

This happens when

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$B = \frac{b}{2m}$

and $\omega'^2 = \frac{k}{m} - \frac{b^2}{4m^2}$ (check)

$\Rightarrow \omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$ as before

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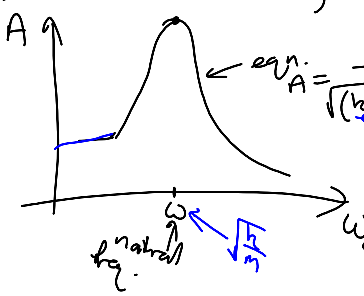
Example periodic driving & damping



eqn. $A = \frac{F_{max}}{\sqrt{(k - m\omega_d^2)^2 + b^2\omega_d^2}}$

driving ω_d

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Intro to Chaos

Everyday use "chaotic" means "unpredictable"

We observe this type of behavior (complex/unpredictable). Sometimes due to lots of components (people, atoms, stars in galaxy)

This is not what we mean by chaos.

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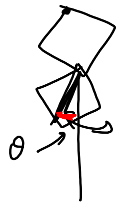
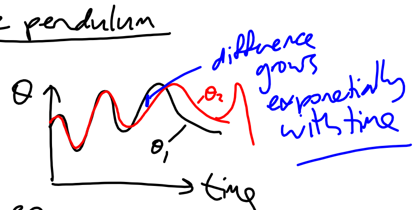
Double pendulum





θ_1 & θ_2 are two different experiments with slightly different initial conditions

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Goal Ⓐ describe chaotic behavior (how do we recognize it? features?)

Ⓑ understand when it will occur. (necessary ingredients?)

Start with Ⓐ

defining feature of chaos is sensitive dependence on initial conditions. – exponentially

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Let's derive A

$$(10) \qquad m\frac{d^2z}{dt^2} = -kz - b\frac{dz}{dt} + F_{max}\,\cancel{\cos(\omega_d t)}\ e^{i\omega_d t}$$

driving

guess soln:

$$z(t) = A\,e^{i(\omega_d t + \phi)}$$

Recall $x(t) = \mathrm{Re}(z(t))$

This is a soln of (10)

Diff. & sub in.

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$$\frac{dz}{dt} = i\omega_d\, z(t)$$

$$\frac{d^2z}{dt^2} = (i\omega_d)^2 z = -\omega_d^2 z$$

$$\Rightarrow \quad z\left(-m\omega_d^2 + ib\omega_d + k - \frac{F_{max}}{A}e^{i\phi}\right) = 0$$

must be zero

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The complex modulus of ______
must be zero

$$\Rightarrow \quad A = \frac{F_{max}}{\sqrt{\text{as before}}}$$

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